Passage of Charged Particles through Plasma

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The plasma has been treated phenomenologically as a homogeneous dispersive medium characterized by a "dielectric constant" which is a function not only of the frequency of the applied field (as in conventional dispersive media) but also on its wave number. The representation of plasma as a dispersive medium is subject to certain validity criteria which are satisfied for such typical cases as the ionosphere and electrical discharge through gases but is not satisfied for electrons in the conduction band of a metal. The passage of charged particles through plasma is investigated by means of straightforward application of Maxwell's equations for a dispersive medium. The Debye screening, which is applicable to the potential of an incident particle having velocity $V \ll \langle v^2\rangle^{\frac{1}{2}}$ (where $\langle v^2\rangle^{\frac{1}{2}}$ is the root mean square velocity of plasma electrons), loses its significance when $V \gg \langle v^2\rangle^{\frac{1}{2}}$; and in the latter case, the potential decreases with the distance in accordance with an inverse cube law. The stopping power has been calculated for slow incident charged particles having $V \ll \langle v^2\rangle^{\frac{1}{2}}$ and for fast particles having $V \gg \langle v^2\rangle^{\frac{1}{2}}$ in a plasma comprising electrons distributed in accordance with Maxwell-Bolzmann and Fermi-Dirac statistics. For slow particles the results represent an extension of the formula of Fermi and Teller. An expression has been derived for the distribution of the polarization density in the space surrounding a moving particle.

I. INTRODUCTION

WITHIN the last several years, considerable interest has been shown in the study of an organized assembly of charged particles designated as "plasma," some of the important properties of which had been described earlier by Rayleigh,[1] Langmuir,[2] and Langmuir and Tonks.[3] A large portion of the recent literature devoted to this subject appeared in the U.S.S.R., initiated by A. Vlasov and his associates.[4–7] In the United States, the properties of plasma were studied by Bohm, Pines, Gross, and others.[8–12]

Plasma in its normal equilibrium state is characterized by a substantially equal density distribution of positive and negative charge. Consequently, the volume distribution of charge is practically zero, and the potential within the medium is determined by Poisson's equation $\nabla^2\varphi = 0$. The positive charges are heavy and their motion may be neglected. Any disturbance in the plasma will tend primarily to disturb the distribution of electrons and produce polarization in the medium. To facilitate our problem, we assume that the positive nuclei have their charges spread our uniformly through the medium.

Our purpose is to study the classical interaction between an incident charged particle and a surrounding plasma. This subject has been treated extensively in the literature and several methods have been applied to determine specific aspects of the problem. Kronig and Korringa[13–15] and Vlasov[7] treated the problem by means of a hydrodynamical model of an electron gas. Vlasov[7] and Akhiezer and Sitenko,[16] applying a method originated by Vlasov, treated the electron collisions by means of a generalization of the Boltzmann transport equation which has been modified to take into account the long-range Coulomb forces. Fermi and Teller,[17] and Kwal[18,19] applied the method of binary collisions and took into account the plasma aspects of the problem by including in their treatment either the Debye length or the natural frequency of the plasma. Kramers[20] used the conventional form of the dielectric constant of an assembly of free stationary electrons in a classical electrodynamic calculation of the stopping power.

Our purpose is to re-examine the stopping power problem and to use a phenomenological approach in which the microscopic behavior of individual electrons in the plasma is ignored, adopting instead a macroscopic point of view and describing the electron assembly as a homogeneous, isotropic, dispersive medium. In conventional dispersive media, the dielectric constant depends on the frequency of the applied field. The dielectric constant of a plasma is characterized, however, by essentially different properties since it depends not only on the frequency but also on the wave number of the applied field. The dielectric properties of plasma have been considered by Gertenshtein[21,22] in the study of longitudinal waves passing through ionized media

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[19] B. Kwal, J. phys. radium **12**, 805 (1951).
[20] H. A. Kramers, Physica **13**, 401 (1947).
[21] M. E. Gertsenshtein, Zhur. Eksptl. i Teoret. Fiz. **22**, 303 (1952).
[22] M. E. Gertsenshtein, Zhur. Eksptl. i Teoret. Fiz. **23**, 678 (1952).

and in an investigation on the scattering of radio-waves by inhomogeneities in the ionosphere. Gertsenshtein used Vlasov's approach and treated electron collisions by means of a generalized form of the Boltzmann transport equation. However, the phenomenological treatment, which is based on the straightforward application of Maxwell's equations properly modified to take into account the motion of the medium, clarifies the physical aspects of plasma behavior and facilitates the treatment of problems involving the interaction of charged particles with plasma.

In part II, using the phenomenological approach, we examine the behavior of the plasma and determine its response to an external disturbance. In part III we consider a specific case of a disturbance created by a moving point charge, and in part IV we derive the potential for a slowly moving point charge. The validity of the phenomenological approach is discussed in part V. In parts VI and VII we consider the field and the polarization charge density in the surrounding medium produced by an incident particle and the effectiveness of the plasma in stopping the particle.

II. ELECTRODYNAMICS OF THE PLASMA

We now consider the effect on the plasma of an impressed electric field $\mathbf{E}(\mathbf{r},t)$. We assume that the field perturbs the equilibrium motion of the electron assembly only slightly and that during the period of time in which the electrons may be considered as responding collectively to the applied field they execute approximately straight-line motion. This condition will be discussed in greater detail in a later section.

Let $nf(\mathbf{v})d\mathbf{v}$ be the number of electrons per unit volume possessing velocities in $d\mathbf{v}$ at $\mathbf{v}$, and $\int f(\mathbf{v})d\mathbf{v}=1$. As a result of the impressed electric field, the medium becomes polarized, and we designate by $\mathbf{P}_\mathrm{v}(\mathbf{r},t)d\mathbf{v}$ the polarization vector associated with the electrons having velocities in a region $d\mathbf{v}$ about $\mathbf{v}$. We shall place ourselves in the reference frame of the moving electrons and determine the equation of motion of the polarization vector. Neglecting the small Lorentz force due to the magnetic field, the equation of motion for the nonrelativistic case can be written as follows[23]:

$$\left(\frac{d^2}{dt^2}+g\frac{d}{dt}\right)\mathbf{P}_\mathrm{v}(\mathbf{r}_0+\mathbf{v}t,t)=\frac{ne^2f(\mathbf{v})\mathbf{E}(\mathbf{r}_0+\mathbf{v}t,t)}{m}, \quad (1)$$

where $\mathbf{r}_0$ is the position vector in the reference frame of the observer and m and e are the mass and charge of the electron, respectively. Damping of the motion of the polarization vector due to electron-ion and electron-electron collisions is represented by the damping constant g. Since the exact nature of this damping is not important to our discussion, it will be neglected at a later point.

[23] J. C. Slater and N. H. Frank, *Electromagnetism* (McGraw-Hill Book Company, Inc., New York, 1947), Chap. 9.

$\mathbf{P}_\mathrm{v}$ and $\mathbf{E}$ are now expressed as Fourier integrals,

$$\begin{Bmatrix}\mathbf{P}_\mathrm{v}(\mathbf{r},t)\\ \mathbf{E}(\mathbf{r},t)\end{Bmatrix}=\int d\mathbf{k}d\omega e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\begin{Bmatrix}\mathbf{P}_\mathrm{v}(\mathbf{k},\omega)\\ \mathbf{E}(\mathbf{k},\omega)\end{Bmatrix}, \quad (2)$$

and we find from Eq. (1) the following relation between $\mathbf{P}_\mathrm{v}(\mathbf{k},\omega)$, and $\mathbf{E}(\mathbf{k},\omega)$,

$$\mathbf{P}_\mathrm{v}(\mathbf{k},\omega)=-\frac{ne^2f(\mathbf{v})}{m}\frac{\mathbf{E}(\mathbf{k},\omega)}{(\omega-\mathbf{k}\cdot\mathbf{v})^2+ig(\omega-\mathbf{k}\cdot\mathbf{v})}. \quad (3)$$

Maxwell's equations in a medium moving with velocity $\mathbf{v}$ are well known[24] and may be written, for a nonmagnetic medium

$$\operatorname{curl}\mathbf{E}=-(1/c)\dot{\mathbf{H}}, \quad (4a)$$

$$\operatorname{div}\mathbf{H}=0, \quad (4b)$$

$$\operatorname{curl}\mathbf{H}=\frac{1}{c}\dot{\mathbf{E}}+\frac{4\pi}{c}\int[\dot{\mathbf{P}}_\mathrm{v}+\operatorname{curl}(\mathbf{P}_\mathrm{v}\times\mathbf{v})]f(\mathbf{v})d\mathbf{v}+\frac{4\pi\mathbf{j}}{c}, \quad (4c)$$

$$\operatorname{div}\mathbf{E}=4\pi\left(\rho_1-\operatorname{div}\int\mathbf{P}_\mathrm{v}f(\mathbf{v})d\mathbf{v}\right), \quad (4d)$$

where $\mathbf{j}$ and ρ_1 are applied current and charge densities, respectively, relative to the fixed reference system.

In the following we shall refer to various magnitudes such as $\mathbf{E}$, $\mathbf{H}$, etc., which are functions of $\mathbf{r}$, t and to their Fourier transforms which vary with $\mathbf{k}$, ω. In the first case, they will be identified as $\mathbf{E}(\mathbf{r},t)$, $\mathbf{H}(\mathbf{r},t)$, etc., and in the second case as $\mathbf{E}$, $\mathbf{H}$, etc. Expanding $\mathbf{H}(\mathbf{r},t)$, $\mathbf{j}(\mathbf{r},t)$, and $\rho_1(\mathbf{r},t)$ as was done in Eq. (2), using the relation (3) to eliminate $\mathbf{P}_\mathrm{v}$ from these equations and summing over all velocities of the plasma electrons, and dividing $\mathbf{E}$ into components $E_\parallel$ and $E_\perp$ parallel and perpendicular, respectively, to the wave vector $\mathbf{k}$, we find

$$i\mathbf{k}\times\mathbf{E}=i\omega/c\mathbf{H}, \quad (5a)$$

$$\mathbf{k}\cdot\mathbf{H}=0, \quad (5b)$$

$$i\mathbf{k}\times\mathbf{H}=-\frac{i\omega}{c}[\epsilon_\perp(\mathbf{k},\omega)\mathbf{E}_\perp+\epsilon_\parallel(\mathbf{k},\omega)\mathbf{E}_\parallel]+\frac{4\pi\mathbf{j}}{c}, \quad (5c)$$

$$i\mathbf{k}\cdot\mathbf{E}\epsilon_\parallel(\mathbf{k},\omega)=4\pi\rho_1, \quad (5d)$$

where

$$\epsilon_\perp=1-\frac{\omega_0^2}{\omega}\int\frac{f(\mathbf{v})d\mathbf{v}}{(\omega-\mathbf{k}\cdot\mathbf{v}+ig)}, \quad (6)$$

$$\epsilon_\parallel=1-\omega_0^2\int\frac{f(\mathbf{v})d\mathbf{v}}{(\omega-\mathbf{k}\cdot\mathbf{v})^2+ig(\omega-\mathbf{k}\cdot\mathbf{v})} \quad (7)$$

are the dielectric constants of the transverse and longitudinal electric fields, respectively. The expressions (6) and (7) agree with those obtained by Gertsenshtein[21] who, as stated previously, used a rather cumbersome

[24] M. Abraham and R. Becker, *Theorie d. Electrizitat* (B. G. Teubner, Leipzig, 1933), sixth edition, Vol. 2, p. 242.

method based on Boltzman transport equations modified by Vlasov.

In the expressions (6) and (7) the term

$$\omega_0=(4\pi ne^2/m)^{\frac{1}{2}}$$

is the "plasma frequency."

It is clear that the foregoing dielectric constants may be easily generalized to include both the effect of bound electrons which may be present in the medium as well as the effect of the motion of ions in the case of an ionized gas.

III. RESPONSE OF THE PLASMA TO A MOVING PART CHARGE

Assume that a particle having charge Ze and velocity V moves in a plasma. The corresponding charge and current densities are, therefore, as follows:

$$q(\mathbf{r},t)=Ze\delta(\mathbf{r}-\mathbf{V}t), \tag{8a}$$

$$\mathbf{j}(\mathbf{r},t)=Ze\mathbf{V}\delta(\mathbf{r}-\mathbf{V}t). \tag{8b}$$

We are working in a gauge in which the vector potential is pure transverse, so that the longitudinal field is derivable solely from the scalar potential φ. Therefore

$$-i\mathbf{k}\varphi=E_{\text{II}}, \tag{9}$$

and we obtain from (5d) and (9)

$$\varphi=\frac{Ze}{2\pi^2}\frac{\delta(\omega-\mathbf{k}\cdot\mathbf{V})}{k^2\epsilon_{\text{II}}}, \tag{10}$$

where

$$\epsilon_{\text{II}}=\left[1-\omega_0^2\int\frac{f(v)d\mathbf{v}}{(\omega-\mathbf{k}\cdot\mathbf{v})^2+ig(\omega-\mathbf{k}\cdot\mathbf{v})}\right], \tag{11}$$

and $v\equiv|\mathbf{v}|$.

The expression (11) can be represented as

$$\epsilon_{\text{II}}=1-\frac{4\pi\omega_0^2}{k^2}\int_0^\infty\frac{\zeta d\zeta}{(\omega/k-\zeta)}f(\zeta), \tag{12}$$

where we have set the damping constant $g=0$.

Substituting (10) in

$$\varphi(\mathbf{r},t)=\int d\mathbf{k}d\omega e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\varphi, \tag{13}$$

we obtain

$$\varphi(\rho,z,t)=\frac{Ze}{\pi V}\int_0^\infty\kappa d\kappa J_0(\kappa\rho)\int_{-\infty}^{\infty}\frac{e^{i\omega(z/V-t)}d\omega}{k^2\epsilon_{\text{II}}}, \tag{14}$$

where z and ρ are the cylindrical coordinates of the field point, and ϵ_{II} is expressed by (12) in which

$$k^2=\kappa^2+\omega^2/V^2. \tag{15}$$

IV. DEBYE LENGTH

One of the most striking features of plasma behavior is the screening of an electric charge in the plasma. When $V\ll\langle v^2\rangle^{\frac{1}{2}}$, where $\langle v^2\rangle^{\frac{1}{2}}$ designates the root mean square of the plasma velocity, we obtain an exponentially screened Coulomb potential, characterized by the Debye length.[25] However, when $V\gg\langle v^2\rangle^{\frac{1}{2}}$, we obtain an essentially different behavior as will be discussed later.

For $V\ll\langle v^2\rangle^{\frac{1}{2}}$, and since $\omega=\mathbf{k}\cdot\mathbf{V}$ we approximate the expression (12) as follows:

$$\epsilon_{\text{II}}=1+\omega_0^2/s^2k^2, \tag{16}$$

where

$$s^2=\left[2\pi\int_{-\infty}^{\infty}f(\zeta)d\zeta\right]^{-1}. \tag{17}$$

Substituting (16) and (17) in (14), we obtain

$$\varphi(\mathbf{r},t)=Ze\exp(-r/D)/r, \tag{18}$$

where $r=[\rho^2+(z-Vt)^2]^{\frac{1}{2}}$ is the distance from the moving particle to the point at which the potential is measured and

$$D=s/\omega_0 \tag{19}$$

is the Debye length representing the screening of the potential.

We shall now examine in more detail the electron velocity distributions $f(v)d\mathbf{v}$ and consider the "Maxwell-Boltzmann plasma" and the "Fermi-Dirac plasma" separately. Various concepts applicable to the Maxwell-Boltzmann plasma shall be designated by subscripts "MB" (i.e., s_{MB}, D_{MB}) and the corresponding concepts applicable to the Fermi-Dirac plasma shall be designated by subscripts "FD" (i.e., s_{FD}, D_{FD}).

In the MB plasma, the temperature T and the electron density n are such that the electrons are distributed in accordance with Maxwell-Boltzmann statistics. Thus

$$f(v)=\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}\exp(-mv^2/2k_BT), \tag{20}$$

where k_B is the Boltzmann constant.

As examples of the MB plasma, we may cite an ionosphere for which $n\sim10^6$ cm^{-3}, $T\sim300$°K and a gaseous discharge for which $n\sim10^{11}$ cm^{-3}, $T=10\,000$°K. Substituting (20) in (17), we obtain

$$s_{MB}^2=\tfrac{1}{3}\langle v^2\rangle. \tag{21}$$

Taking into account (20), we can express the Debye length in its usual form,[25] i.e.,

$$D_{MB}=(k_BT/4\pi ne^2)^{\frac{1}{2}}, \tag{22}$$

The FD plasma represents an electron gas obeying Fermi-Dirac statistics. Thus

$$f(v)=\frac{2m^3}{nh^3}\left\{\exp\left[\frac{1}{k_BT}\left(\frac{mv^2}{k}-E_F\right)\right]+1\right\}^{-1}, \tag{23}$$

[25] P. Debye and E. Hueckel, Physik. Z. 24, 185 (1923).

where E_F designates the "Fermi energy," i.e.,

$$E_F=\frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}}=\frac{mv_F{}^2}{2}, \tag{24}$$

where v_F is the maximum velocity for $T=0$.

Substituting (23) in (17), we obtain

$$s_{FD}{}^2=\tfrac{1}{3}v_F{}^2, \tag{25}$$

and taking into account (19), we can express the Debye length as follows:

$$D_{FD}=(\tfrac{1}{4}\pi a_0\lambda\!\!\!^{-}_F)^{\frac{1}{2}}, \tag{26}$$

where

$$a_0=\hbar^2/me^2;\quad \lambda\!\!\!^{-}_F=\hbar/mv_F. \tag{27}$$

V. VALIDITY OF THE METHOD

We shall now examine in more detail the restrictions that have to be imposed to justify the present treatment, and we shall consider the following: (A) the relative importance of close binary collisions and long-range interactions, (B) the applicability of a macroscopical picture to a microscopical electron assembly, and (C) the applicability of the classical (orbital) representation.

A. Close Collisions and Long-Range Interactions

The expressions (6) and (7) are based on an assumption that during the period of time in which the electrons respond collectively to the applied fields, they execute approximately straight-line motions. In reality, however, each electron undergoes continual collisions with positive charges and other electrons, and, as a result of such collisions, it deviates progressively from its path and continually loses energy. Any electron in a plasma will follow its rectilinear path at a constant velocity as long as the accidental encounters with other electrons have no appreciable influence. However, as time passes, the collisions with other electrons will have a cumulative effect as a result of which the direction of the electron motion changes and its velocity becomes appreciably different from its initial velocity. We shall define as the relaxation time τ_D the time interval, during which as a result of the cumulative effect of binary collisions the trajectory of the electron has deviated by an angle of $\pi/2$ from its original direction. Similarly the relaxation time τ_R will designate the time interval during which the energy exchanged in electronic collisions becomes of the same order of magnitude as the initial energy of the electron. The values τ_D and τ_R do not differ appreciably from each other and can be expressed as [26,27]

$$\tau_D\sim\tau_R\sim\frac{m\langle v^2\rangle^{\frac{3}{2}}}{9ne^4\ln(D_{MB}m\langle v^2\rangle/2e^2)}. \tag{28}$$

[26] L. D. Landau, Zhur. Eksptl. i Teoret. Fiz. **7**, 203 (1936).

[27] S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago, 1942), pp. 48–79.

The values τ_R and τ_D can be determined approximately by means of the following rough arguments which we are giving since they provide a better physical insight into the problem than more rigorous calculations.

The energy loss of an incident electron can be expressed as[28]

$$-\frac{dW}{dz}=\frac{4\pi ne^4}{mv^2}\ln\frac{\rho_{\max}}{\rho_{\min}}\sim-\frac{\Delta W}{\Delta z}, \tag{29}$$

where ΔW represents the energy loss over a path Δz. Taking as the maximum impact parameter $\rho_{\max}=D_{MB}$ and as the minimum impact parameter the collision diameter $\rho_{\min}=2e^2/m\langle v^2\rangle$, we obtain

$$\Delta x\sim-\frac{\Delta Wm\langle v^2\rangle}{4\pi ne^4\ln(D_{MB}m\langle v^2\rangle/2e^2)}. \tag{30}$$

Taking $\Delta W=-m\langle v^2\rangle/2$, we obtain

$$\tau_R=\frac{\Delta x}{\langle v\rangle}\sim\frac{m^2\langle v^2\rangle^{\frac{3}{2}}}{8\pi ne^4\ln(D_{MB}m\langle v^2\rangle/2e^2)}. \tag{31}$$

The relaxation time for a deviation through an angle of $\pi/2$ from the initial direction may be estimated from the multiple scattering theory of Williams[29] which gives for the mean square angle of deviation after traversing a path length X through a medium containing n scattering centers of charge Ze per cm^3,

$$\langle\theta^2\rangle\sim\frac{8\pi Z^2e^4nX}{m^2\langle v^2\rangle^2}\ln\left(\frac{181}{Z^{\frac{1}{3}}}\right)^2. \tag{32}$$

Setting $\langle\theta^2\rangle\sim\pi^2/4$, $\tau_D=X/\langle v^2\rangle^{\frac{1}{2}}$, and $Z=1$, We obtain

$$\tau_D=\frac{\pi m^2\langle v^2\rangle^{\frac{3}{2}}}{32Z^2e^4n\ln(181/Z^{\frac{1}{3}})^2}. \tag{33}$$

The expressions (31) and (33) for τ_R and τ_D give values of the same order of magnitude as the expression (28) derived by more rigorous considerations.

We shall now consider a term appearing in the denominator of (28) which for the specific examples of the MB plasma previously cited is

$$A=\ln(D_{MB}m\langle v^2\rangle/2e^2)\sim10. \tag{34}$$

The individual electron collisions can be neglected if

$$\omega_0{}^{-1}\ll\tau_R. \tag{35}$$

Taking into account (29) and (34), the above expression (31) leads to the following inequality:

$$n\ll0.05(k_BT/e^2)^3\sim10^7T^3. \tag{36}$$

[28] See for instance: E. Fermi, *Nuclear Physics* (University of Chicago Press, Chicago, 1950), p. 28.

[29] E. J. Williams, Proc. Roy. Soc. (London) **169**, 531 (1939).

The close interactions are, therefore, negligible for an electron gas of a relatively low density. For a high-density gas such that the close interaction becomes dominant, the assembly loses its "plasma" character and becomes similar to a gaseous assembly of uncharged particles.

B. Macroscopic Treatment

In our description of plasma, we were concerned with the macrosopic behavior of an electron assembly and the Debye length was derived as a macroscopic concept. In order to have an agreement with a microscopic picture in which there are n discrete particles per cm^3, we must require that

$$D > n^{-\frac{1}{3}}, \tag{37}$$

(1) *MB Plasma*

Substituting (29) in (37), we obtain

$$n < (k_B T/4\pi e^2)^3 \sim 10^5 T^3. \tag{38}$$

(2) *FD Plasma*

Substituting (26) in (37), we obtain

$$v_F > 12.18 v_0, \tag{39}$$

where $v_0 = e^2/\hbar$.

C. Orbital Representation

(1) *Validity Criterion*

We have found in a preceding paragraph that any electron having velocity $V \ll \langle v^2 \rangle^{\frac{1}{2}}$ produces in a plasma a potential $\varphi = -e\exp(-r/D)/r$. We may generalize our argument by stating that each individual electron in the plasma creates the potential φ. This argument is strictly applicable to all electrons having $v \ll \langle v^2 \rangle^{\frac{1}{2}}$. For other electrons, the distribution of the potential does not change essentially and qualitatively the potential is still screened at a distance of the order of magnitude of D. It will be shown later that a fundamental change in the screening potential occurs only for electrons having $v \gg \langle v^2 \rangle^{\frac{1}{2}}$, and the number of these electrons is negligible in a plasma in equilibrium.

We shall now determine whether or not the motion of an electron in a potential φ can be described by means of the classical (orbital) representation on which our treatment is based. According to Williams,[30] the orbital representation is valid if the following two requirements are fulfilled: (*a*) The uncertainty in the momentum of a perturbed electron is much smaller than the classical value of the momentum transfer caused by the perturbing field φ, and (*b*) the wavelength of the perturbed electron is much smaller than the radius of the perturbing field, which is assumed to be equal to D.

The criterion (*a*) leads to the following inequality:

$$\frac{|\varphi| er}{\hbar v} = \frac{e^2 \exp(-r/D)}{\hbar v} \gg 1. \tag{40}$$

[30] E. J. Williams, Revs. Modern Phys. **17**, 217 (1945).

The condition (40) can never be satisfied for the whole region of space and consequently the orbital picture cannot be generally applied. We may associate with each value of v a radius

$$R = D\ln(e^2/\hbar v), \tag{41}$$

such that only for $r \ll R$ the orbital picture is valid. If, however, the region defined by the radius R is sufficiently large so as to include most of the space occupied by the scattering potential, i.e.,

$$R \gg D, \quad \text{or} \quad \ln(e^2/\hbar v) \gg 1, \tag{42}$$

then we may assume that the orbital picture applies to the whole space. The inequality (42) represents the Williams requirement (*a*).

The requirement (*b*) leads to the following inequality:

$$(\hbar/mv)/D \ll 1. \tag{43}$$

(2) *MB Plasma*

We shall apply now the inequalities (42) and (43). The first of these inequalities corresponds to $v \ll v_0$ or to

$$T \ll 13.5 \times 1.16 \times 10^4 = 15.7 \times 10^4\ {}^\circ\text{K}. \tag{44}$$

Substituting (22) in the inequality (43), we obtain

$$v^2 \gg 4\pi\hbar^2 ne^2/m^2 kT. \tag{45}$$

We have from (36) and (38) that $n \ll n_{\max}$ where $n_{\max} = (k_B T/4\pi e^2)^3$. Substituting in (45) $n_{\max}$ in place of n, we obtain

$$v \gg v_{\max} = \hbar k_B T/4\pi me^2. \tag{46}$$

Since v has a range from zero to infinity, not all the electrons satisfy the inequality (46). However, if

$$\tfrac{1}{2} m v_{\max}^2 \ll k_B T, \tag{47}$$

the number of electrons that do not satisfy the inequality (46) may be neglected, and it can be assumed that the inequality (43) holds true.

Substituting (46) in (47), we obtain

$$T \ll 10^7\ {}^\circ\text{K}. \tag{48}$$

(3) *FD Plasma*

The inequality (42) requires that

$$v_F \ll v_0, \tag{49}$$

and substituting (26) in (43), we obtain

$$v^2 \gg (4/\pi) v_0 v_F. \tag{50}$$

Since v has a range of values from zero to v_F, the inequality (50) is not satisfied for all the electrons and the necessary requirement is

$$v_F \gg (4/\pi) v_0. \tag{51}$$

D. Conclusion

Our treatment of the MB plasma is justified if the following inequalities are satisfied: (36), (38), (44), and (48). The inequality (38) is more stringent than (36), and (44) is more stringent than (48). Consequently, the criterion for the applicability of our method is expressed by (38) and (44). This criterion is satisfied for the two representative cases of MB plasma cited above.

The validity of the treatment of the FD plasma is contingent on the inequalities (39), (49), and (51). The inequality (49) is not compatible with (39) and (51) and, therefore, our validity criterion is not satisfied.

The phenomenological treatment of plasma as a dispersive medium can, therefore, be applied rigorously to such cases as electrical gaseous discharges, or the ionosphere. The criterion for our treatment is, however, rigorously not applicable to such cases as electrons in the conduction band of a metal.

VI. HIGH-VELOCITY INCIDENT PARTICLE

A. General

We shall now determine the response of the medium for an incident particle having charge Ze and velocity $V\gg\langle v^2\rangle^{\frac{1}{2}}$. We are again confronted with the problem whether the space surrounding the particle track can be represented as a homogeneous dispersive medium characterized by the dielectric constant expressed by (6) and (7). Whether or not such a representation is acceptable depends upon the relative importance of close binary collisions and long-range interactions.

It is apparent that in the immediate neighborhood of the incident particle the electrons undergo very violent collisions. Consequently, the phenomenological representation cannot be used since it applies only to a plasma under the effect of a small perturbation. We shall designate the portion of the space close to the particle track as the region of "binary collisions" since in this region we take into account only the direct interactions of the incident particle with individual electrons. On the other hand, at large distances from the track, we deal with the "plasma region" since the perturbation exerted by the incident particle is small and the collective effect of the electrons is the dominant factor.

The binary collisions will produce a dominant effect if the momentum gain $p_\perp$ of each electron by the passage of the incident particle is large compared to the average momentum $m\langle v^2\rangle^{\frac{1}{2}}$ of the electron in the undisturbed state, i.e.,

$$p_\perp=2Ze^2/\rho V\gg m\langle v^2\rangle^{\frac{1}{2}}, \tag{52}$$

from which

$$\rho\ll\frac{2Ze^2}{mV\langle v^2\rangle^{\frac{1}{2}}}\sim\frac{Z}{D^2n}\frac{\langle v^2\rangle^{\frac{1}{2}}}{V}. \tag{53}$$

In the plasma region, the following criteria should be satisfied: (1) the momentum gain of each electron by the passage of the incident particle is small compared to the average momentum of the electron in the undisturbed state, and (2) the momentum gains of two neighboring electrons should not differ substantially one from the other.

According to the criterion (1) we should have

$$\rho\gg\frac{2Ze^2}{mV\langle v^2\rangle^{\frac{1}{2}}}\sim\frac{5}{18\pi}\frac{Z}{D^2n}\frac{\langle v^2\rangle^{\frac{1}{2}}}{V}. \tag{54}$$

The criterion (2) requires that

$$(dp_\perp/d\rho)\Delta\rho\ll p_\perp, \tag{55}$$

where $\Delta\rho=n^{-\frac{1}{3}}$ is the average distance between two electrons in the plasma. Equation (55) yields

$$\rho\gg n^{-\frac{1}{3}}. \tag{56}$$

Consequently, the plasma region corresponds to impact parameters that satisfy both inequalities (54) and (56).

Our problem is complicated by the fact that the inequalities (54) and (56) do not limit precisely the extent of the plasma region. Even if we assume that the limit defining the plasma region is known, the question of how to treat the problem would arise. A similar situation arose in the study of fast charged particles passing through conventional dispersive media, and in this connection we shall mention the treatment by Fermi[31,32] and Huybrechts and Schönberg.[33] Fermi, assuming that the homogeneous dispersive medium occupies the whole space, calculated the energy loss in the region $\rho>\rho_1$ from the flux of the Poynting vector through a cylindrical surface of radius ρ_1 having its axis on the path of the particle. The energy loss for $\rho<\rho_1$ was calculated on the basis of conventional collision theory. Huybrechts and Schönberg modified the Fermi treatment by assuming that there are no electrons up to the distance ρ_1 and, therefore, the dielectric constant within the cylinder of radius ρ_1 has been taken to be equal to 1 and has a determined value $\epsilon(\omega)$ outside of this cylinder. They have dealt, therefore, with an inhomogeneous dielectric medium having a discontinuity at the surface of the cylinder.

Our treatment will be similar to the one by Fermi since it is simpler and seems to be more applicable to our problem as we do not know the value ρ_1 definining the discontinuity of the medium.

B. Potential Distribution in Plasma

We shall now return to the expression (14) and determine the potential produced by the incident particle. As stated above, we assume that the homogeneous dispersive medium surrounds the particle track for all impact parameters and we attribute a physical meaning only to the solution for φ for values $\rho>\rho_1$. We shall

[31] E. Fermi, Phys. Rev. **56**, 1242 (1939).
[32] E. Fermi, Phys. Rev. **57**, 485 (1940).
[33] M. Huybrechts and M. Schönberg, Nuovo cimento **9**, 764 (1952).

determine φ also for small values of ρ since it is of some interest.

Since $V \gg \langle v^2\rangle^{\frac{1}{2}}$, the expression (11) for ϵ_{11} can be expanded as follows

$$\epsilon_{11}=1-\frac{\omega_0^2}{\omega^2}\left[1+\langle v^2\rangle\frac{k^2}{\omega^2}+\langle v^4\rangle\frac{k^4}{\omega^4}+\cdots\right], \qquad (57)$$

where

$$\langle v^{2n}\rangle=\int v^{2n}f(v)d\mathbf{v}.$$

Substituting (57) in (14), we obtain

$$\varphi(\rho,z,t)=\frac{ZeV}{\pi}\int_0^\infty \kappa d\kappa J_0(\kappa\rho) \times\int_{-\infty}^{\infty}\frac{e^{i\omega(z/V-t)}\omega^2 d\omega}{(V^2\kappa^2+\omega^2)(\omega^2-\omega_0^2[1+\langle v^2\rangle k^2/\omega^2+\cdots])}.$$

When $\langle v^2\rangle^{\frac{1}{2}} \ll V^2$ the term multiplying ω_0^2 in the denominator will be nearly unity. We observe that the roots of the denominator in addition to the pair $\omega=\pm i\kappa V$ are given by $\omega^2=\omega_0^2[1+\langle v^2\rangle k^2/\omega^2+\cdots]$, and correspond to the existence of plasma oscillations. The ω contour should be chosen so as to pass above these singularities. Then for positions ahead of the incident particle, $z>Vt$, the ω contour may be closed in the upper half plane and the oscillations will exist only behind the particle. Since there is only the singularity $\omega=i\kappa V$ in the upper half plane, the potential ahead of the particle is represented as

$$\varphi(\rho,z,t)=Ze\int_0^\infty J_0(\kappa\rho)e^{-\kappa(z-Vt)}\frac{\kappa^2 d\kappa}{(\kappa^2+\omega_0^2/V^2)}. \qquad (58)$$

To examine the behavior of this function, we note that if $(z-Vt)\ll V/\omega_0$, we may consider the integrand only for large κ. In this case,

$$\varphi(r)=Ze\int_0^\infty d\kappa J_0(\kappa\rho)e^{-\kappa(z-Vt)}=Ze/r, \qquad (59)$$

where r is the distance from the charged particle in a coordinate system moving with the charged particle. When $r\gg V/\omega_0$, we look at the integrand when κ is small. We find, approximately,

$$\varphi(r)=Ze\frac{V^2}{\omega_0^2}\frac{d^2}{dz^2}\left(\frac{1}{r}\right), \qquad (60)$$

which is identical with the potential due to an axial quadrupole. We see that the collective effect of the plasma is to cause the potential to fall off approximately as the inverse cube of the distance for large values of r. This result raises some question concerning the procedure of Kwal[18,19] and Van der Ziel[34] of employing an exponentially screened Coulomb potential which is valid for a slowly moving ($V\ll\langle v^2\rangle^{\frac{1}{2}}$) particle when calculating individual energy transfer to electrons in plasma by fast charged particles. The screening parameter for a low velocity incident particle is of the order of the Debye length which in metals may be less than 1 A. The screening parameter V/ω_0 for a relativistic incident particle may be much larger, e.g., $\sim 10^2$ A.

For positions behind the incident particle, $Z<Vt$, the contour of integration must be closed in the lower half of the ω plane. There will then be the contributions from the pair of poles on the real axis and the pole at $\omega=-i\kappa V$. We assume that to a sufficient approximation the integral may be written

$$\varphi(z,\rho,t)=\frac{ZeV}{\pi}\int_0^\infty \kappa J_0(\kappa\rho)d\kappa \times\int_{-\infty}^{\infty} e^{i(\omega/V)(z-Vt)}\frac{\omega^2 d\omega}{(V^2\kappa^2+\omega^2)(\omega^2-\omega_0^2)}, \qquad (61)$$

and

$$\varphi(z,\rho,t)=\varphi_{\rm I}(z,\rho,t)+\varphi_{\rm II}(z,\rho,t). \qquad (62)$$

Corrections to the position of the poles on the real ω axis will be considered in the next section. Evaluation of the residue at the pole at $\omega=-i\kappa V$ gives an expression $\varphi_{\rm I}$ which is the analytical continuation of Eq. (58) into the region $z<Vt$, i.e., $\varphi_{\rm I}$ is again Coulombian for points close to the incident charge and falls off roughly as the inverse cube of the distance for large distances. That part of φ arising from the residue at the poles at $\omega=\pm\omega_0$ is purely oscillatory in $(z/V-t)$, i.e.,

$$\varphi_{\rm II}(z,\rho,t)=2(Ze\omega_0/V)\sin\omega_0(z/V-t)K_0(\rho\omega_0/V). \qquad (63)$$

This represents the potential due to the polarization charge which is set into oscillation by the passage of the charged particle. In our neglect of damping, the polarization charge continues to oscillate with the plasma eigen-frequency ω_0. The logarithmic singularity of $\varphi_{\rm II}$ at the track of the particle is due to the erroneous use of an infinite upper limit in the integration. This will be examined in the next section.

C. Polarization Charge Density

The polarization charge density p may be expressed as

$$p=-i\mathbf{k}\cdot\mathbf{P}, \qquad (64)$$

where $\mathbf{P}$ is the polarization vector. Further, since

$$i\mathbf{k}\cdot(\mathbf{E}+4\pi\mathbf{P})=(Ze/2\pi^2)\delta(\omega-\mathbf{k}\cdot\mathbf{V}), \qquad (65)$$

and $-i\mathbf{k}\cdot\mathbf{E}=k^2\varphi$,

$$p=-\frac{Ze}{(2\pi)^3}\delta(\omega-\mathbf{k}\cdot\mathbf{V})+\frac{k^2\varphi}{4\pi}, \qquad (66)$$

[34] A. Van der Ziel, Phys. Rev. 92, 35 (1953).

or, in configuration space

$$p(\rho,z,t)=\frac{Ze\omega_0^2}{(2\pi)^2V}\int_0^\infty \kappa d\kappa J_0(\kappa\rho)$$
$$\times\int_{-\infty}^{\infty} e^{i\omega(z/V-t)}\frac{d\omega(1+\langle v^2\rangle k^2/\omega^2+\cdots)}{\{\omega^2-\omega_0^2(1+\langle v^2\rangle k^2/\omega^2+\cdots)\}}. \quad (67)$$

We have postulated $\langle v^2\rangle^{\frac{1}{2}}\ll V$ so we find the roots of the denominator by successive approximation. The ω contour is chosen to pass above the poles on the real axis, to correspond to zero polarization charge ahead of the particle. Then, the roots of the denominator are approximately

$$\omega=\pm[\omega_0^2+\langle v^2\rangle(\omega_0^2/V^2+\kappa^2)]^{\frac{1}{2}}. \quad (68)$$

Evaluating the residues, we find for $(z/V-t)<0$

$$p(\rho,z,t)=\frac{Ze\omega_0}{2\pi V}\int_0^\infty \kappa d\kappa J_0(\kappa\rho)\left[\omega_0^2\left(1+\frac{\langle v^2\rangle}{V^2}\right)+\langle v^2\rangle\kappa^2\right]$$
$$\times\sin\left\{\left(\frac{z}{V}-t\right)\left[\omega_0^2\left(1+\frac{\langle v^2\rangle}{V^2}\right)+\langle v^2\rangle\kappa^2\right]^{\frac{1}{2}}\right\}. \quad (69)$$

The integral as written does not exist. However, we note that the upper limit may not be infinite since the dielectric is not capable of supporting oscillations whose wavelength is shorter than the average spacing between electrons in the medium.

To obtain an approximate solution we assume that the integral may be cut off at some finite upper limit designated by κ_m. Then

$$p(\rho,z,t)=-\frac{Ze}{2\pi V}\frac{d^2}{d\tau^2}\int_0^{\kappa_m}\kappa d\kappa J_0(\kappa\rho)$$
$$\times\frac{\sin\tau[\omega_0^2(1+\langle v^2\rangle/V^2)+\langle v^2\rangle\kappa^2]^{\frac{1}{2}}}{\omega_0^2(1+\langle v^2\rangle/V^2)+\langle v^2\rangle\kappa^2}, \quad (70)$$

where $\tau=(z/V)-t$. Now if ρ and τ are both large, the bulk of the integral will originate from reion of small κ, and we tentatively let $\kappa_m\to\infty$ and using an identity proven by Lamb,[35] write

$$p(\rho,z,t)=-\frac{Ze}{2\pi}\frac{d^2}{d\tau^2}\begin{cases}\dfrac{\cos\omega_0[\tau^2-\rho^2/\langle v^2\rangle]^{\frac{1}{2}}}{[\tau^2-\rho^2\langle v^2\rangle]^{\frac{1}{2}}} & \tau\langle v^2\rangle^{\frac{1}{2}}>\rho \\ 0 & \tau\langle v^2\rangle^{\frac{1}{2}}<\rho.\end{cases} \quad (71)$$

The singularity on the surface of the cone

$$z-Vt=(V/\langle v^2\rangle^{\frac{1}{2}})\rho$$

is fictitious; the polarization charge density actually has a finite maximum on this surface.

[35] H. Lamb, Proc. London Math. Soc. Sec. 2, **7**, 122 (1909).

D. Stopping Power

(1) *General*

The space surrounding the particle track is subdivided into three regions: (a) The region of binary collisions within a cylinder having as axis the track of the particle and a radius ρ_1 satisfying the inequality (53), (b) the plasma outside of a similar cylinder having radius ρ_2, and satisfying the inequality (54), and (c) the intermediate region between cylinders having radii ρ_1 and ρ_2. The intermediate region includes the portion of space for which no adequate theory exists at present.

(2) *Plasma Region*

We wish to determine the energy lost by the incident particle to the portion of the plasma in the region $\rho>\rho_2$ to which the dielectric concept rigorously applies. To accomplish this, we will consider the force with which the polarization in the "far" region ($\rho>\rho_2$) reacts upon the incident particle. We imagine that the cylinder $\rho<\rho_2$ is removed from the infinite medium leaving the polarization in the far region unchanged. The stopping power is then given by the force exerted on the incident particle by the surface charge density (which is equal to the normal component of the polarization vector) on the inside of this cylinder and by that portion of the wake of volume polarization charge which lies in the far region.

Now

$$\mathbf{P}=\frac{iZe}{(2\pi)^3}\delta(\omega-\mathbf{k}\cdot\mathbf{v})\cdot\frac{1-\epsilon_{\text{II}}}{\epsilon_{\text{II}}}\frac{\mathbf{k}}{k^2}. \quad (72)$$

The surface charge density is

$$\sigma=-\hat{n}\cdot\mathbf{P}(\rho,z,t), \quad (73)$$

where $\hat{n}$ is a unit vector in the direction of the outward normal to the cylinder. The expression for the polarization charge is given in Eq. (64). Then

$$-\frac{dW}{dz}=2\pi Ze\left\{\rho_2\int_{-\infty}^{\infty}\frac{zdz}{(z^2+\rho_2^2)^{\frac{3}{2}}}\sigma(\rho_2,z)\right.$$
$$\left.+\int_{\rho_2}^{\infty}\rho d\rho\int_{-\infty}^{\infty}\frac{zdz}{(z^2+\rho^2)^{\frac{3}{2}}}p(\rho,z),\right. \quad (74)$$

where the variable z is understood to have its origin at the position of the incident particle. Inserting the Fourier integral representations for σ and p, and using the approximate form for the dielectric constant for $\langle v^2\rangle^{\frac{1}{2}}\ll V$ one finds after carrying out the integrations

$$-\frac{dW}{dz}=\frac{Z^2e^2\rho_2}{V(1+\beta_1^2)}\int_0^\infty\frac{\kappa d\kappa J_0(\kappa\rho)}{(\omega_0^2+\kappa^2V^2)}\omega'^3K_1(\rho_2\omega'/V), \quad (75)$$

where $\beta_1=\langle v^2\rangle^{\frac{1}{2}}/V$ and $\omega'=[\omega_0^2(1+\beta_1^2)+\beta_1^2V^2\kappa^2]^{\frac{1}{2}}$.

This may be shown to reduce to

$$-\frac{dW}{dz}=\frac{Z^2e^2}{(1+\beta_1^2)}\frac{\omega_0^2}{V^2}a\left\{K_0(a)K_1(a)+\frac{2\beta_1}{a}K_1\left(\frac{a}{\beta}\right)-\frac{a}{2}\int_{a/\beta_1}^{\infty}K_1(x)dx/x^3\right\},\quad(76)$$

where $K_0(a)$ and $K_1(a)$ are modified Bessel functions of the second kind of the order 0 and 1, respectively,[36] and $a=\omega_0\rho_2/V$. Now if $a/\beta_1\gg1$,

$$-\frac{dW}{dz}=\frac{Z^2e^2a}{1+\beta_1^2}\frac{\omega_0^2}{V^2}\left\{K_0(a)K_1(a)+(2\pi)^{\frac{1}{2}}(\beta_1/a)^{7/2}e^{-a/\beta_1}\right\}.\quad(77)$$

Taking into account that

$$(2\pi)^{\frac{1}{2}}(\beta_1/a)^{7/2}e^{-a/\beta_1}\ll K_0(a)K_1(a),\quad(78)$$

we obtain

$$-\frac{dW}{dz}=\frac{Z^2e^2\rho_2\omega_0^2}{V^3(1+\beta_1^2)}K_0(\rho_2\omega_0/V)K_1(\rho_2\omega_0/V).\quad(79)$$

If $\rho_2\omega_0/V\ll1$, one may replace K_0 and K_1 by the first terms in their series expansions, obtaining:

$$-\frac{dW}{dz}=\frac{Z^2e^2\omega_0^2}{V^2(1+\beta_1^2)}\ln\frac{1.123V}{\omega_0\rho_2},\quad(80)$$

which in the limit $\beta_1\to0$ reduces to

$$-\frac{dW}{dz}=\frac{Z^2e^2\omega_0^2}{V^2}\ln\frac{1.123V}{\omega_0\rho_2}.\quad(81)$$

It may be of interest to compare the above formula with the corresponding expression derived by Pines.[11] Using the present notation, the stopping power obtained by Pines can be expressed as follows:

$$-\frac{dW}{dz}=\frac{Z^2e^2\omega_0^2}{V^2}\left\{\ln\frac{k_0V(1-\beta_1^2)^{\frac{1}{2}}}{\omega_0}+\frac{\langle v^2\rangle^{\frac{1}{2}}\beta_1^2(1-\beta_1^2)k_0^2}{4\omega_0^2}\right\},\quad(82)$$

where k_0 is a cut-off wave number.

Assuming $(1-\beta_1^2)^{\frac{1}{2}}\sim1$, we obtain from (82)

$$-\frac{dW}{dz}=\frac{Z^2e^2\omega_0^2}{V^2}\left\{\ln\frac{V}{\omega_0\rho_2}+\frac{\beta_1^2\langle v^2\rangle^{\frac{1}{2}}k_0^2}{4\omega_0^2}\right\}.\quad(83)$$

For the limiting case of $\beta_1=0$ the expression (83) is identical to (81) except for the factor 1.123 in the logarithm. However, for $\beta_1\neq0$ the expression (80) gives a decreased value for the stopping power, whereas according to the Pines formula (83) the stopping power increases by a much smaller amount.

[36] For tables of K_0 and K_1 functions, see for instance: *British Association for the Advancement of Science, Mathematical Tables* (Cambridge University Press, Cambridge, 1937), Vol. 6, pp. 264–271.

(3) *Binary Collisions*

The contribution of binary collisions to the stopping power is

$$-\frac{dW}{dz}=\frac{Z^2e^2\omega_0^2}{V^2}\ln\left(\frac{\rho_1}{b_{\min}}\right),\quad(84)$$

where

$$b_{\min}=Ze^2/mV^2\quad\text{if}\quad Ze^2/\hbar V\gg1,\quad(85)$$

and

$$b_{\min}=h/2mV\quad\text{if}\quad Ze^2/\hbar V\ll1.\quad(86)$$

(4) *Total Stopping Power*

The contribution to the stopping power in the intermediate region for $\rho_1<\rho<\rho_2$ remains undetermined. In this intermediate region, both individual and collective interactions take place. By interpolating, it is seen that the contribution to the stopping power of any portion of the intermediate region comprising distances $d\rho$ about ρ is the same if calculated on the basis of collective interaction or individual interaction. We shall assume, therefore, that $\rho_1=\rho_2$ and is located somewhere in the intermediate region. By adding the contributions (81) and (84), we obtain for the total stopping power

$$-\frac{dW}{dz}=\frac{4\pi Z^2e^4}{mV^2}\ln\left(\frac{1.123V}{\omega_0b_{\min}}\right),\quad(87)$$

where $b_{\min}$ is given by (86).

VII. LOW-VELOCITY INCIDENT PARTICLE

A. Polarization Charge Density

In Sec. IV we found that to a first approximation a slowly moving incident particle in plasma gives rise to an exponentially screened Coulomb potential. We shall now determine the form of the polarization charge density to second order, taking into account in this treatment the damping of a disturbance in plasma due to the random motion of the medium.

To determine p we shall need an approximate expression for the dielectric constant for the case $\langle v^2\rangle^{\frac{1}{2}}\gg V$. It may be written

$$\epsilon_{\text{II}}=1+\frac{2\pi\omega_0^2}{k^2}\left[\int_{-\infty}^{\infty}f(u)du+\frac{\omega}{k}\int_{-\infty}^{\infty}\frac{f(u)du}{(u-\omega/k)}\right]\quad(88)$$

$$\simeq1+\frac{\omega_0^2}{k^2s^2}\left[1+i\frac{B}{s}\frac{\omega}{k}+O\left(\frac{\omega^2}{k^2}\right)\right],\quad(89)$$

where the constant B depends upon the kind of statistics assumed for the plasma, i.e., $B_{MB}=(\pi/2)^{\frac{1}{2}}$, $B_{FD}=\sqrt{3}\pi/6$. The imaginary term arises because the path of integration in the u-plane must be deformed so as to pass below the singular point $u=\omega/k$ as the damping constant g approaches zero. As in Sec. VI we write the Fourier integral expression for p, using the approximation given above for ϵ_{II}. We then express the result as a power series in $\beta=V/s$, retaining only the first two terms. We find then

$$p=\frac{Ze\omega_0^2}{4\pi s^2}\frac{e^{-r/D}}{r}+\frac{Ze\beta B\omega_0}{4\pi^2 s}\frac{Z}{r^3}$$
$$\times\{(1-a^2)[e^{-a}\,\mathrm{Ei}(a)-e^{a}\,\mathrm{Ei}(-a)]$$
$$+a[e^{a}\,\mathrm{Ei}(-a)+e^{-a}\,\mathrm{Ei}(a)+1]\},\quad (90)$$

where

$$\mathrm{Ei}(a)=\int_{-\infty}^{a} e^{x}dx/x$$

and $a=r/D=\omega_0 r/s$. The origin of coordinates ($r=0$) is taken at the position of the incident particle and moves with the particle. The z coordinate is measured along the path of the particle and is taken positive in the direction of motion of the particle.

B. Stopping Power

(1) *General Considerations*

We shall now determine the stopping power of an incident particle gaving charge Ze and velocity $V\ll\langle v^2\rangle^{\frac{1}{2}}$.

The energy loss per unit path length may be expressed in a way similar to that used for the high-velocity particle. We shall calculate the energy delivered to the plasma in the "far" region, i.e., to the region outside of a sphere of radius R whose center coincides with the position of the incident particle. Thus

$$-\frac{dW}{dz}=Ze\left\{2\pi\int_0^{\pi}\sin\theta\,\cos\theta\sigma(\theta,R)d\theta\right.$$
$$\left.+2\pi\int_0^{\pi}\sin\theta\,\cos\theta d\theta\int_R^{\infty}drp(\theta,r)\right\},\quad (91)$$

where the coordinate system is chosen to move with the incident particle and to have its origin at the particle position. In this expression $\cos\theta=z/r$, $\sigma=-\hat{n}\cdot\mathbf{P}$ where $\hat{n}$ is a unit vector normal to the sphere of radius R in the outward direction, and p is the polarization charge density.

Using the above approximation for ϵ_{II} in the Fourier integral expressions for $\mathbf{P}$ and p and expanding the resulting formulas in a power series in β we find eventually, to the first power in β,

$$-\frac{dW}{dz}=\frac{Z^2e^2\omega_0^2\beta B}{3\pi V^2 a}\left\{\left(1+\frac{1}{a}+\frac{1}{a^2}\right)e^{-a}\,\mathrm{Ei}(a)\right.$$
$$\left.-\left(1-\frac{1}{a}+\frac{1}{a^2}\right)e^{a}\,\mathrm{Ei}(-a)-\frac{2}{a}\right\},\quad (92)$$

where $a=r/D$. If $a\ll 1$,

$$-\frac{dW}{dz}=\frac{Z^2e^2\beta B\omega_0^2}{3\pi s^2}\ln\left(\frac{s^2}{3.17R^2\omega_0^2}\right).\quad (93)$$

In this phenomenological treatment we have excluded the region inside a sphere of radius R centered at the particle. This is the region of close collisions and the value of R depends upon whether we deal with MB or FD plasma.

(2) *MB Plasma*

In this case we choose R to be the average spacing between electrons in the medium,

$$R\simeq n^{-\frac{1}{3}},$$

or

$$a^2=4\pi e^2 n^{\frac{1}{3}}/k_BT.\quad (94)$$

Thus the nonuniformity of the medium is taken into account in an approximate way. The stopping power is found by substituting (94) in either (92) or (93).

(3) *FD Plasma*

We have previously found that the classical phenomenological representation is not strictly applicable to an FD plasma. However, we shall apply the expression (92) to an FD plasma, keeping in mind that in the past a classical treatment has been not infrequently used with satisfactory results in quantum-mechanical problems. We may cite the Thomas-Fermi statistical model of electron gas in an atom as well as various hydrodynamical models of perturbed electron gas, for instance, the one used by Bloch.[37] It is of interest to mention the calculation made by Bloch[38]of the stopping power of a particle having charge Ze and velocity V, such that $Ze^2/\hbar V\ll 1$. Although the interaction of the particle with the surrounding electrons is not subject to an orbital representation, the stopping power has been calculated by using an impact parameter method and the quantum mechanical aspects of the problem have been taken into account by assuming that the minimum impact parameter is equal to $\hbar/mV$. We shall follow the general procedure used by Bloch.

Returning now to our problem we consider the close collision of a particle having velocity V with an electron gas having isotropically distributed velocities in the range from zero to v_F. The wavelength of most of the electrons as seen from the moving particle will be somewhat smaller, but of the same order of magnitude as $2\pi\lambda$. We shall subdivide the space surrounding the incident particle into two regions: region I within a sphere having its center at the position of the particle and radius $\pi\lambda_F$; and region II comprising the space

[37] F. Bloch, Z. Physik. **81**, 363 (1933).
[38] F. Bloch, "Lecture Series in Nuclear Physics," U. S. Atomic Energy Commission Report MDDC-1175 (U. S. Government Printing Office, Washington, D. C., 1947), p. 26.

outside of the sphere. The contribution to the stopping power from the region I is negligible, since in this region the incident particle traverses a wave packet that exerts on it electrical forces from various directions, thus giving partial cancellation. Therefore, we shall assume that the total contribution to the stopping power is due to the presence of plasma in the region II.

The expression for the stopping power can, therefore, be obtained by substituting $R=\pi\lambda_F$ in (94). Of particular interest is the case of $a_{FD}\ll 1$ which has been treated by Fermi and Teller[39] in a different manner. To determine the stopping power for this case, we substitute $R=\pi\lambda_F$ in (93), obtaining

$$-\frac{dW}{dz}=\frac{2}{3\pi}\frac{Z^2e^4m^2V}{\hbar^3}\ln\frac{v_F}{3.17\times 4\pi v_0}. \tag{95}$$

[39] E. Fermi and E. Teller, Phys. Rev. **72**, 399 (1947); see also N. Mott, Proc. Phys. Soc. (London) **137**, 1462 (1949).

Assuming that $v_F/v_0\gg 3.17\times 4\pi$, the formula (95) can be expressed as

$$-\frac{dW}{dz}=\frac{2}{3\pi}\frac{Z^2e^4m^2V}{\hbar^3}\ln\left(\frac{v_F}{v_0}\right). \tag{96}$$

This formula is identical to the corresponding formula by Fermi and Teller.

The Fermi-Teller formula does not apply to the conduction band of metals since it is derived under the assumption that $v_F\gg v_0$. In order to determine the stopping power for the conduction band of metals, the formula (92) would seem to be applicable.

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Microwave Study of Positive Ion Collection by Probes*

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The theory of the positive ion saturation region of probes at low pressure is modified to take into account the directed current at the sheath edge. This drift current explains the discrepancy between the theoretical and experimental ratio of the electron saturation probe current to the positive ion saturation probe current. The theory is checked by using microwave methods to obtain an independent measurement of the ion density. Probe and microwave measurements are compared in the pressure region of 0.05–6 mm Hg in hydrogen, argon, and helium.

I. INTRODUCTION

SINCE the original work of Langmuir and Mott-Smith,[1] probes have been used extensively for the study of plasma properties in low-pressure gas discharges. The electron and positive ion densities, the electron temperature, and the plasma potential are determined from the volt-ampere characteristics of the probe, called the "probe curve." It is customary to divide the probe curve into three parts: the positive ion saturation, the region of partial collection of electrons, and the electron saturation.

When the probe potential is sufficiently negative with respect to the plasma in which the probe is inserted, the probe will attract positive ions and repel all electrons and thus the probe current consists only of positive ions. This is called the positive ion saturation. In the vicinity of the probe, only positive ions contribute to the space charge and the potential is a steep function of position; this region is called the sheath. At larger distances from the probe, both electrons and positive ions contribute to the space charge and the electric fields are small. The plasma in this region is disturbed by the withdrawal of positive ions to the probe.

As the magnitude of the voltage on the probe with respect to plasma potential is decreased, the probe starts collecting electrons. When the probe is near plasma potential, the probe curve exhibits a bend. At plasma potential, the probe is not covered by a sheath and electrons and ions reach the probe by diffusion. Electron densities may be determined from Langmuir's theory[1] when the bend in the probe curve is sharp.

When the gas pressure is high or when the probes draw a large current, the probe curve does not exhibit a sharp bend at plasma potential and no electron

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[1] I. Langmuir and H. M. Mott-Smith, Gen. Elec. Rev. **27**, 449, 538, 762, 810 (1924).