# Infrared Divergence in Bound State Problems

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The proof, in the iteration solution, that all infrared divergences exactly cancel for all physical processes and to all orders in the coupling constant, leads to apparent difficulties in bound state problems. These difficulties are resolved.

IN a recent paper by Jauch and the author[1] a proof was presented concerning the cancellation of all infrared divergences in physically meaningful expressions. This cancellation was shown to hold for all types or processes in quantum electrodynamics and to all orders in the coupling constant. Part of the proof consisted in a calculation in the interaction picture which lends itself to a simple physical interpretation of the details of this cancellation. However, when the same physical picture is used for certain bound state problems, the argument seems to fail. This leads to a contradiction, since the infrared divergences must clearly cancel also in bound state problems. In fact, the proof could have been carried out in the bound interaction picture, since a transformation from this picture to the free interaction picture cannot introduce infrared divergences.[2]

## THE PROBLEM

In order to exhibit the difficulty clearly, it is sufficient to consider a typical case, *viz.*, the radiative corrections to the two-photon annihilation of positronium. As is well known, this annihilation can take place only from states of even charge-parity. Furthermore, these same states cannot decay into three photons because of conservation of parity.

But, as was shown in IRD, radiative corrections always give rise to infrared divergences and these are always exactly canceled by soft photon emission in addition to the original process. Applied to our case, this would mean that the radiative corrections to two-photons annihilation are infrared divergent, and that these divergences are exactly canceled by the annihilation into three photons, one of which is very soft. But since the latter process is forbidden by selection rules, this infrared divergence seems to remain uncompensated.

Similar difficulties arise with the compensation of the radiative corrections of three-photon annihilation by four-photon annihilation including one very soft photon, etc.

Clearly, there is no difficulty in pair annihilation in flight, since then the parity is not a good quantum number and no such selection rules arise.

## THE SOLUTION

The soft photon emission gives rise to a divergence which can be expressed as a factor $b$ multiplying the original transition probability [see IRD, Eq. (5)],

$$b=\frac{e^2}{(2\pi)^3}\sum\int\frac{d^3k}{2\omega}\left|\frac{p\cdot e}{p\cdot k}-\frac{p'\cdot e}{p'\cdot k}\right|^2, \tag{1}$$

where $p$ and $p'$ are the initial and final momenta of the emitting electron path, and the summation is to be carried out over the two *transverse* polarization directions of the emitted soft photon.

The divergence in $b$ is exactly canceled by the term

$$r=2(\rho_1+\rho_2), \tag{2}$$

which arises from the radiative corrections. One finds [see IRD Eqs. (12) and (20)]:

$$\rho_1=-\frac{i\alpha}{4\pi^3}\sum\int\frac{p'\cdot e p\cdot e}{(p'\cdot k)(p\cdot k)}\frac{d^4k}{k^2} \tag{3}$$

and

$$\rho_2=\frac{\alpha}{\pi}\int\frac{d\omega}{\omega}. \tag{4}$$

In Eq. (3), the summation is to be carried out over all polarization directions of the virtual photon, i.e., over transverse, longitudinal, and scalar photons.

Consider now the nonrelativistic limit of the process in question. If $p$ refers to a negaton and $p'$ to a positon, this limit is exactly the lowest order approximation of a positronium annihilation process. The corresponding state vector of the initial state can be regarded as the first term in an expansion of the exact state vector in free particle states.

In this limit $b$ vanishes, so that $r$ cannot contain contributions from *transverse* photons. That this is indeed the case follows easily from Eq. (3). In the limit only the longitudinal and scalar photons contribute to the sum and yield

$$\lim\rho_1=-\frac{\alpha}{\pi}\int\frac{d\omega}{\omega},$$

which exactly cancels $\rho_2$. We conclude that the infrared divergences which arise from longitudinal and scalar virtual photons in the radiative corrections always cancel *within* $r$.

[1] J. M. Jauch and F. Rohrlich, Helv. Phys. Acta **27**, 613 (1954). This paper will be quoted as IRD.

[2] In this connection it must be noted that convergence problems of the iteration solution are here irrelevant, since the proof must hold in each power of $\alpha$ separately.

It follows, therefore, that the description of positronium in terms of the bound interaction picture, but without radiative corrections, does not involve infrared divergences, since the Coulomb wave functions take the place of the nontransverse photons.

In this picture, the radiative corrections to positronium annihilation can only involve transverse photons. These correspond to retardation effects, and they do not give rise to infrared divergences as a whole.

These arguments show that the absence of the term $b$ because of selection rules is not only *possible* without affecting the correct cancellation of the divergences, but is indeed necessary. There is thus no further contradiction.

It should be remarked that the vanishing of $b$ in the nonrelativistic limit is well known in the three-photon decay of positronium. The cross section for this process *vanishes* when the energy of one of the photons approaches zero. This fact, in turn, may seem to be in contradiction with the substitution law, according to which the matrix elements of the double Compton effect and of three-photon annihilation differ only in the meaning of the momentum four-vectors involved. The cross section for the double Compton effect is known to show an *infrared divergence* when one of the emitted photons vanishes. However, this apparent contradiction is also easily resolved when one observes that, although the matrix elements of these two processes are related by the substitution law, the corresponding cross sections differ in their densities of final states which accounts for the difference.

Finally, a word may be added about bound state problems involving external fields. These take a position in a certain sense intermediate between the scattering problem described by the iteration solution and the positronium problems discussed above. Again, there exist no terms of the type $b$, but instead of them there occur certain diagrams in the expansion in powers of the external field, which may be called higher Born approximations, and which contain infrared divergences. These again exactly cancel the divergences arising from radiative corrections. Examples of this situation are known in calculations of level shifts in electrostatic and magnetostatic fields. The proof of the exact cancellation of infrared divergences in external field problems follows from the proof given in IRD and the remark that external fields are classical limits of quantized fields; external field problems can thus be regarded as special cases of problems involving the interaction of quantized fields only.[3]

I would like to thank Professor L. M. Brown for an interesting discussion, in which he convinced me that some of the implications of the IRD paper are not obvious.

[3] Methods of computation can be devised which differ from the usual Born expansions in external field problems and which avoid the appearance of infrared divergences. The existence of such methods may be thought of as explicit proof of the spurious character of these divergences and their eventual disappearance in the final expressions for these problems, no matter which methods are adopted. Compare R. G. Newton, Phys. Rev. **96**, 523 (1954).