PHYSICAL REVIEW VOLUME 98, NUMBER 6 JUNE 15, 1955

Symmetry Effects in Gas Kinetics. I. The Helium Isotopes

O. HALPERN, % *American Express Company, Zurich, Switzerland*

AND

R. A. BUCKINGHAM, *University College, London, England*

(Received January 26, 1955)

Continuing earlier investigations, we discuss in this paper the influence of the process of symmetrization of the wave functions of the colliding atoms upon various gas-kinetic cross sections. Through the choice of a suitable interaction potential it thus becomes possible to derive the temperature dependence of the viscosities of the two He isotopes near absolute zero; it is also possible to show that the differences between these two quantities is largely caused by the fact that the nuclei of He^3 and He^4 follow different statistics. The agreement between theory and experiment is not quite as satisfactory in the case of the viscosity of mixtures of the two isotopes. Finally, curves are presented showing the effect of symmetrization on various possible scattering experiments.

I. THEORETICAL DEVELOPMENT

IT was first established through observations on the collisions of charged particles that one needs to symmetrize the wave function when the interacting particles are identical. This principle of symmetrization was applied by Massey and Mohr[1] to the collision processes of neutral atoms and molecules in a gas. They considered in particular the collision cross sections determining viscosity, heat conductivity, and diffusion, showing the characteristic differences to be expected for similar and dissimilar particles. The results were applied to the interpretation of the observed temperature dependence of the transport properties of helium and hydrogen.

To apply these calculations to actual gases it was of course necessary to assume a law of force governing the collision processes. Since the effects due to symmetrization were rather small, and since furthermore these laws of force were not known precisely, a comparison of the theoretical results with observations did not give wholly convincing evidence for the need to symmetrize the wave function.

A subsequent investigation by Halpern and Gwathmey[2] modified the theoretical results just mentioned in several respects, and suggested an experimental approach which should make the observer more independent of the actual law of force. It was first pointed out that for H_2—as in general for all molecules and almost all atoms, excepting those with no electronic angular momentum and no nuclear spin—symmetrization must be carried out with care. In fact, apart from exceptional conditions, a better approach to reality would be to omit all symmetrization. This is so because all states with unequal internal quantum numbers must be considered as different. Alternatively one may say that since in a large number of states the wave function is to be symmetrized, while in a slightly smaller number of states it has to be antisymmetrized, the total result is almost equivalent to the use of Boltzmann statistics. For hydrogen in particular, except for a case mentioned below, this reduces the symmetrization effect very much.

It was further shown that the cross section for diffusion must not be calculated with a symmetrized wave function for any type of gas.

Now the fact that symmetrization should be carried out so much more carefully and rarely, led to a different approach which, it was hoped, would be less sensitive to assumptions concerning the law of force. Thus, consider the viscosity of H_2 at temperatures so low that almost all molecules occupy only the zeroth (para) or first (ortho) rotational levels. In para-H_2 no internal degrees of freedom need to be considered, so here the full principle of symmetrization holds true. In ordinary H_2 the ortho states have weight nine, so that in collisions between ortho molecules, as well as between ortho and para molecules, the symmetrization effect is practically negligible. We should therefore expect a noticeable difference in the viscosity of ordinary H_2 and para-H_2 at low temperatures.

In the paper quoted attention was also called to an isotope effect which according to the present theory could be expected to appear in the gas kinetic quantities. Different isotopes have in general different nuclear spins; the statistics of the electronic shell being the same, one should therefore expect (apart from any mass effect) characteristic differences in the expressions for viscosities, etc., caused by changes in symmetrization of the wave functions. As an illustration the case of neon was considered, neon being at that time the only element of which the isotopes were available and which had no electronic spin. However, the large mass and high boiling point of neon made the effect very small, and accordingly attention was later[3] called to the much more favorable example of He^4 and He^3. Since He^4 has no spin and follows Bose statistics, the orbital part of the wave function must be symmetrized in all collisions; He^3 has spin $\frac{1}{2}$ and follows Fermi statistics, and hence the orbital function must be anti-

---

[1] H. S. W. Massey and C. B. O. Mohr, Proc. Roy. Soc. (London) **A141**, 434 (1933); **A144**, 188 (1934).

[2] O. Halpern and E. Gwathmey, Phys. Rev. **52**, 944 (1937).

[3] O. Halpern, Phys. Rev. **82**, 561 (1951).

symmetrized in $\frac{3}{4}$ of all collisions of He^3 atoms, and symmetrized in the remaining $\frac{1}{4}$. Assuming for illustration that so low a temperature of observation could be used that most collisions are *S*-collisions, then the scattering cross section of He^4 (apart from the mass factor) must be expected to be four times that of He^3. This very unrealistic case is mentioned only to make evident that quite large differences can be expected in the two He isotopes.

Following the original contribution by Halpern and Gwathmey, several theoretical papers by other authors have given quantitative results for the temperature dependence of various properties of these isotopes, using more or less realistic forms for the law of force. Thus an early investigation by Buckingham and Massey[4] showed that symmetry effects in the second virial coefficient of He^4 might be significant below 1–2°K. This was substantiated later for He^3 by van Kranendonk, Compaan, and de Boer,[5] and in greater detail in the recent extensive calculations of Kilpatrick *et al.*[6] In connection with the gaseous viscosity of the He isotopes, measurable effects below 3°K were shown to occur by Buckingham and Temperley,[7] de Boer and Cohen,[8] and Buckingham and Scriven.[9] These are discussed in detail in later sections.

II. EXPERIMENTAL WORK

The theoretical views here summarized have only recently attracted the attention of experimentalists. Becker[10] and his collaborators have published several investigations of the differential effects in the viscosities of normal and para-H_2, (up to 90°K) normal and para-D_2 (at 15° and 20°K), and also of the two He isotopes at much lower temperatures. The observations showed that the collision cross sections in para-H_2 are very slightly *smaller* than those in ortho-H_2, when both are observed in their lowest rotational state. Now the assumption of a rigid sphere model, with the same collision radius for all molecules, leads to the result that the para-H_2 cross section should always be *larger* than that for the ortho-H_2 molecules. For interaction potentials with an attractive part it is true that this difference is appreciably modified in magnitude, and at some low temperatures may even be reversed in sign. Nevertheless, it seems unlikely that this behavior is reached at the temperatures at which Becker *et al.*, made their observations, and hence there is a clear indication that even for differential observations the law of force should not be assumed the same for all collisions. A slight admixture of noncentral forces, which can be expected to be present for ortho-H_2 molecules, would possibly lead to the observed effect and hide the small effect due to symmetrization. It is doubtful, however, whether theoretical calculations of the effect of such noncentral forces can be very useful; the parameters in the intermolecular potential are not well defined at present, so that any agreement between the few observations and the theory would likely be of more algebraic than physical significance.

The difficulties caused by noncentral forces do not occur in the case of the He isotopes; here the experiments have led to large differences between the viscosities of the kind expected from the theory. It is also probable that symmetrization effects in the collision of hydrogen molecules can be made evident by a differential experiment which can hardly be influenced by the presence of noncentral forces. We are referring to the scattering of para-H_2 molecules by para-H_2 and normal H_2, respectively. Since the symmetrization effect may be quite considerable for scattering through small angles, its detection becomes a question of intensity only. We have learned that Dr. Becker has started an experiment for the purpose of observing such an effect.

In the remainder of this paper we consider some properties of the He isotopes which can be derived from the usual quantal phases connected with collisions between He atoms. The viscosity results are presented in a form such that the quantum effects of symmetrization are most clearly distinguishable from the other quantum effects arising from the fact that the de Broglie wavelength associated with the collision is comparable with the atomic dimensions. In deriving the viscosity of He^3, appropriate use is made of the Enskog-Chapman theory for mixtures, the components of the mixture in this case being the atoms in various spin states. This procedure is further applied to find the viscosity of mixtures of the He isotopes. Finally, the effect of symmetry on the cross section for scattering of He atoms is considered in more detail than hitherto.

In a subsequent paper we shall return to the problem of hydrogen and deuterium, and shall present data which may be useful in relation to scattering and viscosity experiments with these gases.

III. VISCOSITY OF PURE He^3 AND He^4

We must first summarize the essential formulas used in deriving the viscosity η. In what follows it is assumed that the interatomic potential energy has the usual form corresponding to a steep repulsion at small distances combined with the usual van der Waals attraction at larger distances. It can be generally represented by

$$V(r) = -\epsilon f(\sigma),$$

where $\sigma = r/r_m$, ϵ and r_m being physical parameters repre-

---

[4] R. A. Buckingham and H. S. W. Massey, Proc. Roy. Soc. (London) **A168**, 378 (1938).

[5] van Kranendonk, Compaan, and de Boer, Phys. Rev. **76**, 998 and 1728 (1949).

[6] Kilpatrick, Keller, Hammel, and Metropolis, Phys. Rev. **94**, 1103 (1954).

[7] R. A. Buckingham and H. N. V. Temperley, Phys. Rev. **78**, 482 (1950).

[8] J. de Boer and E. G. D. Cohen, Physica **17**, 993 (1951).

[9] R. A. Buckingham and R. A. Scriven, Proc. Phys. Soc. (London) **65**, 376 (1952).

[10] E. W. Becker and O. Stehl, Z. Physik **133**, 615 (1952). Becker, Misenta, and Stehl, Z. Physik **136**, 457 (1953); Becker, Misenta, and Schmeissner, Z. Physik **137**, 126 (1954).

senting the depth and position of the minimum in $V(r)$, and $f(\sigma)$ a suitable shape function such that $f(1)=1, f'(1)=0$.

Using this interaction the viscosity cross section can be expressed in terms of the familiar phase shifts δ_L, obtained by solving the Schrodinger equation for different values of the collision parameter $k(=\pi Mv/h)$ and for integral values of the azimuthal quantum number L. Thus

$$Q_\eta=\frac{4\pi}{k^2}\sum_L \omega_L\frac{(L+1)(L+2)}{(2L+3)}\sin^2(\delta_{L+2}-\delta_L), \quad (1)$$

where ω_L takes different values according to the type of statistics used:

Statistics	Bose	Fermi	Classical
L even	$\omega_L=2$	0	1
L odd	$\omega_L=0$	2	1

For a given shape of interaction $f(\sigma)$, it is convenient to regard Q_η as a function of two variables K and Λ_m, where K $(=Mv^2/4\epsilon)$ is the relative kinetic energy of collision in units of ϵ, and $\Lambda_m[=h/(M\epsilon r_m^2)^{\frac{1}{2}}]$ is the ratio of the de Broglie wavelength for collision energy ϵ to the atomic diameter r_m. In place of Λ_m we sometimes use $\lambda=2\pi/\Lambda_m$.

We now define the following reduced quantities:

Reduced temperature $\qquad T^*=\kappa T/\epsilon, \quad (2)$

Reduced viscosity $\qquad \eta^*=[r_m^2/(M\epsilon)^{\frac{1}{2}}]\eta, \quad (3)$

Reduced viscosity cross section

$$S_\eta(K,\Lambda_m)=Q_\eta/(\tfrac{2}{3}\pi r_m^2), \quad (4)$$

Reduced integrated cross section

$$\bar{S}_\eta(T^*,\Lambda_m)=\frac{1}{3!}\int_0^\infty x^3e^{-x}S_\eta dx, \quad (5)$$

where $x=K/T^*$. Both S_η and $\bar{S}_\eta$ would be unity in the classical theory of a gas of rigid spheres of diameter r_m. The integrated cross section $\bar{S}_\eta$ (sometimes written in the general theory of transport phenomena as $\Omega^{*(2,2)}$) is effectively the reciprocal of the reduced viscosity, since

$$\eta^*(T^*)^{-\frac{1}{2}}=\frac{5(1+q)}{16\pi^{\frac{1}{2}}\bar{S}_\eta}, \quad (6)$$

q here being a small higher-order correction which at low temperatures does not exceed 0.005.

The foregoing formulas, which involve the substitution of an appropriate quantal cross section in the Chapman-Enskog expression for η, are quite unambiguous when the gas consists of identical particles of mass M. Thus, if these are He^4 atoms, which have no net nuclear or electronic spin and should therefore follow Bose statistics, the viscosity can be obtained by calculating phases for even L only. The position is less simple for a pure He^3 gas, consisting of atoms with nuclear spin $\pm\frac{1}{2}$ in equal proportions. In several papers[7,8] it has been tacitly assumed that the correct quantal cross section to insert in (5) is

$$S_\eta(\mathrm{He}^3)=\tfrac{1}{2}\{S_\eta(\mathrm{Fermi})+S_\eta(\mathrm{class.})\}$$
$$=\tfrac{1}{4}\{S_\eta(\mathrm{Bose})+3S_\eta(\mathrm{Fermi})\}, \quad (7)$$

since one-half of all collisions are between unlike atoms (with opposing spins) and for these no symmetrization is necessary.

To substantiate this procedure it is advisable to examine the extended form of the Chapman-Cowling and Enskog theory, developed by Curtiss and Hirschfelder[11] for gaseous mixtures. It is unnecessary to quote their general result here, as this is lengthy, but we consider first the simple case in which the gas contains two types of atom, of equal mass, in proportions x_1 and $x_2(=1-x_1)$. The viscosity of the mixture is then expressed in terms of the viscosities η_1 and η_2 of pure gases of the two types, and also of the viscosity η_{12} of a fictitious gas in which all collisions are assumed to occur between unlike atoms. In calculating η_1 and η_2, therefore, the proper symmetry must be applied; for η_{12} no symmetrization is necessary. In He^3 gas, in which atoms are distinguished by the orientation of their spin, we have the further simplification that $\eta_1=\eta_2$ and $x_1=x_2=\frac{1}{2}$. Leaving x_1 unspecified for the moment, we find from the Curtiss-Hirschfelder formula that

$$\frac{1}{\eta_{\mathrm{mix}}}=\gamma+(\alpha-\gamma)\frac{x_1^2+x_2^2+cx_1x_2}{1+2cx_1x_2}, \quad (8)$$

where $\alpha=1/\eta_1$, $\gamma=1/\eta_{12}$, $\beta=(5/6A_{12}^*)\gamma$, and $c=(\alpha-\gamma)/(\beta+\frac{1}{2}\gamma)$. The quantity A_{12}^* is here the ratio of two collision integrals closely connected with diffusion of the two types of atom; in magnitude it is not far from unity.

Let η_{cl}, η_F, and η_B refer to the viscosities calculated using classical, Fermi, and Bose statistics, respectively, satisfying the relation $1/\eta_{\mathrm{cl}}=\frac{1}{2}(1/\eta_F+1/\eta_B)$. Now for He^3, we must take $\eta_1=\eta_F$ and $\eta_{12}=\eta_{\mathrm{cl}}$, whence it follows that $\alpha-\gamma=\frac{1}{2}(1/\eta_F-1/\eta_B)$. It is then clear that the term in $(\alpha-\gamma)$ is essentially a low temperature correction, since $\alpha-\gamma$ tends to zero at high temperatures, leaving $\eta_{\mathrm{mix}}=\eta_{12}$ in this limit. However, when we insert $x_1=x_2=\frac{1}{2}$ as in the actual gas, we obtain at all temperatures,

$$1/\eta_{\mathrm{mix}}=\gamma+\tfrac{1}{2}(\alpha-\gamma)=\tfrac{1}{4}(3/\eta_F+1/\eta_B).$$

This is exactly what is assumed in (7), but it should be noticed that the result depends on the two spins being equally abundant in the mixture. For any other distribution the value of η_{mix} depends on c, and therefore on A_{12}^*.

---

[11] C. F. Curtiss and J. O. Hirschfelder, J. Chem. Phys. **17**, 550 (1949).

Figure 1 shows the behavior of the viscosity cross sections defined previously for He^4 and He^3. These results have been calculated by using an interatomic potential of the form

$$f(\sigma)=f_1\sigma^{-6}(1+b\sigma^{-2})-f_2e^{-(\sigma-1)}, \tag{9}$$

where

$$f_1=a/\{a(1+b)-6-8b\{, \quad f_2=-1+(1+b)f_1,$$

and the parameters a, b have been assigned the values 13.5, 0.2, respectively. For He^4, even-order phases using this type of potential have been given by Buckingham, Hamilton, and Massey,[12] and for He^3 they have been calculated by Scriven and Buckingham,[13] in each case for values of K sufficient to give the viscosity up to 5°K. For the present purpose, it was necessary to estimate odd order phases for He^4, and it was found possible to do this adequately without resorting to numerical integration of the wave equation. The values of λ^2 for He^4 and He^3 were taken to be 7.27 and 5.48 ($\Lambda_m=2.33$, 2.68, respectively). These correspond to a value of 122×10^{-32} erg cm^2 for $\epsilon r_m{}^2$, known from earlier work[12] to give reasonable agreement with the observed second virial coefficient of He^4 gas below 5°K.

In both Figs. 1(a) and 1(b) two pairs of curves have been given. The full-line curves correspond to the quantal statistics expected to hold [Bose for He^4, the mixture (7) for He^3]; the broken curves to the classical Boltzmann statistics. Since the classical mass factor ($M^{\frac{1}{2}}$) does not enter at this stage, the difference between the broken curves is a measure of the quantal mass effect, corresponding to the two values of λ^2 in the aforementioned. The departure of each full-line curve from its accompanying broken curve then shows how the cross section is affected by using the correct statistics, i.e., the symmetrization effect.

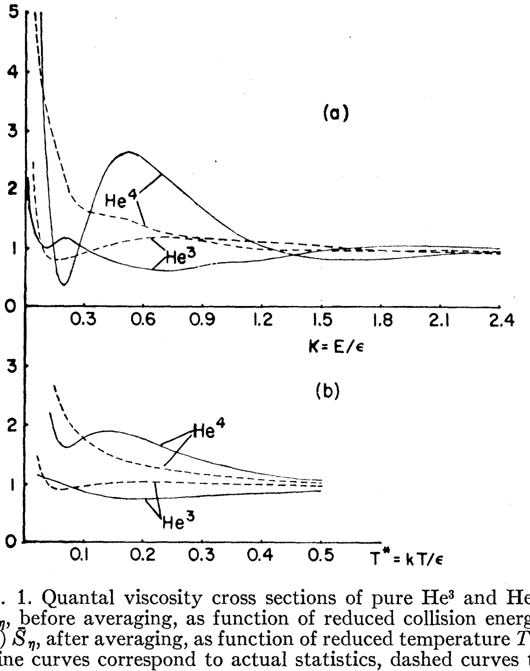

FIG. 1. Quantal viscosity cross sections of pure He^3 and He^4. (a) S_η, before averaging, as function of reduced collision energy K; (b) $\bar{S}_\eta$, after averaging, as function of reduced temperature T^*. Full-line curves correspond to actual statistics, dashed curves to classical Boltzmann statistics.

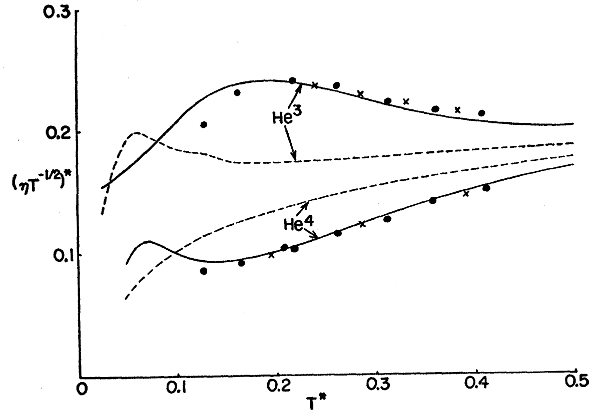

FIG. 2. Comparison of theoretical and observed viscosities of pure He^3 and He^4. Theoretical full-line curves correspond to actual statistics, dashed curves to classical statistics. Experimental points, due to Becker *et al.*, have been reduced by assuming $\epsilon/\kappa=10.2°$, $r_m=2.94$ A.

The chief features of the S_η curves are the rapid rise at small energies, and the considerable oscillations of the quantal statistical curves about those for classical statistics. These oscillations soon become very small for larger values of K, and as one would expect they are also much less marked in the variation of $\bar{S}_\eta$, which represents the result of averaging S_η with respect to the Maxwell velocity distribution. It should be noticed that the K and T^* scales are chosen so that for equal lengths, $K=3T^*$; this corresponds to the fact that the weighting function x^3e^{-x} has its maximum when $x=K/T^*=3$.

In Fig. 1(b), it is seen that when $T^*=0.5$ the curves are all fairly close to unity, which is the classical value for rigid spheres. However, for large T^* the value of $\bar{S}_\eta<1$ because the effective collision radius $<r_m$. For He^4 and $T^*>0.1$, the Bose curve is above the Boltzmann curve; this again is similar to the result for rigid spheres, but it should be noticed that when $T^*\simeq0.1$, a reversal occurs which is directly connected with the attractive part of the potential. For He^3, the behavior is complicated by the mixture of Bose and Fermi statistics. The other important point is that when $T^*>0.1$ the symmetrization effects, being in opposition for the two isotopes, enhance the mass effect, and indeed over much of the temperature range the difference in viscosity of the isotopes is due more to the different statistics than to the different masses.

In plotting S_η against K, and $\bar{S}_\eta$ against T^*, it has not been necessary to specify the energy parameter ϵ. This can in fact be chosen to give the best over-all agreement with observed viscosity values. Such a

---

[12] Buckingham, Hamilton, and Massey, Proc. Roy. Soc. (London) **A179**, 103 (1941).

[13] R. A. Scriven and R. A. Buckingham (to be published).

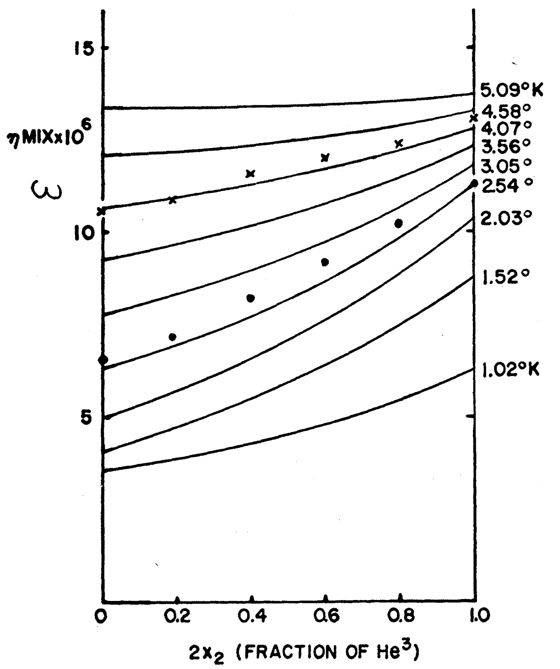

FIG. 3. Viscosity of mixtures of He^3 and He^4. Theoretical curves, calculated from formula (11) using potential with $\epsilon/\kappa=10.2°$K, $r_m=2.94$ A. Experimental points are due to Becker *et al.*; × at 4.15°K, ● at 2.64°K.

comparison is made in Fig. 2, where the theoretical values of $\eta^*(T^*)^{-\frac{1}{2}}$ for He^4 and He^3 are shown against the experimental values of Becker *et al.*, the temperatures being reduced by assuming $\epsilon/\kappa=10.2°$K. The excellent agreement for both isotopes is a clear vindication of the theory in general, of the method of symmetrization, and to some extent, of the choice of interaction.

IV. VISCOSITY OF HE^3–He^4 MIXTURES

Having considered the pure isotopes, it is natural to ask how the viscosity of a mixture of them depends on their relative concentration. In terms of the Curtiss-Hirschfelder mixture formula we can examine this by considering a mixture which contains a molar fraction x_1 of He^4 atoms, and fractions x_2 of each spin-type of He^3 atom, so that $x_1+2x_2=1$.

The first approximation to η_{mix} is given[11] by the ratio of two determinants,

$$\frac{1}{\eta_{\text{mix}}}=-\begin{vmatrix} h_1 & k_1 & k_1 \\ k_1 & h_2 & k_2 \\ k_1 & k_2 & h_2 \end{vmatrix} \div \begin{vmatrix} h_1 & k_1 & k_1 & x_1 \\ k_1 & h_2 & k_2 & x_2 \\ k_1 & k_2 & h_2 & x_2 \\ x_1 & x_2 & x_2 & 0 \end{vmatrix}$$

$$=\frac{h_1(h_2+k_2)-2k_1^2}{(h_2+k_2)x_1^2-4k_1x_1x_2+2h_1x_2^2}, \tag{10}$$

where

$$h_1=\alpha_1x_1^2+(2\beta_1+\gamma_1/s)x_1x_2,$$
$$h_2=\alpha_2x_2^2+(\beta_1+\tfrac{1}{2}s\gamma_1)x_1x_2+(\beta_2+\tfrac{1}{2}\gamma_2)x_2^2,$$
$$k_1=-(\beta_1-\tfrac{1}{2}\gamma_1)x_1x_2,$$
$$k_2=-(\beta_2-\tfrac{1}{2}\gamma_2)x_2^2,$$

and

$$\alpha_1=1/\eta_1 \qquad \alpha_2=1/\eta_2,$$
$$\beta_1=5p^2/(6A_{12}^*\eta_{12}) \qquad \beta_2=5/(6A_{23}^*\eta_{23}),$$
$$\gamma_1=p^2/\eta_{12} \qquad \gamma_2=1/\eta_{23},$$
$$p^2=4M_1M_2/(M_1+M_2)^2 \qquad s=M_1/M_2.$$

M_1 and M_2 are the masses of He^4 and He^3 atoms respectively; η_1 is the viscosity of a (real) Bose gas of He^4 atoms, and η_2 that of a (fictitious) Fermi gas of He^3 atoms; η_{12} and η_{23} are also fictitious viscosities referring to collisions between one He^4 and one He^3 atom, and between two He^3 atoms respectively, calculated without any symmetrization.

By algebraic reduction, using the relation between x_1 and x_2, the formula for $1/\eta_{\text{mix}}$ becomes

$$\frac{1}{\eta_{\text{mix}}}=\frac{(s+\mu)\alpha_1x_1^2+4\gamma_1x_1x_2\{1/s-(1-\alpha_1\alpha_2'/\gamma_1^2)+\frac{1}{4}\mu(s+2+1/s)\}+(1/s+\mu)4\alpha_2'x_2^2}{(s+\mu)x_1^2+4x_1x_2\{(\alpha_1+\alpha_2')/\gamma_1-1+\mu\}+(1/s+\mu)4x_2^2}, \tag{11}$$

where $\mu=2\beta_1/\gamma_1$ and $\alpha_2'=\frac{1}{2}(\alpha_2+\gamma_2)$. It will be observed that β_2, which involves interdiffusion of the two spin components of He^3, has vanished from these expressions as it did for pure He^3.

All the quantities required to calculate η_{mix} are available, including A_{12}^* which is related to the diffusion coefficient of He^3 and He^4, and which has been evaluated for the above potential by Buckingham and Scriven.[9]

Figure 3 shows how the viscosity of the mixture depends on the molar fraction of He^3 present, for temperatures up to 5°K. The same values of the potential parameters, Λ_m and ϵ/κ, have been used as before. Also shown in the figure are the experimental values of Becker *et al.*[10] for 2.64° and 4.15°K. It will be seen that the theoretical curves show departures from linearity which, up to 3.5°K at least, are of the order of 5 percent when $x_1=2x_2=0.5$. The experimental results on the other hand do not indicate any very significant departure from linearity. One would expect that a deviation of 5 percent should be observable, since in general the experimental results of pure He^3 and pure He^4 appear to be consistent to within 2–3 percent and moreover Becker *et al.* claim to measure the fraction of He^3 present in mixtures to 2 percent. The point seems to deserve further investigation.

V. SCATTERING OF He^4 ATOMS

A further test of symmetrization effects would appear to be possible in the direct scattering of He atoms. The most sensitive comparison, and one not likely to depend critically on the chosen law of force, would be that of the differential cross section for scattering through 90° in the center-of-mass system, at fairly low thermal energies. Thus, since collisions involving antisymmetrization make no contribution to scattering through 90°, the cross sections corresponding to Bose statistics, classical statistics, and a 3:1 mixture of Fermi and Bose statistics, would be in the ratios 4:1:1, regardless of the collision energy. Differential cross sections which have been calculated for He^4-He^4, He^3-He^4 and He^3-He^3 collisions, assuming the appropriate statistics, although modified to some extent by mass effects, do show ratios which are substantially the same as the above and are not very sensitive to collision energy, at least when the energy parameter K exceeds about 0.2 (corresponding to 2°K).

For an observational test of the symmetrization effect therefore, the scattering near 90° is valuable because the effect is not obscured if the collision energies cover a fairly wide range. This is not necessarily so for scattering through small angles; although the effects of symmetry are still substantial, they are much more sensitive to the collision energy, and tend to average out over a range of energies. That this is likely can easily be seen by considering the variation with energy of the total scattering cross section, and its dependence on the statistics assumed, for say He^4-He^4 collisions. The analysis is essentially the same as for the viscosity cross section, though in place of (1) we have

$$Q_{sc}(K,\Lambda_m)=\frac{4\pi}{k^2}\sum_L \omega_L(2L+1)\sin^2\delta_L, \tag{12}$$

and as a suitable reduced cross section, $S_{sc}=Q_{sc}/\pi r_m^2$. In addition to S_{sc} we have evaluated an average cross section $\bar{S}_{sc}$, by integrating S_{sc} with respect to energy and assuming a Maxwell distribution. Thus

$$\bar{S}_{sc}(T^*,\Lambda_m)=\int_0^\infty xe^{-x}S_{sc}dx, \tag{13}$$

where $x=K/T^*$ as before.

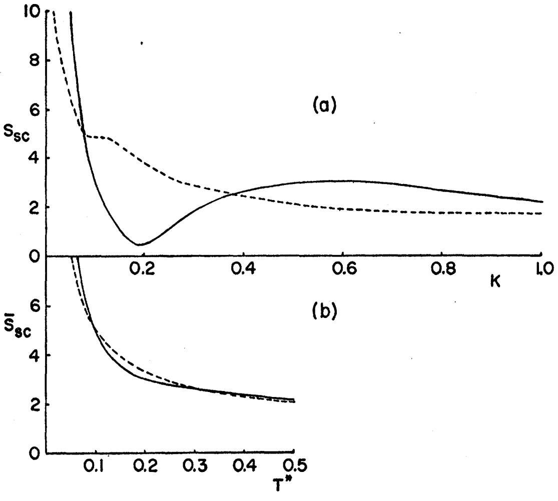

FIG. 4. Total scattering cross sections for He^4-He^4 collisions, (a) S_{sc}, before averaging, as a function of the reduced collision energy K; (b) $\bar{S}_{sc}$, after averaging, as a function of the reduced temperature T^*. Full-line curve corresponds to Bose statistics, dashed curve to classical statistics.

The presentation of these cross sections in Figs. 4(a) and 4(b) is exactly similar to that of the viscosity cross sections, except that the scales of K and T^* are now taken to be the same, since the weighting function xe^{-x} has its peak when $x=1$. The significant points are first, the oscillatory behavior of the S_{sc} curves for Bose and classical statistics with respect to each other, and secondly, the very marked diminution of these oscillations after the averaging carried out in $\bar{S}_{sc}$. This makes the detection of any symmetry effect from the total scattering cross section very difficult if the distribution of relative velocities is anything like Maxwellian; hence it is desirable that the collisions should be as nearly monoenergetic as possible. The same applies to measurements of the differential cross section at small angles, as indicated above. On the other hand it must be remembered, in connection with angular scattering near 90°, that, although the energy dependence is much less critical, the differential cross section at 90° is relatively small, about 5–10 percent of that near 0°.