Relation between Multiple Coulomb Scattering and Residual Range in Nuclear Emulsion*

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Measurements of multiple Coulomb scattering were carried out on 101 artificially produced stopping protons from the end of the track to a point where the residual range was 2500μ. These measurements yield a new relation between range and multiple scattering which can be represented by a power law. The mass dependence was derived theoretically and checked by measuring 20 positive pions. The general relationship is:

$$\langle|\eta|\rangle_{\text{Av}} = (19.0\pm0.3)R^{-(0.607\pm0.016)}(M_P/M)^{(0.393\pm0.016)}(t/50)^{\frac{3}{2}},$$

where $\langle|\eta|\rangle_{\text{Av}}$ is the mean absolute sagitta due to multiple scattering, R is the range, t the cell length, M is the particle mass, and M_P the mass of the proton. All lengths are measured in microns. This relationship leads to a set of schemes for making scattering measurements on stopping tracks in order to determine the mass of the particle with the maximal efficiency. The precision possible on an individual track is limited, but a sequence of tracks, each of which need not be too long, can give a very satisfactory precision, free from the systematic errors inherent in ionization measurements.

INTRODUCTION

IN recent years the attempt to clarify the nature and relationships of the various heavy mesons has assumed great importance. The mass of the heavy mesons themselves is one of the important experimentally definable parameters of the particles, and methods for determining this in photographic emulsions are receiving renewed interest at the present time. These techniques had been somewhat neglected ever since the artificial production of pions made possible more powerful techniques for determining the mass of both pions and muons by combining the photographic emulsion with other instruments. The techniques applicable in emulsion normally use the range as one parameter in the two-parameter system needed to specify the particle mass. The other parameter used is then either ionization, measured by photometric track density, gap number density or gap length density, or multiple Coulomb-scattering. The latter method may not be capable of giving as precise statistical information on individual tracks as some of the others, but it has the advantage that it is free from some of the systematic errors due to plate condition and processing techniques that are often inherent in the ionization measurements. Corrections for these effects involving the making of careful calibrations on each individual plate and even on each region of the plate can sometimes be made. The number of cases of the new particles are few enough, and the number of postulated particles involved are numerous enough, to make desirable as many independent measurements on each track as possible.

The method of multiple scattering has received extensive theoretical[1–9] and experimental[5,10–16] treatment as a means of momentum determination, with special relevance to fast tracks. Some measurements have been made on slow tracks,[17] but the information obtained is for only a limited velocity range, and the measurements do not easily lend themselves to accurate mass determination. Bose and Choudhuri[18] used scattering measurements to distinguish mesons from protons. Lattimore used the range-scattering method for quantitative mass determination of pions.[19] More recently Biswas, George, and Peters[20] and Dilworth, Goldsack, and Hirschberg[21] have proposed a scattering scheme for mass determination, using available range-energy relations and the theoretical law of multiple scattering. It is of interest to check the validity of scattering formulas as applied to tracks of stopping particles. This investigation was undertaken to determine directly the range-scattering relation for stopping particles, with special reference to particles in the heavy meson and hyperon mass ranges.

Artificially accelerated protons provided a means of unambiguously identifying the particles without con-

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fusion with other particles, such as deuterons, tritons, and α particles. This condition can only be attained with difficulty in cosmic ray plates with the large number of particles necessary. The positive pions were identified by their decay into muons of definite range. This could also be done in cosmic ray plates, but the probability of having all particles in the plate and the pions still nearly parallel to the surface of the plate is very small. The range-scattering relation was obtained to an accuracy consistent with its possible use in multiple scattering measurements for mass determination. Beyond the points checked by this experiment, the velocity of the particles is high enough that one should be able to reliably use range-energy relationships and the multiple scattering relationships which are well established to extend the measurements to longer tracks.

The relationship thus established can be most effectively used by extracting the maximum information about the multiple scattering from each track. This can be done most effectively by a method of varying the cell-length, so as to make the mathematical expectation of the sagitta the same in all cells. This allows a great increase in the number of cells over the number obtainable by using constant cell lengths. Even with this refinement, the errors involved in a mass measurement on an individual track are comparatively large; however, applied to a small group of tracks, whose lengths need not be excessively long, a very reliable mass estimate can be obtained which is independent of the estimates which include the inherent uncertainties in ionization measurements.

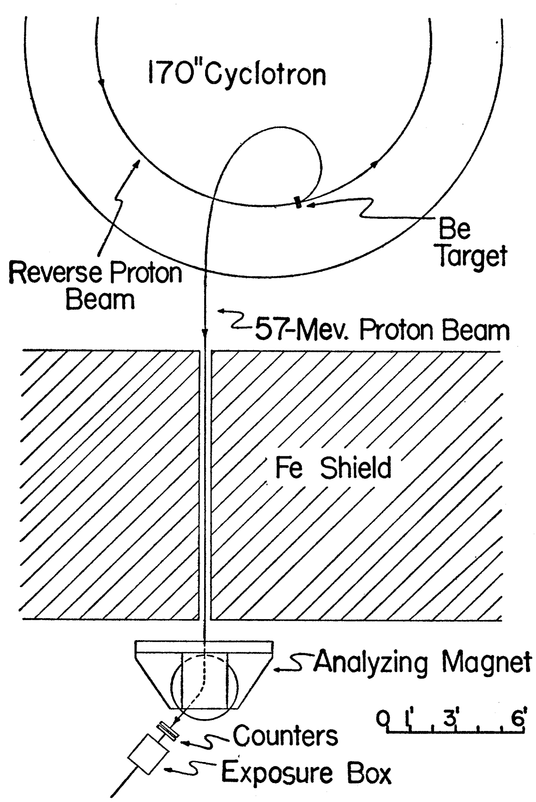

FIG. 1. Schematic drawing of arrangement for exposing plates to 57-Mev protons. Counters and plates not to same scale as cyclotron.

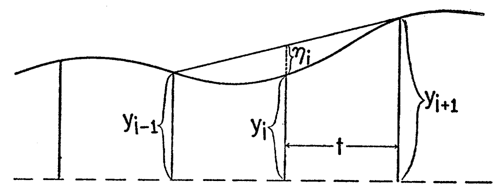

FIG. 2. Projection of a track on the plane of the emulsion. y is the coordinate measured. η is the sagitta, t is the cell length.

EXPERIMENTAL PROCEDURE TO DETERMINE THE SCATTERING RELATION

Protons, scattered off the beryllium target in the beam of the University of Chicago 460-Mev synchrocyclotron were bent first in the fringing field of the cyclotron and then in a "steering" magnet with the current adjusted so that it corresponded to a proton energy of 57 Mev, in an arrangement first attempted by Fermi, Sugarman, and Haber (see Fig. 1).[22] The beam then entered the 600-micron Ilford G-5 emulsion where many of the protons stopped. Since the identity of the particles was uniquely determined by looking for particles which stopped in a region of the emulsion which corresponds to the range of protons of the momentum given by the magnetic field, it was possible to pick out protons unambiguously.

Measurements of multiple Coulomb scattering were carried out on 101 of these protons which stopped in the emulsion and which had a projected length in the emulsion of more than 3000 microns. This criterion for the measured proton tracks does not bias the results, since the scattering in the plane perpendicular to the plate is independent of the scattering in the plane parallel to the plate; only the former can affect the probability of the particle scattering out of the emulsion, while only the latter is measured in this experiment.

For the purpose of establishing accurately the relation between range and multiple scattering, the multiple scattering was measured with a constant cell length, chosen to be 50 microns, which is a cell length such that the scattering all along the track is significantly above noise along the total measured length. The measurements were made from the stopping end of the proton track, back for a distance of 2500 microns. These measurements by the coordinate method were made on the precision scattering stage of Professor Schein's laboratory, which has been used and described previously.[5,23] Figure 2 shows a drawing representing the projection of a section of a track on the plane of the

[22] Fermi, Sugarman, and Haber, University of Chicago, Institute for Nuclear Studies Accelerator Progress Report, 1953 (unpublished).

[23] Lord, Fainberg, and Schein, Phys. Rev. 80, 970 (1950).

emulsion. As indicated in Fig. 2, the coordinate, y_i, of the track perpendicular to the direction of motion of the stage in the plane of the emulsion, is measured at equal intervals of cells. To eliminate the linear trend of the track, which is determined only by the direction in which the plate is lined up, and get a number characteristic of the scattering alone, the sagittae of the track in successive cells,

$$\eta_i = y_i - \tfrac{1}{2}(y_{i-1} + y_{i+1}), \qquad (1)$$

were obtained. From these data it was possible to obtain the correct empirical multiple scattering law for protons, as will be explained later.

To get a check on the mass dependence, similar measurements were carried out for positive pions. These were produced in the 46-Mev positive pion beam of the University of Chicago synchrocyclotron and were unambiguously identifiable as pions by the fact that each had a muon of range 600±30 microns coming from its stopping, with an electron from the end of the muon track. Since these were to be used only as a check on the predicted mass dependence of the new scattering law, 20 pions gave sufficient statistical accuracy. Because pions have less kinetic energy at a given residual range and hence scatter more at the same range than protons, a cell length of only 30 microns was used for these pions compared to the cell length of 50 microns for protons.

We may define the noise as the apparent residual scattering measured, in the absence of any true Coulomb scattering. It is mostly due to error in determination of the actual position of the path of the particle, due to the fluctuations in the position of the grains of the emulsion. For most of the range of the measurements, the true scattering was very large compared with the noise-level. The noise on these tracks was measured by using a very short cell length, 10 microns, such that after a distance of 250 microns from the end of the track the actual scattering in this distance became negligible, and the measurements gave only an estimate of the spurious contributions, which are the noise. The noise level estimate thus obtained was $\langle|\eta_{\text{noise}}|\rangle_{\text{Av}} = 0.09\mu$, as compared with 0.10μ obtained by measuring tracks of particles of such high energy that no true scatter was detectable. It is reasonable to have a lower value for these slow tracks, since the greater grain density means that more information is available to set the position of the microscope filar by. Hence, the value of 0.09μ was adopted and used as a correction for noise.

In all these measurements it was decided to ignore any effect such as the changing dip and inclination of the track to the direction of motion of the stage. The dip angle was necessarily less than 6° and the inclination of the track direction to the direction of motion of the stage was held under 20° at all points. Any variation within this limit was accepted as something to be averaged into all tracks, both those used in determining the range and those whose masses are to be measured by this means, so that any systematic error should automatically cancel out.

ANALYSIS OF THE DATA

In general, the true range-energy relation cannot be exactly represented by any simple analytic expression. In a limited region of ranges and energies it can be represented for a given particle in the form

$$E = kR^{-n}, \qquad (2)$$

where E is the energy, R the range, and k an empirical constant. In a region where the nonrelativistic approximation momentum×velocity$=2E$ is valid, the multiple scattering relation verified at higher energies can be represented for a given particle and a given cell length by

$$\langle|\eta|\rangle_{\text{Av}} = lE^{-1}, \qquad (3)$$

where l is the sagitta of the track and l is approximately constant. Then the direct scattering-range relation is obtained in the form

$$\langle|\eta|\rangle_{\text{Av}} = KR^{-n}, \qquad (4)$$

where K is a new constant. A relation of the same form as (4) may be approximately valid for slow particles even though some of the assumptions used in deriving it may not be exactly correct. For this reason it was decided to try to fit the data for the protons to such an equation. For protons and a cell length of 50 microns, the use of the accepted range energy relation in G-5 emulsion in the form $R=10.6E^{1.68}$, and the scattering constant of 25.0 commonly accepted at higher energies, leads to $K=15.7$, $n=0.595$, for energies measured in Mev and ranges measured in microns.

To fit the data, first the absolute average of the sagittae (η_i) was computed at each range. Following the well-known procedure for applying a cutoff, those values larger than four times the average were eliminated and a new absolute average computed until no sagittae larger than the cutoff remained. By this procedure only 1.5 percent of the data were dropped, but the fluctuations in the mean scattering were greatly reduced. Since each value of $\langle|\eta_i|\rangle_{\text{Av}}$ was the average of 101 independent measurements, the relative statistical error of each value was approximately 10 percent. The noise correction was also made to all values. Let us denote by x_i the quantity $\langle|\eta_i|\rangle_{\text{Av}}$ at the ith position and R_i the range at this position. Then we are trying to fit a set of values of x_i which have a reasonably great precision to a law of the form

$$x_i = KR_i^{-n}. \qquad (5)$$

If we take the logarithm of both sides in Eq. (5), we obtain

$$\log x_i = \log K - n \log R_i. \qquad (6)$$

This represents a familiar problem[24] in linear regression

[24] Harold Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1946), p. 548.

of $\log x_i$ on $\log R_i$. Since the distribution of $\log x_i$ is a rather complicated one, the exact solution is difficult. However, for our purposes this distribution can be approximated well enough by a normal or Gaussian distribution if the distribution of x_i is narrow enough. One has

$$\log x_i = \log E(x_i) + \frac{\log e}{E(x_i)}[x_i - E(x_i)] - \frac{\log e}{2E^2(x_i)}[x_i - E(x_i)]^2 + \cdots, \quad (7)$$

where $E(x)$ represents the mathematical expectation of x. Then

$$E(\log x_i) = \log E(x_i) - \frac{\log e}{2E^2(x_i)} E([x_i - E(x_i)]^2) + \cdots. \quad (8)$$

The second term on the right side of (8) represents a shift in the $\log x_i$ axis of all points by 0.00123 (as computed from the experimental data) which represents the multiplication of all values of x_i by 1.003. A correction for this will be made but all higher terms will be still smaller and hence can be disregarded. From (7) the following relation can be derived:

$$E([\log x_i - E(\log x_i)]^2) = \frac{\log^2 e}{E^2(x_i)} E([x_i - E(x_i)]^2) + \cdots. \quad (9)$$

Relations (8) and (9) will be used later.

Under the assumption that $\log x_i$ is approximately normally distributed, one can then try to fit the data to a law of the form (6), by the method of least squares.[25] The result of this fitting gives

$$\log x_i = 1.275 - 0.607 \log R_i, \quad (10)$$

which by the use of (8) is equivalent to

$$\langle |\eta_i| \rangle_{\text{Av}} = 18.9 R_i^{-0.607}, \quad (11)$$

or, if one keeps in mind the correction factor of 1.003 for the second term in (8) mentioned above, the constant becomes 19.0. We can estimate the errors in the coefficient in (10) by the well known estimates for the linear regression problem, as given by Cramér[24]:

$$s(1.275) = \sigma^* N^{-\frac{1}{2}},$$

$$s(0.607) = \sigma^* N L^{-\frac{1}{2}},$$

$$\sigma^{*2} = \frac{1}{N-2} \sum_{i=1}^{N} (\langle \log x_i \rangle_{\text{Av}} - 1.275 + 0.607 \log R_i)^2, \quad (12)$$

$$L = \frac{1}{N} \sum_{i=1}^{N} (\log R_i - \langle \log R_i \rangle_{\text{Av}})^2,$$

where $s(x)$ is the estimate of the standard deviation in the quantity represented by x. We get from this that the proton range-scattering relation can be expressed in the form

$$\langle |\eta_i| \rangle_{\text{Av}} = (19.0 \pm 0.3) R_i^{-(0.607 \pm 0.016)}. \quad (13)$$

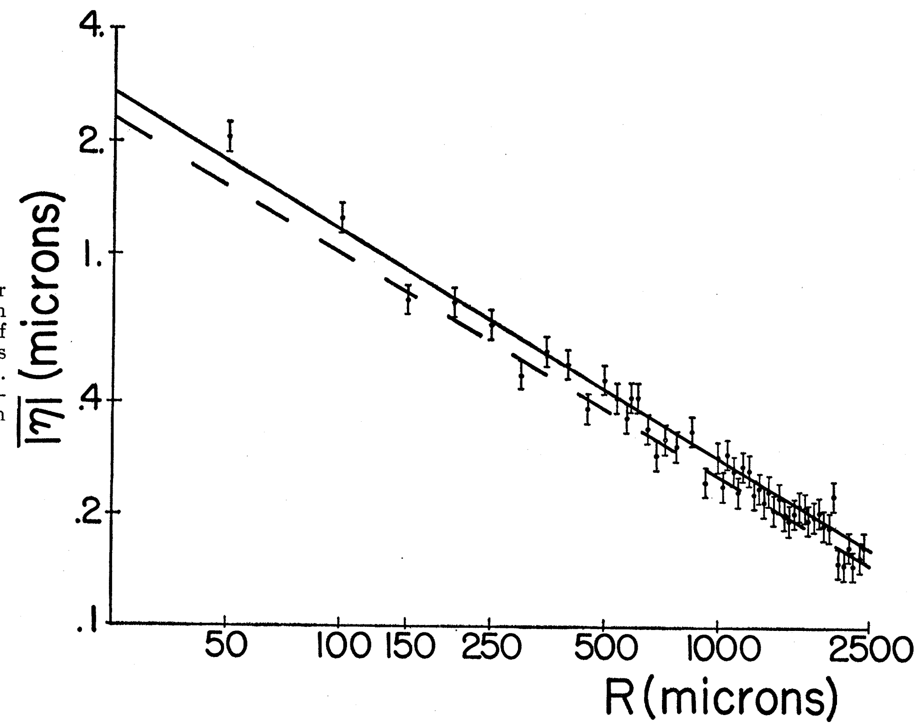

FIG. 3. Scattering *vs* range for protons. Cell length is 50μ. Each point represents the average of 101 measurements. Full curve is the new range-scattering relation. Dashed curve is the old semi-theoretical relation. Errors shown are only the statistical errors.

[25] P. G. Hoel, *Introduction to Mathematical Statistics* (John Wiley and Sons, New York, Inc., 1947), p. 78.

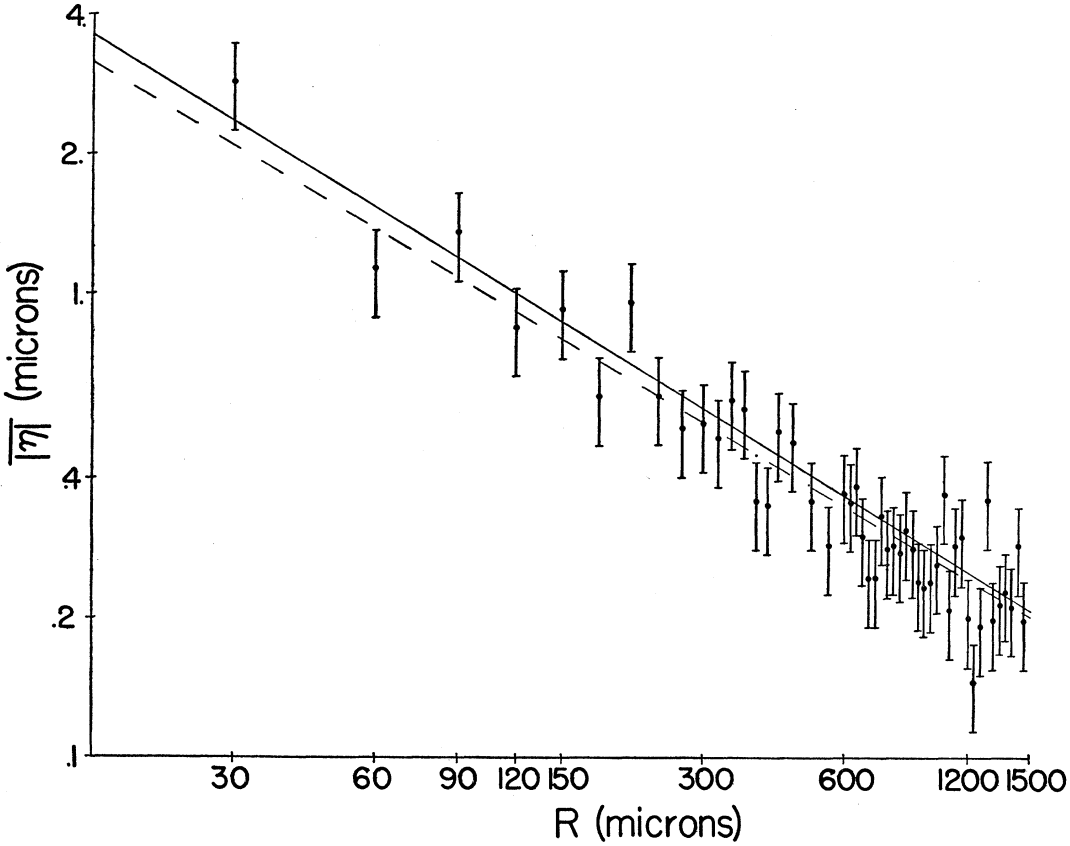

FIG. 4. Scattering *vs* range for pions. Cell length is 30μ. Each point represents the average of 20 measurements. Full curve is the new relation. Dashed curve is the old relation. Errors shown are only the statistical errors.

In Fig. 3 are shown the proton sagittae at each range, the curve obtained from the new relation (13), and the old curve obtained from the equation resulting from the argument given at the beginning of this section.

We can express the general range energy relation in the form

$$R = Mf(v), \tag{14}$$

where $f(v)$ is a function only of the particle velocity v, and M is the mass of the particle. The scattering relation can be expressed to a good approximation in the form

$$\langle|\eta|\rangle_{\text{Av}} = g(v)t^{3/2}/M, \tag{15}$$

where $g(v)$ is another function of velocity alone. Then we have

$$\langle|\eta|\rangle_{\text{Av}} = M^{-1}t^{3/2}k(R/M), \tag{16}$$

from the two preceding relations. If, as we have seen, we can represent the relation for protons with a 50-micron cell length by an equation of the form of (4), we will have for the general relation, since the functional form of $k(x)$ is determined by this relation:

$$\langle|\eta|\rangle_{\text{Av}} = K\left(\frac{M_P}{M}\right)^{1-n}\left(\frac{t}{50}\right)^{3/2}R^{-n}, \tag{17}$$

where M_P is the proton mass. Inserting the values of K and n given by (13), we have

$$\langle|\eta|\rangle_{\text{Av}} = (19.0\pm0.3)\left(\frac{M_P}{M}\right)^{(0.393\pm0.016)} \times\left(\frac{t}{50}\right)^{3/2}R^{-(0.607\pm0.016)}. \tag{18}$$

Formula (18) then should represent, for a given mass, the relation between range and multiple scattering.

The first of Eqs. (12) gives an estimate of the standard deviation of $\log x_i$. Equation (9) gives the relation of this estimate to the relative deviation of x_i (standard deviation of x_i divided by the expectation of x_i). From these two values we get the relative deviation 0.097. Since this is based on 101 measurements at each point, the estimate of the relative deviation of $\langle|\eta|\rangle_{\text{Av}}$, based on N measurements, is $(\sqrt{101})\times0.097/\sqrt{N} = 0.98/\sqrt{N}$. This estimate includes any effect due to correlation of the successive measurements on a given track and is larger than given by treating the η_i as statistically independent quantities.

The mass dependence of Eq. (18) was checked by the measurements on the pions. The data were treated in the same way as for the proton measurements to obtain the sagittae. Figure 4 shows the pion points, the curve predicted by (18) and the curve predicted by the use of the range-energy relation and an extrapolation of the usual multiple scattering law. The fit of the curve obtained from (18) was tested by the method of least squares, and a χ^2 value (M value) of 41 was obtained. If the fit were perfect, a χ^2 value of 49 with a standard deviation of 7 would be expected, so that the fit is very good. Another way of demonstrating that the mass dependence predicted is correct is to determine the mass of the pion from these measurements. The mass ratio of the pion to the proton yielded a value of 0.147 $\pm$0.012, in excellent agreement with the established value of 0.149.[26]

CONSTANT SAGITTA METHOD

Constant cell-length measurements, as used in the establishment of the multiple scattering-range relation will not give as much information from a given track as will the constant sagitta method of varying the cell length. In this method the cell length is varied in such a manner that the expected sagitta due to scattering is in every cell the same. As can be seen from Eq. (18) the relation $t=\text{const}\times R^{0.405}$ will accomplish this, where the constant is chosen to give the desired cell length. The measurement was started at a point of the track such that $R=t_0$, where t_0 is the initial cell length. That is, the cell length-range relation is used in the form

$$t=t_0(R/t_0)^{0.405}, \tag{19}$$

where t_0 is the initial cell size and the first cell begins at a distance t_0 from the end of the track.

To determine the optimum cell size a statistical analysis was made of the error as a function of cell length. The limiting factor in the precision of measurement is the sagitta due to noise, that is, the contribution when the effect due to true scattering is negligible. It was assumed for the purposes of this computation that both noise and true scattering distributions can be represented by independent Gaussian distributions. The non-Gaussian character of the distribution should produce errors which will be negligible compared to the statistical errors involved. It then turns out that, just as for fast tracks, the optimum cell size is that for which $\langle|\eta|\rangle_{\text{Av}}=(\sqrt{6})\langle|\eta_{\text{noise}}|\rangle_{\text{Av}}$, where η is the correct sagitta (due to true scattering plus noise) and η_{noise} is the sagitta due to just noise (as determined in this experiment). From (18) and (19) we get for the true scattering

$$\langle|\eta|\rangle_{\text{Av}}=0.0535(M_P/M)^{0.393}t_0^{0.893}, \tag{20}$$

or, since $\langle|\eta_{\text{noise}}|\rangle_{\text{Av}}=0.09\mu$,

$$t_0=4.4(M/M_P)^{0.44}. \tag{21}$$

For protons, formula (21) gives 4.4μ; for pions, 1.9μ; and for a particle of mass $1000m_e$, 3.4μ as the optimum

[26] Smith, Birnbaum, and Barkas, Phys. Rev. **91**, 765 (1953).

TABLE I. Ranges at which measurements are to be made for constant sagitta measurements and the mass estimates based on these measurements.

1. Initial cell length (t_0) $=8\mu$						
8.0	189.1	550.0	1059.4	1699.3	2458.7	3330.7
16.0	217.9	594.4	1118.3	1769.9	2541.0	3423.9
26.6	248.4	640.2	1178.5	1841.7	2624.4	3518.1
39.6	280.6	687.4	1239.1	1914.7	2708.9	3613.4
54.9	314.5	736.1	1301.0	1988.9	2794.5	3709.8
72.4	350.0	786.2	1364.2	2064.3	2881.2	3807.3
92.0	387.0	837.7	1428.7	2140.9	2968.9	3905.9
113.5	425.5	890.8	1494.5	2218.6	3057.7	4005.7
136.9	465.5	945.5	1561.6	2297.5	3147.6	
162.1	507.0	1001.8	1629.9	2377.5	3238.6	
$M/M_P=(0.344/\langle|\eta|\rangle_{\text{Av}})^{2.54}$						
2. Initial cell length (t_0) $=5\mu$						
5.0	175.2	522.1	1010.0	1622.3	2347.9	3179.5
10.0	196.3	544.9	1052.9	1674.2	2408.2	3247.7
16.6	218.4	588.5	1096.5	1726.8	2469.1	3316.5
24.8	241.4	622.9	1140.8	1780.0	2530.6	3385.8
34.3	265.4	658.1	1185.8	1833.9	2592.7	3455.7
45.2	290.3	694.1	1231.5	1888.4	2655.5	3526.2
57.4	316.2	730.9	1277.9	1943.6	2718.9	3597.3
70.8	343.0	768.5	1325.0	1999.4	2782.9	3668.9
85.4	370.7	806.9	1372.8	2055.9	2847.5	3741.1
101.2	399.3	846.0	1421.3	2113.0	2912.7	3813.9
118.1	428.7	885.9	1470.5	2170.8	2978.5	3887.3
136.1	459.0	926.5	1520.4	2229.2	3044.9	3961.2
155.1	490.1	967.9	1571.0	2288.2	3111.9	
$M/M_P=(0.226/\langle|\eta|\rangle_{\text{Av}})^{2.54}$						
3. Initial cell length (t_0) $=4\mu$						
4.0	174.5	527.7	1026.0	1650.5	2391.1	3229.0
8.0	193.0	556.7	1063.8	1696.3	2444.4	3289.2
13.3	212.3	586.3	1102.1	1742.7	2498.2	3349.8
19.8	232.3	616.5	1141.0	1789.6	2552.4	3410.9
27.4	253.1	647.3	1180.5	1837.0	2607.1	3472.4
36.1	274.6	678.7	1220.6	1884.9	2662.3	3534.4
45.8	296.8	710.7	1261.2	1933.3	2717.9	3596.8
56.5	319.7	743.4	1302.4	1982.2	2764.0	3659.7
68.2	343.4	776.7	1344.1	2031.6	2820.5	3723.0
80.8	367.7	810.6	1386.3	2081.5	2877.5	3786.8
94.3	392.7	845.1	1429.0	2131.9	2934.9	3851.0
108.7	418.4	880.1	1472.3	2182.8	2992.8	3915.7
123.9	444.7	915.7	1516.1	2234.2	3051.2	3980.8
139.9	471.7	951.9	1560.4	2286.0	3110.0	
156.8	499.4	988.7	1605.2	2338.3	3169.3	
$M/M_P=(0.185/\langle|\eta|\rangle_{\text{Av}})^{2.54}$						
4. Initial cell length (t_0) $=3\mu$						
3.0	174.7	532.7	1038.8	1674.4	2428.4	3292.6
6.0	190.2	557.2	1070.8	1713.3	2473.6	3343.7
10.0	206.3	582.1	1103.2	1752.5	2519.1	3395.1
14.8	222.9	607.5	1136.0	1792.1	2565.0	3446.9
20.6	240.1	633.3	1169.2	1832.1	2611.2	3499.0
27.1	257.8	659.5	1202.8	1872.4	2657.7	3551.4
34.4	276.0	686.2	1236.8	1913.1	2704.6	3604.1
42.5	294.7	713.3	1271.2	1954.1	2751.8	3657.1
51.3	313.9	740.8	1306.0	1995.5	2799.3	3710.4
60.8	333.6	768.8	1341.2	2037.2	2847.2	3764.0
71.0	353.8	797.2	1376.7	2079.3	2895.4	3818.0
81.8	374.5	826.0	1412.6	2121.7	2943.9	3872.3
93.3	395.7	855.2	1448.9	2164.5	2992.7	3926.9
105.4	417.4	884.8	1485.6	2207.6	3041.9	3981.8
118.1	439.5	914.8	1522.6	2251.1	3091.4	
131.4	462.1	945.2	1560.0	2294.9	3141.2	
145.3	485.2	976.0	1597.8	2339.1	3191.3	
159.7	508.7	1007.2	1635.9	2383.6	3241.8	
$M/M_P=(0.146/\langle|\eta|\rangle_{\text{Av}})^{2.54}$						

TABLE II.

Particle	Track length used	t_0[a]	No. of cells	Mass (units m_e)
Chicago τ	1040μ	4μ	48	850 ± 400
Brookhaven K^- (1)	2400	3	106	1350 ± 400
	2400	4	93	1350 ± 400
	2400	8	45	1400 ± 500
Brookhaven K^- (2)	3700	3	115	1250 ± 350
	3800	5	86	1000 ± 300
Brookhaven K^+	3300	4	90	1050 ± 300

[a] In the table, t_0 is the initial cell used as explained in the text. The Brookhaven particles referred to were kindly lent to the author by J. Hornbostel and E. O. Salant of Brookhaven National Laboratories [Phys. Rev. **93**, 902 (1954)]. These authors quote mass values for $K^-(1)$ of 1050 ± 150, 1200 ± 300, and 1080 ± 220 from gap-length *vs* range, scattering *vs* range, and magnetic rigidity *vs* range. Hill, Salant, and Widgoff also give mass values for some Brookhaven particles in a later article [Phys. Rev. **95**, 1699 (1954)].

value of t_0. Table I gives a set of values of ranges at which measurements should be made for the first 4000μ of a constant sagitta scheme following Eq. (19) for $t_0=3$, 4, 5, and 8μ.

Application of the method to mass determination on heavy mesons studied in this laboratory is in progress. Some of the mass values obtained are shown in Table II.

Other schemes for constant sagitta measurement which have been proposed, such as those of the Bombay[20] and Brussels[21] groups, have been based on reasoning similar to that following Eq. (4). They give a different dependence of cell length on range for constant sagitta. The difference is not large but may introduce a significant bias into measurements on groups of tracks of various lengths.

CONCLUSION

A new range-multiple scattering relationship in photographic emulsion has been determined which has a precision adequate for use in mass measurements on stopping particles. The scattering at low energies increases beyond the amount predicted by use of a constant scattering factor. We define the scattering factor as

$$\frac{180}{\pi}\frac{p\beta\langle|\eta|\rangle_{\text{Av}}}{t^{\frac{3}{2}}}.$$

If one assumes the range-energy relation represented by $R=10.6E^{1.68}$, as in the first paragraph of the section on "Analysis of the Data," then the scattering factor would increase with decreasing energy and would be about 28 at 18 Mev and approach 30 at very low energies. At the lower energies second order effects, such as the increase in path length due to scattering in a given cell and due to an inclination of the track to the average direction (which determines the position of the base line), certainly affect the value of the scattering factor. An additional factor which must be considered is the question whether the present theories correctly estimate the cross section for single scattering at low energies. The numerical estimate which can be made on the basis of the various theories does not seem to account correctly for the increase in the scattering factor. The fact that the cause of this increase is not understood is, of course, irrelevant to the use of the relation established here for mass measurement.

The precision which can be given by use of a scattering measurement has previously been slightly overestimated by using idealized models to represent the statistical situation. The main contribution to the increased variance in a scattering measurement over that predicted by the simple models is the correlation between successive values of η because they contain a common scattering. There is also some contribution from the non-Gaussian character of the distribution, although most of this is removed through applying the cutoff.

The formulation given in this paper allows one to make mass measurements of reasonable precision, even on the comparatively short tracks which are very common and very hard to deal with. On a long track, as frequently observed in pellicle stacks, it enables one to increase the precision of the measurement since the slower a track is the more information is contained in its scattering per unit length. The precision resulting from various numbers of cells and the corresponding track length are given in Table III.

TABLE III. Percentage statistical error in mass measurements based on multiple scattering on stopping tracks of mass $M=1000m_e$.

Available range (microns)	No. of cells	Error on one track	Error on 25 tracks
1000	45	45%	8.9%
3000	81	33%	6.7%
6000	131	26%	5.3%
10 000	178	23%	4.5%
50 000	350	14%	3.0%

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Note added in proof.—C. C. Dilworth [Nuovo cimento **11**, 203 (1954) discusses the constant sagitta method as used by various groups and compares its effectiveness as estimated by the various authors with other methods.