Variational Principle for Scattering with Tensor Forces*

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A reformulation of the variational principle in differential form for the phase shift δ_l of a central-force scattering problem is presented. This is considered to be the most simple that can be formed in terms of the inside-wave function representing the difference between the wave function and an appropriately chosen asymptotic form. It is then generalized by means of matrix notation so as to provide corresponding variational principles for the three parameters that arise in scattering with tensor forces, the two phase shifts δ_α, δ_β and the mixture parameter ϵ. These all develop from the differential formulation of Schrödinger's equation and do not depend on the integral equation formulation as does the one originally presented by Schwinger for the phases δ_α and δ_β, and the extension of it by Blatt and Biedenharn to the parameter ϵ.

INTRODUCTION

THE variational principle for the phase shift in scattering problems based upon the integral equation formulation of Schrödinger's equation was first given by Schwinger.[1] This could be applied to problems with either central forces,[2] or tensor forces.[3] Recently, Biedenharn and Blatt[4] have shown how the principle may be extended so as to be applicable to the mixture parameter that arises in the latter case.

Both Schwinger[5] and Hulthén[6] have also introduced variational principles for the phase shift which are based on Schrödinger's equation in differential form with central forces. These are often more feasible for numerical computation than the integral equation formulation, and have in fact been improved upon in recent years.[7–12] In the present note, we wish to give the analogous extension to tensor forces of this differential formulation of the variational principle. This requires three stationary expressions for the three parameters (two phase shifts plus mixture parameter) that arise in this case. These will be found in Secs. II and III.

Before proceeding to this, we first provide in Sec. I a new formulation of the variational prlnciple for the phase shift which is slightly simpler than ones now in use. The point of view adopted will also help indicate why so many formulations have appeared in the literature. The procedure is then generalized in a natural way to tensor forces with the aid of matrix notation.

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I. REFORMULATION OF THE VARIATIONAL PRINCIPLE FOR $k\cot\delta_l$

Our work here is an extension in many ways of the point of view adopted in reference 12, but, for completeness, we repeat the necessary equations and definitions.

The radial part u/r of the Schrödinger wave function for a state of orbital angular momentum l satisfies the equation,

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}+U(r)\right]u(r)=0, \tag{1}$$

with the usual notation. The boundary conditions on $u(r)$ are that

$$\left.\begin{aligned} &u(0)=0,\\ &u(\infty)=\cot\delta_l\sin(kr-\tfrac{1}{2}l\pi)+\cos(kr-\tfrac{1}{2}l\pi)\end{aligned}\right\}. \tag{2}$$

It is desirable to formulate the problem in terms of a wave function which is essentially nonzero only in the region where the potential $U(r)$ is important. To this end, introduce the asymptotic function u_∞ defined by the equation

$$u_\infty=\cot\delta_l F_l+G_{l\infty}, \tag{3}$$

where $F_l=krj_l(kr)$, $G_l=-krn_l(kr)$, with j_l, n_l the well-known spherical Bessel functions. These functions have the required asymptotic property at infinity, i.e.,

$$\begin{aligned}&\lim_{r\to\infty}F_l=\sin(kr-\tfrac{1}{2}l\pi),\\ &\lim_{r\to\infty}G_l=\cos(kr-\tfrac{1}{2}l\pi)\equiv G_{l\infty}.\end{aligned} \tag{4}$$

The use of $G_{l\infty}$ instead of G_l in the definition of u_∞ avoids dealing later on with the objectionable singular behavior of G_l in the neighborhood of the origin. It may be noted that

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right]F_l=0, \tag{5}$$

and that

$$\left[\frac{d^2}{dr^2}+k^2\right]G_{l\infty}=0, \tag{6}$$

while, of course,

$$\lim_{r\to\infty} u = \lim_{r\to\infty} u_\infty.$$

Now introduce the inside-wave function y defined by

$$y = u_\infty - u. \tag{7}$$

Then the equation for y is

$$\frac{d^2y}{dr^2} + k^2 y - \frac{l(l+1)}{r^2}(y - G_{l\infty}) + U(y - u_\infty) = 0. \tag{8}$$

The boundary conditions in terms of y become

$$y(0) = G_{l\infty}(0) = \cos(\tfrac{1}{2}l\pi),$$
$$y(\infty) = 0.$$

Also useful is the equation for $(y - G_{l\infty})$,

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} + U\right](y - G_{l\infty}) - \cot\delta_l U F_l = 0. \tag{9}$$

There are two different expressions that may be formed directly for $k\cot\delta_l$, neither of which is, by itself, stationary. However, most of the variational principles given may be looked on as an appropriate combination of these expressions (an exception is Eq. (10) of Kato's article[10]). Our present point of view is to form them at the very outset in terms of the inside-wave function y so that there will be no explicit dependence on $k\cot\delta_l$ of the expression to be varied. The first of these may be obtained from Eq. (9) by multiplying by $(y - G_{l\infty})$ and integrating. There results in a straightforward manner after some partial integrations, the equation

$$0 = (k\cot\delta_l)B - C, \tag{10}$$

where

$$B = \frac{1}{k}\int_0^\infty dr U F_l (y - G_{l\infty}), \tag{11}$$

$$C = \int_0^\infty dr \left\{ -\left(\frac{dy}{dr}\right)^2 + k^2 y^2 + \left[U(r) - \frac{l(l+1)}{r^2}\right](y - G_{l\infty})^2 \right\}. \tag{12}$$

On the other hand, if we multiply Eq. (9) by F_l, Eq. (5) by $(y - G_{l\infty})$, integrate, and subtract, we obtain after partial integration a second equation for $k\cot\delta_l$:

$$k\cot\delta_l = k\cot\delta_{lB}(1+B), \tag{13}$$

where $k\cot\delta_{lB}$ is just the Born approximation,

$$\frac{1}{k\cot\delta_{lB}} = \frac{1}{k^2}\int_0^\infty dr U F_l^2. \tag{14}$$

Equations (10) and (13) may be looked on as a fundamental set of equations for $k\cot\delta_l$, and, in fact, they were already given for the case of the S-state phase shift ($l=0$) by Hulthén.[8] However, the starting point of most of the variational principles has been to multiply Eq. (8) by $(y - u_\infty)$ and integrate (or, as originally, to work with the equation for u). If, rather, Eqs. (10) and (13) are considered directly, it is observed that they have the important property that their variations with respect to variation in the wave function y are proportional to each other. Thus, from Eq. (10),

$$\delta(k\cot\delta_l) = \frac{\delta C}{B} - \frac{C}{B^2}\delta B.$$

It follows from Eq. (12) that, in view of the differential Eq. (8) for y,

$$\delta C = 2\cot\delta_l \int_0^\infty dr U F_l \delta y = (2k\cot\delta_l)\delta B. \tag{15}$$

Consequently,

$$\delta(k\cot\delta_l) = (k\cot\delta_l)\delta B/B.$$

On the other hand, it follows directly from Eq. (13) that

$$\delta(k\cot\delta_l) = (k\cot\delta_{lB})\delta B.$$

In both cases the variation in $k\cot\delta_l$ is proportional to δB. Therefore, it should not be surprising if we could discover very many combinations of Eqs. (10) and (13) such that the contributions from each to $\delta(k\cot\delta_l)$ cancel each other, i.e., these combinations form stationary expressions. This is indeed the case. The question of choosing among them appears to be merely a matter of selecting the one most feasible for numerical computation. The simplest combination appears to be the following one:

$$k\cot\delta_l = (k\cot\delta_{lB})(1+B)^2 - C. \tag{16}$$

It is easy to verify that, in view of Eqs. (13) and (15), this is a stationary expression for $k\cot\delta_l$. From the present point of view, previously given variational principles for $k\cot\delta_l$ represent somewhat more cumbersome combinations of Eqs. (10) and (13).

II. GENERALIZATION TO TENSOR FORCES

We now wish to extend the foregoing procedure to the case of tensor forces. A general discussion of the scattering theory with spin-orbit coupling has been given in reference 4. It suffices here to restrict ourselves to the mathematical details. Consider a tensor force interaction,

$$V = V_c + V_t(r) S_{12},$$

coupling the $^3S_1$ and $^3D_1$ states of a neutron-proton system, where S_{12} is the usual tensor operator,

$$S_{12} = \frac{3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})}{r^2} - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2.$$

The extension of the analysis for a spin-orbit interaction coupling any two orbital angular momentum states is trivial. Let the $^3S_1$ and $^3D_1$ state radial wave functions be denoted by u and w, respectively.

These are coupled by the following two equations[13]:

$$\frac{d^2u}{dr^2}+k^2u+W_cu+2\sqrt{2}W_tw=0,$$
$$\frac{d^2w}{dr^2}+k^2w-\frac{6}{r^2}w+(W_c-2W_t)w+2\sqrt{2}W_tu=0. \tag{17}$$

The boundary conditions at the origin are

$$u(0)=w(0)=0,$$

while, at $r=\infty$, there are two linear combinations of the asymptotic forms of u and w that are possible. These are denoted by the letters α and β. Thus,

$$\left.\begin{aligned}\lim_{r\to\infty}u&=\cos\epsilon[\cot\delta_\alpha \sin(kr)+\cos(kr)],\\ \lim_{r\to\infty}w&=\sin\epsilon[\cot\delta_\alpha \sin(kr-\pi)+\cos(kr-\pi)],\end{aligned}\right\}\text{solution } \alpha;$$

$$\left.\begin{aligned}\lim_{r\to\infty}u&=-\sin\epsilon[\cot\delta_\beta \sin(kr)+\cos(kr)],\\ \lim_{r\to\infty}w&=\cos\epsilon[\cot\delta_\beta \sin(kr-\pi)+\cos(kr-\pi)],\end{aligned}\right\}\text{solution } \beta.$$

The quantity ϵ is called the mixture parameter as it determines the amount of admixture of pure $^3S_1$ and pure $^3D_1$ waves at infinity.

The problem at hand is to derive variational principles for δ_α, δ_β, and ϵ analogous to Eq. (16). This can be accomplished most easily with the use of matrix notation. The equations will be numbered so as to indicate the corresponding equations of Sec. I. First, introduce the asymptotic functions,

$$u_{\alpha\infty}=\cos\epsilon[\cot\delta_\alpha F_0+G_{0\infty}],\quad u_{\beta\infty}=-\sin\epsilon[\cot\delta_\beta F_0+G_{0\infty}],$$
$$w_{\alpha\infty}=\sin\epsilon[\cot\delta_\alpha F_2+G_{2\infty}],\quad w_{\beta\infty}=\cos\epsilon[\cot\delta_\beta F_2+G_{2\infty}].$$

Furthermore, define inside-wave functions normalized to unity at the origin, i.e.,

$$\cos\epsilon y_\alpha=u_{\alpha\infty}-u_\alpha,\quad -\sin\epsilon y_\beta=u_{\beta\infty}-u_\beta,$$
$$-\sin\epsilon v_\alpha=w_{\alpha\infty}-w_\alpha,\quad -\cos\epsilon v_\beta=w_{\beta\infty}-w_\beta.$$

Then

$$y_{\alpha,\beta}(0)=v_{\alpha,\beta}(0)=1,$$
$$y_{\alpha,\beta}(\infty)=v_{\alpha,\beta}(\infty)=0.$$

Now introduce the vector U by the equation

$$U=\begin{pmatrix}u\\ w\end{pmatrix},$$

and define the scalar product of two vectors U_1 and U_2 by the bracket notation

$$\{U_1,U_2\}=\int_0^\infty dr(u_1{}^*u_2+w_1{}^*w_2).$$

In addition, let

$$F_\alpha{}^\infty=\begin{pmatrix}\cos\epsilon F_0\\ \sin\epsilon F_2\end{pmatrix},\qquad G_\alpha{}^\infty=\begin{pmatrix}\cos\epsilon G_{0\infty}\\ \sin\epsilon G_{2\infty}\end{pmatrix},$$
$$F_\beta{}^\infty=\begin{pmatrix}-\sin\epsilon F_0\\ \cos\epsilon F_2\end{pmatrix},\quad G_\beta{}^\infty=\begin{pmatrix}-\sin\epsilon G_{0\infty}\\ \cos\epsilon G_{2\infty}\end{pmatrix}.$$

Then we may define, in analogy with Eq. (3), the asymptotic vector

$$U^\infty=\cot\delta F^\infty+G^\infty, \tag{3A}$$

with either subscript α or β applicable. Further, let

$$Y=U^\infty-U \tag{7A}$$

or, explicitly,

$$Y_\alpha=\begin{pmatrix}\cos\epsilon y_\alpha\\ -\sin\epsilon v_\alpha\end{pmatrix},\quad Y_\beta=\begin{pmatrix}-\sin\epsilon y_\beta\\ -\cos\epsilon v_\beta\end{pmatrix}.$$

With the following definitions of operators,

$$T=\begin{pmatrix}d^2/dr^2 & 0\\ 0 & d^2/dr^2-6/r^2\end{pmatrix},\quad T_2=\begin{pmatrix}0 & 0\\ 0 & -6/r^2\end{pmatrix},$$
$$W=\begin{pmatrix}W_c & 2\sqrt{2}W_t\\ 2\sqrt{2}W_t & W_c-2W_t\end{pmatrix},$$

Eq. (17) may be written as

$$TU+k^2U+WU=0. \tag{1A}$$

Also,

$$TF^\infty+k^2F^\infty=0, \tag{5A}$$
$$(T-T_2)G^\infty+k^2G^\infty=0. \tag{6A}$$

These last two equations combined yield

$$TU^\infty-T_2G^\infty+k^2U^\infty=0,$$

and consequently, the equation for Y is

$$TY-T_2G^\infty+k^2Y+W(Y-U^\infty)=0. \tag{8A}$$

For convenience, it may be indicated that the equation for $(Y-G^\infty)$ is

$$T(Y-G^\infty)+k^2(Y-G^\infty)+W(Y-G^\infty)-(\cot\delta)WF^\infty=0. \tag{9A}$$

All the operations necessary for obtaining Eq. (16) may now be carried out in completely analogous fashion so that it is sufficient to write down the corresponding equations. Thus,

$$0=(k\cot\delta)B-C, \tag{10A}$$

in which we use the same symbols as before but they

[13] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics*, (John Wiley and Sons, Inc., New York, 1952), Chap. II, Sec. 5D.

have the following extended meaning:

$$B=\frac{1}{k}[F^\infty, W(Y-G^\infty)]. \quad (11A)$$

$$C=[Y, (T-T_2)Y]+[(Y-G^\infty), T_2(Y-G^\infty)] +k^2(Y,Y)+[(Y-G^\infty), W(Y-G^\infty)]. \quad (12A)$$

In addition,

$$k\cot\delta=(k\cot\delta_B)(1+B), \quad (13A)$$

with

$$\frac{1}{k\cot\delta_B}=\frac{1}{k^2}(F^\infty,WF^\infty). \quad (14A)$$

The term in C involving second derivatives of the wave function has been symmetrized by means of a partial integration. The variational principle for the two phase shifts δ_α and δ_β is given by

$$k\cot\delta=(k\cot\delta_B)(1+B)^2-C. \quad (16A)$$

It is to be understood that where a subscript is omitted, either α or β may be inserted. The equation (16A) for $k\cot\delta_\alpha$ is stationary with respect to simultaneous variations of the wave functions y_α, v_α and similarly with β replacing α. It is an analog in differential form of the Schwinger variational principle[1,3] for the phase shifts with tensor forces. It may be easily seen that Eq. (16A) reduces to Eq. (16) if the tensor coupling is zero by letting W_t and $\epsilon\to 0$. Then δ_α reduces to the ordinary S-wave phase shift δ_0 and δ_β becomes the D-wave phase shift δ_2. For the benefit of completeness, we write out the expressions for B and C in long form.

$$B_\alpha=\cos^2\epsilon\cdot\frac{1}{k}\int_0^\infty dr[W_cF_0(y_\alpha-G_{0\infty}) +\tan\epsilon\cdot 2\sqrt{2}W_tF_2(y_\alpha-G_{0\infty}) -\tan\epsilon\cdot 2\sqrt{2}W_tF_0(v_\alpha+G_{2\infty}) -\tan^2\epsilon\cdot(W_c-2W_t)F_2(v_\alpha+G_{2\infty})],$$

$$C_\alpha=\cos^2\epsilon\int_0^\infty dr\left\{-\left(\frac{dy_\alpha}{dr}\right)^2+k^2y_\alpha^2+W_c(y_\alpha-G_{0\infty})^2 +\tan^2\epsilon\left[-\left(\frac{dv_\alpha}{dr}\right)^2+k^2v_\alpha^2 +\left(W_c-2W_t-\frac{6}{r^2}\right)(v_\alpha+G_{2\infty})^2\right] -2\tan\epsilon\cdot 2\sqrt{2}W_t(y_\alpha-G_{0\infty})(v_\alpha+G_{2\infty})\right\},$$

$$\frac{1}{k\cot\delta_{B\alpha}}=\cos^2\epsilon\cdot\frac{1}{k^2}\int_0^\infty dr[W_cF_0^2+2\tan\epsilon\cdot 2\sqrt{2}W_tF_0F_2 +\tan^2\epsilon\cdot(W_c-2W_t)F_2^2],$$

$$B_\beta=\cos^2\epsilon\cdot\frac{1}{k}\int_0^\infty dr[\tan^2\epsilon\cdot W_cF_0(y_\beta-G_{0\infty}) -\tan\epsilon\cdot 2\sqrt{2}W_tF_2(y_\beta-G_{0\infty}) +\tan\epsilon\cdot 2\sqrt{2}W_tF_0(v_\beta+G_{2\infty}) -(W_c-2W_t)F_2(v_\beta+G_{2\infty})],$$

$$C_\beta=\cos^2\epsilon\int_0^\infty dr\left\{\tan^2\epsilon\left[-\left(\frac{dy_\beta}{dr}\right)^2+k^2y_\beta^2 +W_c(y_\beta-G_{0\infty})^2\right]-\left(\frac{dv_\beta}{dr}\right)^2+k^2v_\beta^2 +\left(W_c-2W_t-\frac{6}{r^2}\right)(v_\beta+G_{2\infty})^2 +2\tan\epsilon\cdot 2\sqrt{2}W_t(y_\beta-G_{0\infty})(v_\beta+G_{0\infty})\right\},$$

$$\frac{1}{k\cot\delta_{B\beta}}=\cos^2\epsilon\cdot\frac{1}{k^2}\int_0^\infty dr\{\tan^2\epsilon\cdot W_cF_0^2 -2\tan\epsilon\cdot 2\sqrt{2}W_tF_0F_2+(W_c-2W_t)F_2^2\}.$$

III. VARIATIONAL PRINCIPLE FOR THE MIXTURE PARAMETER ε

In Eq. (16A), it has been assumed that the correct value of the mixture parameter ϵ was inserted in the expressions on the right-hand side. But the value of this parameter is not really known, and we must supplement the stationary equations for δ_α and δ_β with a third stationary expression if we are to render the value of ϵ correct to second order. The possibility of carrying out the same operations as for Eq. (16A) in "mixed" form yields such a stationary expression. Thus, multiply Eq. (9A) for $(Y_\alpha-G_\alpha^\infty)$ by $(Y_\beta-G_\beta^\infty)$ and vice versa, integrate, and add. We obtain in exactly similar fashion to (10A) the following equation:

$$0=(k\cot\delta_\alpha)B_\beta'+(k\cot\delta_\beta)B_\alpha'-2C', \quad (10B)$$

where

$$B_\alpha'=\frac{1}{k}[(Y_\alpha-G_\alpha^\infty), WF_\beta^\infty],$$

$$B_\beta'=\frac{1}{k}[(Y_\beta-G_\beta^\infty), WF_\alpha^\infty],$$

$$C'=[Y_\alpha, (T-T_2)Y_\beta]+k^2(Y_\alpha,Y_\beta) +[(Y_\alpha-G_\alpha^\infty), (W+T_2)(Y_\beta-G_\beta^\infty)],$$

with the term in C' involving second derivatives appropriately symmetrized. In addition, corresponding to Eq. (13A), we obtain

$$k\cot\delta_\alpha=(k\cot\delta_B')B_\alpha',$$
$$k\cot\delta_\beta=(k\cot\delta_B')B_\beta', \quad (13B)$$

in which $k\cot\delta_B'$ is a kind of "mixed" Born approximation,

$$\frac{1}{k\cot\delta_B'}=\frac{1}{k^2}(F_\alpha^\infty,WF_\beta^\infty). \tag{14B}$$

The difference in form of Eqs. (13A) and (13B) arises from the fact that, for Eq. (13B),

$$[F_\beta,\,T(Y_\alpha-G_\alpha^\infty)]-[(Y_\alpha-G_\alpha^\infty),\,TF_\beta]=0,$$

whereas, for Eq. (13A),

$$[F_\alpha,\,T(Y_\alpha-G_\alpha^\infty)]-[(Y_\alpha-G_\alpha^\infty),\,TF_\alpha]=k.$$

Substituting Eqs. (13B) into (10B), we obtain

$$0=(k\cot\delta_B')B_\alpha'B_\beta'-C'. \tag{16B}$$

This expression is stationary with respect to variations in all four wave functions, y_α, y_β, v_α, and v_β, as may be readily verified. Also, it contains no explicit dependence on $k\cot\delta_\alpha$ or $k\cot\delta_\beta$ and consequently may be used to solve for $\tan\epsilon$ (as a function of $\tan\epsilon$, it is a quartic algebraic equation). Therefore, it is proper to consider Eq. (16B) as an implicit stationary expression for $\tan\epsilon$. However, it appears more correct to consider Eqs. (16A) and (16B) as three simultaneous stationary equations for the three scattering parameters. A procedure for using them that suggests itself is to guess a value of $\tan\epsilon$, use Eq. (16A) for obtaining $k\cot\delta_\alpha$ and $k\cot\delta_\beta$, and then use Eq. (16B) as a corrector equation for $\tan\epsilon$, inserting in it the varied wave functions already determined by rendering Eq. (16A) stationary.

We append below the explicit expressions for the primed quantities.

$$\begin{aligned}B_\alpha'=\cos^2\epsilon\cdot\frac{1}{k}\int_0^\infty dr[&-\tan\epsilon\cdot W_cF_0(y_\alpha-G_{0\infty})\\&+2\sqrt{2}W_tF_2(y_\alpha-G_{0\infty})\\&+\tan^2\epsilon\cdot 2\sqrt{2}W_tF_0(v_\alpha+G_{2\infty})\\&-\tan\epsilon\cdot(W_c-2W_t)F_2(v_\alpha+G_{2\infty})],\end{aligned}$$

$$\begin{aligned}B_\beta'=-\cos^2\epsilon\cdot\frac{1}{k}\int_0^\infty dr[&\tan\epsilon\cdot W_cF_0(y_\beta-G_{0\infty})\\&+\tan^2\epsilon\cdot 2\sqrt{2}W_tF_2(y_\beta-G_{0\infty})\\&+2\sqrt{2}W_tF_0(v_\beta+G_{2\infty})\\&+\tan\epsilon\cdot(W_c-2W_t)F_2(v_\beta+G_{2\infty})],\end{aligned}$$

$$\begin{aligned}C'=\cos^2\epsilon\int_0^\infty dr\Big\{&-\tan\epsilon\Big[-\Big(\frac{dy_\alpha}{dr}\Big)\Big(\frac{dy_\beta}{dr}\Big)\\&+k^2y_\alpha y_\beta+W_c(y_\alpha-G_{0\infty})(y_\beta-G_{0\infty})\Big]\\&+\tan\epsilon\Big[-\Big(\frac{dv_\alpha}{dr}\Big)\Big(\frac{dv_\beta}{dr}\Big)+k^2v_\alpha v_\beta\\&+\Big(W_c-2W_t-\frac{6}{r^2}\Big)(v_\alpha+G_{2\infty})(v_\beta+G_{2\infty})\Big]\\&-2\sqrt{2}W_t(y_\alpha-G_{0\infty})(v_\beta+G_{2\infty})\\&+\tan^2\epsilon\cdot 2\sqrt{2}W_t(y_\beta-G_{0\infty})(v_\alpha+G_{2\infty})\Big\},\end{aligned}$$

$$\begin{aligned}\frac{1}{k\cot\delta_B'}=\cos^2\epsilon\cdot\frac{1}{k^2}\int_0^\infty dr\\\times[-\tan\epsilon\cdot W_cF_0^2+(1-\tan^2\epsilon)2\sqrt{2}W_tF_0F_2\\+\tan\epsilon\cdot(W_c-2W_t)F_2^2].\end{aligned}$$

Note that in applying the above expressions to Eq. (16B), the common factor of $\cos^2\epsilon$ may be dropped from all of them.[14]

[14] *Note added in proof.*—The author is indebted to L. Sartori for pointing out to him that T. Regge and M. Verde, Nuovo cimento **10**, 997 (1953) have extended Hulthen's second formulation (see reference 8) to the three independent elements of the K matrix.