Self-Focusing Streams

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Streams consisting of a mixture of ions and electrons in which the average velocity of the electrons in the direction of the stream is different from the average stream velocity of the ions can be magnetically self-focusing. The random velocities of the electrons and ions about their respective average stream velocities do not need to be Maxwellian. The only condition which must be met is that the total electric current in stream must exceed a critical value which can be calculated in terms of the stream velocities and the average kinetic energies of the particles due to components of velocity transverse to the axis of the stream. Streams containing both slow and fast particles of like kind tend to eject the slow particles and keep the fast ones. Streams of particles of one kind only, entering an ionized or ionizable region can be self-focusing. Axial asymmetries grow in arcs but disappear in low-density self-focusing streams.

SINCE the original publication[1] of the basic theory applicable to magnetic self-focusing in streams consisting of mixtures of charged particles of opposite polarities, a number of experiments have been reported on magnetic self-focusing.[2–4] Theoretical papers have also appeared[5–8] which were written apparently in ignorance of the original work and which failed to recognize the more important physical properties of such streams. For this reason it would be desirable at this time to present the theory of magnetic self-focusing in a somewhat more general form in order to point out these properties more clearly and to assist in presenting some new applications of the theory which have recently been recognized.[9] The theory will be developed in a relativistically invariant form and it will be shown: (1) that whether or not mixed low-density ion and electron streams have Maxwellian velocity distributions, there is a critical current which must be exceeded if the stream is to be magnetically self-focusing; (2) that in the evolution of any such stream, the first things to happen are the permanent loss from the stream of some of its components and the rapid attainment of cylindrical symmetry by the remainder of the stream; (3) that such streams are not subject to the "kink" instability of arcs; (4) that the application of a sustained emf to the stream, as done in all reported experiments, makes published theories wholly inadequate to explain those experiments; and (5) that this theory explains magnetic self-focusing in interplanetary proton streams.

The kind of stream considered consists of a stream of ions most of which move in one direction mixed with a stream of electrons most of which move in the opposite direction. Each kind of particle in the stream is assumed to have a distribution in velocity and in radial distance from the axis which is approximately uniform along the stream at any instant and which is symmetric about the same axis at all times. No restrictions will be imposed on the radial variations of these distributions with time. It will be supposed that the ions and electrons may initially each have any distribution in velocity component parallel with the axis, and that each such distribution may be divided into a a large number of subdistributions each of which consists of all the particles of one kind in the stream having velocity component parallel with axis in a small interval of such velocity component. The various subdistributions are mixed with each other in space but not necessarily in uniform proportions because each subdistribution may have a distribution in radial distance from the axis which initially is different from that of the other subdistributions. Each subdistribution is supposed to continue to be symmetric about the axis while its radial distribution may vary with time.

It is essential at the outset to distinguish clearly between "low-density streams" in which the effects of collisions are small during the mean time of travel of charged particles between their positions farthest from and nearest to the axis, and "high-density streams" where the effects of collisions during such a time are large making it necessary that a sustained emf be applied in the direction of the stream to maintain a current or in other words to have any stream at all. The first part of this paper will relate to low-density streams to which no emf is applied in the direction of the stream, after which high-density streams will be discussed.

LOW-DENSITY STREAMS

The force exerted on any particle with charge e_1 and velocity $\mathbf{v}_1$ by any other charge e_2 with velocity $\mathbf{v}_2$ located at a displacement $\mathbf{r}$ from the first particle is

$$\mathbf{E}+(1/c^2)\mathbf{v}_1\times[\mathbf{v}_2\times\mathbf{E}], \tag{1}$$

[1] W. H. Bennett, Phys. Rev. 45, 890 (1934).
[2] S. W. Cousins and A. A. Ware, Proc. Roy. Soc. (London) 64, 159 (1951) and A. A. Ware, Trans. Roy. Soc. (London) 243, 197 (1951).
[3] Bostick, Levine, and Coombes, Gaseous Electronics Conference, Princeton, New Jersey (September 6, 1952) (unpublished).
[4] Thoneman, Cowhig, and Davenport, Nature 169, 34 (1952).
[5] L. Tonks, Phys. Rev. 56, 360 (1939).
[6] P. C. Thoneman and W. T. Cowhig, Nature 166, 903 (1950); Proc. Roy. Soc. (London) 64, 345, 618 (1951).
[7] M. Blackman, Proc. Roy. Soc. (London) 64, 1039 (1951).
[8] A. Schlüter, Z. Naturforsch. 6A, 73 (1951).
[9] For one of these, see W. H. Bennett and E. O. Hulburt, Phys. Rev. 95, 315 (1954) and J. Atmos. Terrest. Phys. 5, 211 (1954).

where the first term is the Coulomb electrostatic force between the two charged particles in the direction of **r** and the second term is the magnetic force[10] not generally in the direction of **r**. By resolving the force on any one charged particle into radial, tangential and axial components and integrating over all the other particles in the assumed radially symmetric and longitudinally uniform stream, it is found that the force on each particle in the stream in which the transverse components of velocity are a lower order of magnitude than the axial components of velocity of at least some of the particles in the stream is approximately the same as it would be if all particles in the stream had velocities equal to only their components of velocity parallel with the axis (axial components of velocity).

If e_α is the charge on each particle in the αth subdistribution, ρ_α is the numerical density of particles in that subdistribution and is a function of radial distance r from the axis, and u_α is the axial component of velocity of the particles in that subdistribution, then the force (radially away from the axis) which the particles in the αth subdistribution exert on any particle, say one in the κth subdistribution, having charge e_κ and axial component of velocity u_κ, is given by

$$\frac{2e_\kappa e_\alpha}{r}\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\int_0^r \rho_\alpha 2\pi r dr,$$

and the total force acting on any particle is

$$F_\kappa=\frac{2e_\kappa}{r}\sum_\alpha e_\alpha\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\int_0^r \rho_\alpha 2\pi r dr, \qquad (2)$$

which is obtained by summing over all the subdistributions in the stream.

In the radially symmetric and longitudinally uniform streams considered here, the forces acting on each particle are radial, that is, the forces are cylindrically central. Except for the effects of collisions, the angular momentum of each particle about the axis is conserved. If the particles have random distribution in angular momentum and in total energy, they will have correspondingly random distributions in period of motion between the positions nearest to and farthest from the axis and they will have random rates of precession about the axis. In a low-density stream, these spreads in periods and in rates of precession produce a mixing in radial distribution approaching a nearly steady state early in the evolution of the stream and long before collisions within each subdistribution have produced Maxwellian velocity distributions within each of the various subdistributions.

The stream will be said to be in its first phase from the time it is first set up with its non-Maxwellina distributions until each subdistribution has become approximately Maxwellian, and will be in its second phase while the kinetic energies due to transverse momenta are slowly increasing due to conversions of axial momenta into transverse momenta.

[10] M. Mason and W. Weaver, *The Electromagnetic Field* (Dover Publications, New York, 1952), p. 299.

FIRST PHASE

The virial of Clausius in cylindrical coordinates can be used for finding the directions of radial acceleration in a stream prior to the arrival of the stream at the quasi-steady state produced by the mixing mentioned above. The virial is derived in Appendix A, and the second time derivative of the moment of inertia per unit length of the stream is obtained:

$$\frac{1}{2}\frac{d^2I}{dt^2}=2N\psi+\frac{i^2}{c^2}-q^2.$$

In this equation, the moment of inertia per unit length of stream,

$$I=\sum_\alpha \int_0^\infty m_\alpha r^2\rho_\alpha(r)2\pi r dr,$$

may be used as a measure of the radial spread of the stream without implying that there will be any rotation of the stream about the axis, which of course there cannot be if the velocity distributions are symmetrical about the axis; N is the total number of particles per unit length of stream; ψ is the mean kinetic energy per particle due to components of velocity transverse to the direction of the axis;

$$i=\sum_\alpha e_\alpha u_\alpha\int_0^\infty \rho_\alpha(r)2\pi r dr$$

is the total electric current in the stream; and

$$q=\sum_\alpha e_\alpha\int_0^\infty \rho_\alpha(r)2\pi r dr$$

is the total charge per unit length of stream. For the stream to concentrate towards the axis, dI/dt must be negative and for this reason, a negative value of d^2I/dt^2 is used as the criterion for self-focusing. This criterion can be written in the form

$$i_c=c(2N\psi+q^2)^{\frac{1}{2}} \qquad (3)$$

for the critical current which the stream must exceed in order to be self-focusing. In a stream in which there is zero net charge per unit length of stream ($q=0$), all subdistributions of positive charges moving in the direction opposite to that of the net current, i, and all negative charges moving with the net current are magnetically repelled everywhere and permanently lost from the stream. Such losses increase the net current, i, in the stream and increase the value of "transverse" kinetic energy ψ which the remainder of the stream may have and still be self-focusing. As will be shown in the following, the changes in the velocity and density distributions produced by mixing and more slowly by collisions do not at any time violate the aforementioned criterion for self-focusing.

Boltzmann's equation can be used in deriving the quasi-steady state equation which must be satisfied by each subdistribution as soon as radial redistributions have been arrested by the mixing specified earlier. The quasi-steady state equation is derived in Appendix B, Eq. (20):

$$e_\kappa \sum_\alpha e_\alpha \left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\int_0^r \rho_\alpha(\xi)2\pi\xi d\xi+\theta_\kappa-\phi_\kappa -\frac{r}{\rho_\kappa}\frac{d}{dr}(\rho_\kappa\phi_\kappa)=0, \quad (4)$$

where θ_κ is the mean kinetic energy at radial distance r of the κth kind of particle due to tangential components of velocity, and ϕ_κ is the mean kinetic energy at r of the same particles due to radial components of velocity. This equation is useful for deriving relationships, which will not be detailed here, for the first-phase steady-state stream prior to the attainment of Maxwellian velocity distributions, and it will also be needed in deriving the equations for the second phase with its Maxwellian distributions in the following.

SECOND PHASE

The effects of collisions in systems consisting principally or entirely of charged particles can most easily be found by applying the equations derived by Thomas.[11] Where a charged particle is moving among charged particles there is no definite mean free path but time rates of change of mean kinetic energy can be calculated. Thomas' equation (4.33) can be rewritten to give the rate of change of mean kinetic energy of a particle with charge e and mass m_1 moving with a velocity v_1 through a density N_2 of particles each with charge e, mass m_2 and root mean square velocity v_2. When $v_1>v_2$ the rate of change of mean kinetic energy ψ due to components of velocity transverse to the direction of v_1 (which multiplied by $\frac{3}{2}$ equals the rate of increase of mean thermal energy of the particle) is given approximately by

$$\frac{d\psi}{dt}=\frac{4\pi e^4 N_2}{m_1 v_1}\log\left(\frac{2A}{\log A}\right), \quad (5)$$

where

$$A=\frac{3m_1^3 m_2^3 v_1^6}{4\pi e^6(m_1+m_2)^3 N}.$$

In this Eq. (5), the N_2 is the density of the particles of the second kind whose mean thermal velocity is less than the velocity of the particle of the first kind relative to the mean velocity (vectorial) of the particles of the second kind. As the particle of the first kind approaches thermal equilibrium with the particles of the second kind, the portion which N_2 is of the density of all the particles of the second kind diminishes and so the expression (5) for $d\psi/dt$ finally approaches zero as it approaches mean random velocity of the particles of the second kind. The value of (5) is comparatively insensitive to the logarithmic term and to an adequate approximation the rate of change of energy of the particle is inversely proportional to the velocity of the particle relative to the other particles with which it is in continuous "remote" collision. For this reason, any particle in a stream approaches thermal equilibrium with the other particles in its own subdistribution and so each subdistribution approaches a Maxwellian distribution with its own temperature. Next in rate of progress is the approach of adjacent subdistributions (that is, subdistributions with adjacent values of velocity component parallel with the axis of the stream) to thermal equilibrium, or more accurately, adjacent subdistributions tend to acquire adjacent values of temperature at the same time that each subdistribution is acquiring a Maxwellian distribution. Least rapid in rate of progress is the conversion of momenta parallel with the axis into momenta transverse to the axis due to collisions between particles having large differences in velocity parallel with the axis.

As each subdistribution approaches a Maxwellian distribution, the temperature T_κ for that distribution (see Appendix C) is related to the mean kinetic energy due to components of velocity for each degree of freedom by

$$\tfrac{1}{2}kT_\kappa=\theta_\kappa=\phi_\kappa.$$

Substituting into Eq. (4) and transposing gives

$$kT_\kappa r\frac{d}{dr}\log\rho_\kappa=2e_\kappa\sum_\alpha e_\alpha\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\int_0^r \rho_\alpha 2\pi r dr. \quad (6)$$

Taking the derivative and dividing by r gives

$$kT_\kappa\nabla^2\log\rho_\kappa=4\pi e_\kappa\sum_\alpha e_\alpha\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\rho_\alpha, \quad (7)$$

where ∇^2 is the Laplacian in cylindrical coordinates.

The variables in these equations can be transformed to the frame of reference of an observer moving parallel with the axis at a velocity v as explained in Appendix C, and the Eqs. (7) become

$$kT_\kappa'\nabla^2\log\rho_\kappa'=4\pi e_\kappa\sum e_\alpha\left(1-\frac{u_\kappa' u_\alpha'}{c^2}\right)\rho_\alpha',$$

which are identical in form with Eqs. (7); that is, the Eqs. (7) are relativistically invariant in form, and any solution will be transformable to any other frame of reference.

[11] L. H. Thomas, Proc. Roy. Soc. (London) **121**, 464 (1928).

A solution of these equations is

$$\rho_\kappa=\rho_{\kappa 0}(1+ar^2)^{-2},\quad T_\kappa=ie_\kappa u_\kappa/2c^2k \qquad (8)$$

where

$$i=\sum_\alpha e_\alpha u_\alpha\int_0^\infty \rho_\alpha 2\pi r dr$$

= total electric current in the stream.

In this solution, the total charge per unit length of stream,

$$q=\sum_\alpha e_\alpha\int_0^\infty \rho_\alpha 2\pi r dr,$$

is zero, which can be true only for an observer with a particular velocity parallel with the axis. This observer will be referred to as the "central observer." The solution as seen by any other observer whose velocity parallel with the axis of the stream is v relative to the central observer has the form

$$\rho_\kappa'=\rho_{\kappa 0}'(1+ar^2)^{-2},\quad T_\kappa'=\frac{i'e_\kappa'u_\kappa'}{2c^2k}-\frac{q'}{2k}e_\kappa, \qquad (9)$$

where i' is the total current in the stream as seen by that observer, and q' is the total charge per unit length of stream as seen by that observer.

In the second phase of the stream described by this solution, space charge everywhere is neutralized as seen by the central observer but the same stream, as seen by any other observer may have either a positive or negative net space charge, depending upon the velocity of the observer.[12]

Substituting the solution (9) into Eq. (2) for the force on any particle in the stream gives

$$F_\kappa'=-\frac{ar}{1+ar^2}\left[\frac{i'e_\kappa u_\kappa'}{c^2}-q'e_\kappa\right]=-\frac{2ar}{1+ar^2}kT_\kappa'.$$

This shows that all the particles in any subdistribution whose axial velocity is such that a negative temperature would be required, are repelled from the axis and permanently lost from the stream. From the view-point of the central observer, all positive charges must move one way along the stream and all negative charges must move the opposite way. This early first-phase departure of those subdistributions moving the wrong way from the view-point of the central observer, has the effect of increasing the current in the stream and hence of increasing the temperatures which the remaining subdistributions may have and still be retained in the stream.

As each subdistribution is approaching a Maxwellian distribution, it is also approaching the well-known density distribution given by

$$\rho_\kappa=\rho_{\kappa R}\exp(-\chi_\kappa/kT_\kappa), \qquad (10)$$

where $\rho_{\kappa R}$ is the density at some radial distance R, and χ_κ is the work done on a κ particle by the fields while it moves from a radius R to r. For large values of R and r, the force on the particle is approximately

$$F_\kappa=\frac{2e_\kappa}{r}\left[q-\frac{iu_\kappa}{c^2}\right],$$

and the energy χ_κ is

$$\chi_\kappa=2e_\kappa\left(q-\frac{iu_\kappa}{c^2}\right)\log\left(\frac{r}{R}\right).$$

The density for large radial distances obtained by substituting this into Eq. (10) is given by

$$\log\rho_\kappa=\left(\frac{2e_\kappa q-2ie_\kappa u_\kappa/c^2}{kT_\kappa}\right)\log r.$$

The number of these particles per unit length of stream lying beyond some large radius R_0,

$$N_{\kappa R0}=\int_{R0}^\infty \rho_\kappa 2\pi r dr,$$

is infinite if

$$ie_\kappa u_\kappa/c^2-e_\kappa q<kT_\kappa. \qquad (11)$$

From this it follows that any subdistribution whose temperature is greater than twice that given by solution (9) loses particles permanently from the stream, and the redistribution of velocities of the remaining particles in that subdistribution results in a reduction in temperature of the remainder of that subdistribution until the temperature is not more than twice that given by solution (9).

The behavior of any subdistribution in a stream having densities like those of solution (9) but not necessarily temperatures given by solution (9) can be found by substituting for the density in the hydrodynamical equation of mass motion derived in Appendix B, Eq. (19):

$$\frac{d\dot{r}_{\kappa 0}}{dt}+\dot{r}_{\kappa 0}\frac{\partial\dot{r}_{\kappa 0}}{\partial r}=\frac{4\pi ar}{m_\kappa(1+ar^2)}\left[T_\kappa-\left\{\frac{ie_\kappa u_\kappa}{2c^2k}-\frac{q}{2k}e_\kappa\right\}\right]. \qquad (12)$$

From this it is seen that those subdistributions with temperatures higher than those given by solution (9) have a mass motion away from the axis and are adiabatically cooled, while those with temperatures too low for solution (9) have a mass motion towards the axis and are adiabatically heated. Such expansions and contractions are not entirely free to proceed however. For example, if the subdistributions of one polarity of charge on the whole are too hot and those of the other

[12] This is not a violation of the relativistic invariance of the charge on any one particle, but it is due to the different ways in which the numerical densities of the different kinds of particles vary depending on their different velocities as discussed in reference 22.

are too cold, the motion of charge produces a space charge near the axis opposing the separation. If more of the subdistributions are too hot than too cold, the effect of the space charge separation is to draw both distributions along and still further heat them, possibly driving some of the temperatures above the criterion of (11) and so driving some of those particles out of the stream.

TWO-COMPONENT STREAM

The system discussed in the original paper[1] consisted of only two distributions. The solution obtained could be written

$$\rho_1=\left[1-\frac{u_2}{c}\frac{T_1u_2+T_2u_1}{(T_1+T_2)c}\right]\rho_0(1+ar^2)^{-2},$$
$$\rho_2=\left[1-\frac{u_1}{c}\frac{T_1u_2+T_2u_1}{(T_1+T_2)c}\right]\rho_0(1+ar^2)^{-2}, \tag{13}$$

where

$$\rho_0=\frac{2ac^2k(T_1+T_2)}{e^2(u_1-u_2)^2}.$$

The velocity of the central observer for this system is such that u_1 has a sign opposite to that of u_2 and the temperatures are related to the two velocities by

$$T_1/u_1=T_2/(-u_2)=K, \tag{14}$$

where K is the proportionality constant. The total current in this stream is

$$i=(2c^2k/e)K, \tag{15}$$

which is independent of the velocities u_1 and u_2 and of the constant a which determines the radial spread of the stream. The expression for the temperatures in Eqs. (8) of course can be written in the same form as Eq. (14); *viz*:

$$i=(2c^2k/e_\kappa)(T_\kappa/u_\kappa)=(2c^2k/e)K,$$

and this is also consistent with Eq. (3).

The total potential energy per unit length of a stream is derived in Appendix D and is

$$V=-2\sum_\alpha\sum_\beta e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{e^2}\right)\int_0^\infty\int_0^r\rho_\beta(\xi)2\pi\xi d\xi \times\log(r/R)\rho_\alpha(r)2\pi r dr,$$

where R is the radius at which the kinetic energy of a particle is zero. Because this is a matter of definition, R may be taken to equal $\sqrt{e}$.

In a two-component stream in its second phase as seen by the central observer, the potential energy per unit length of stream is

$$V=-(i^2/2c^2)\log a.$$

Consider now a stream which initially had radial distributions like those of Eqs. (13)

$$\rho_1=\rho_2=\rho_{00}(1+a_0r^2)^{-2},$$

and Maxwellian velocity distributions with temperatures like Eq. (14),

$$T_1/u_1=T_2/(-u_2)=K_0,$$

but which contained a current different from that given by Eq. (15),

$$i\neq(2c^2k/e)K_0.$$

The total energy per unit length of stream, which is constant, equals

$$h=h_a+NkT_1+NkT_2-(i^2/2c^2)\log a_0,$$

where h_a is the kinetic energy per unit length of stream due to axial components of momenta (which is decreasing only very slowly due to collisions between the two different distributions whose relative velocity is large); and N is the number of each kind of particle per unit length of stream so that NkT_1 and NkT_2 are the kinetic energies of the two distributions due to transverse momenta. The total energy per unit length of stream in its initial form may be written as

$$h=h_a+(ki/e)K_0-(i^2/2c^2)\log a_0,$$

and is to be compared with the equation for the stream after it has reached the quasi-steady state of the second phase:

$$h=h_a+(ki/e)K-(i^2/2c^2)\log a.$$

Subtracting, transposing, and substituting for K from (15),

$$\frac{1}{i}\left\{\frac{2c^2k}{e}K_0-i\right\}=\log\frac{a_0}{a}.$$

The quantities a_0 and a are related to the initial and final mean radial distances of particles $\bar{r}_0$ and $\bar{r}$ respectively by

$$\sqrt{a_0}=\pi/2\bar{r}_0 \quad\text{and}\quad \sqrt{a}=\pi/2\bar{r},$$

and

$$\bar{r}=\bar{r}_0\exp[(i_c-i)/2i], \tag{16}$$

showing that in this case the mean radial distance of particles $\bar{r}$ decreases exponentially with the excess of the stream current i over the value i_c given by Eq. (15).

HIGH-DENSITY STREAMS

When the densities are such that collisions cannot be taken to be infrequent during the travel of a charged particle between positions of maximum and minimum radial distance from the axis, the situation is much different and the methods used in the preceding are inadequate. Firstly, a current cannot be sustained for any appreciable time without applying an emf in the direction of the stream either by applying an electric field in that direction or by electromagnetic induction. The application of such an emf in combination with the self-magnetic field of the stream produces orbits of the charged particles—principally the electrons, because they are the more readily accelerated—which tend to move the particles towards the axis but not in a simple manner. In addition to this, the application of an emf drives the electrons away from a Maxwellian velocity distribution and towards an Allis-Brown distribution[13] in which the mean thermal energy of the electrons is much greater than the mean thermal energy of the ions. The kinetic energy of random motion of the electrons continues to rise until the rate at which energy is fed into the stream by the applied emf equals the rate at which energy is dissipated, principally as bremsstrahlung or characteristic spectra of the gas or in producing ionizations. Even if these large electron thermal energies are related to a corresponding electron "temperature," equations like Eq. (3) or Eq. (12) cannot be invoked to calculate the minimum current for self-focusing because the cooperative action of the emf and the self-magnetic field of the stream in driving electrons towards the axis can in part overcome the tendency of the high electron temperature to disperse the stream as long as the emf is maintained.

In addition to the above, there is another reason that that any discussion of magnetic self-focusing in electric arcs[14] is somewhat academic. It is that arcs are essentially unstable for a quite different reason as has been demonstrated by Kruskal and Schwarzschild.[15] If the stream is initially not perfectly straight, but a short section in it is slightly displaced, the self-magnetic field of the stream current is greater on the concave side of the bend in the middle of the displaced section than on its convex side. The stream is pushed towards the convex side, exaggerating the middle bend while the reverse bends at each end of the middle bend begin to grow towards the other side. Thus, each bend grows and creates new bends which in turn also grow. This kind of instability in an arc is familiar to anyone who has seen the serpentine writhing of very high voltage arcs in air.

This kind of instability cannot exist in a low-density stream because the very rapid mixing in azimuth about the axis produced by the spreads in rates of precession about the axis quickly eliminates any kinks in the stream.

APPLICATIONS

The theory of low-density streams given here may be applied to the proton streams travelling from the sun towards the earth invoked in previous publications[9] to account for the aurorae. In these articles, it was supposed that the protons energe from the sun in a diverging jet at speeds of about 10^{10} cm/sec. Later studies have made it seem likely that the more usual speeds are of the order of 2×10^9 cm/sec so that $u^2 \cong c^2/225$, as will be explained in a later publication. The divergence of the jet makes the mean "transverse" kinetic energy of the solar protons very much greater than the mean thermal energy of either the ions or the electrons in the residual ionized interplanetary matter, and consequently, the central observer is very nearly at rest with respect to the interplanetary matter. The greatest density of fast ions in a 10 000-ampere stream of 2×10^9-cm/sec protons in a stream whose mean radius is 1000 kilometers is one ion per 200 cm^3, while the density of interplanetary matter is generally conceded to be at least one ion and one electron per cm^3.

The interplanetary ions and electrons can be supposed to have Maxwellian velocity distributions both at the same temperature, but the high-velocity protons in the stream probably do not have a Maxwellian velocity distribution. Equation (6) can be applied to the interplanetary ions and electrons, respectively, as seen by an observer at rest relative to the interplanetary matter:

$$kTr\frac{d}{dr}\log\rho_2 = 4\pi e^2 \int_0^r (\rho_1+\rho_2-\rho_3) r dr,$$

$$kTr\frac{d}{dr}\log\rho_3 = -4\pi e^2 \int_0^r (\rho_1+\rho_2-\rho_3) r dr,$$

where T is the temperature of the interplanetary matter; ρ_1, ρ_2, and ρ_3 are the numerical densities of the fast protons, the interplanetary ions and the interplanetary electrons respectively. Adding and integrating: $\rho_2\cdot\rho_3 =$ a constant which equals the square of the numerical density, n, of each kind of particle remote from the stream and so is n^2 everywhere. If $\rho_1 \ll n$ as it is in this case, $|n-\rho_2|$ and $|n-\rho_3|$ are each small, and the net charge density everywhere is small. Substituting for ρ_2 and ρ_3 in turn from $\rho_2\cdot\rho_3 = n^2$ in $\rho_1+\rho_2-\rho_3=0$ gives

$$n-\rho_2 = \tfrac{1}{2}\rho_1 = \rho_3 - n.$$

The proton stream in the plasma will be self-focusing if the current in the stream exceeds the critical current given by Eq. (4) with $q=0$:

$$i = c(2N\psi)^{\frac{1}{2}}, \tag{17}$$

where N is twice the number of fast protons per unit length of stream which is equal to the number of protons plus the excess of electrons over slow ions per unit length of stream; and ψ is the mean energy of those N fast protons and N excess slow electrons due to transverse components of velocity.

If, as assumed in the previous articles,[9] the fast protons come from a jet diverging with a half-angle α, and the temperature of the interplanetary ions and electrons is too low to affect materially the value of ψ, the values of critical current for 2×10^9-cm/sec protons in a cone of half-angle α are given by substituting

[13] W. P. Allis and S. C. Brown, Phys. Rev. **87**, 419 (1952).
[14] Called the "pinch effect" by Tonks. See reference 5.
[15] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. (London) **223**, 348 (1954).

TABLE I. Critical current in amperes for various cone half-angles.

α	1°	2°	5°	10°	20°
i_c	300	1200	7600	31 000	132 000

$\frac{1}{2}Nev=i$ in Eq. (17) where $v=2\times10^9$:

$$i \geqslant 4c^2\psi/ev. \tag{18}$$

ψ is one-half the mean energy of the fast protons due to transverse components of velocity. Some values for various half-angles are tabulated in Table I.

Substitution of numerical values for a typical fast-proton stream as suggested above, in Eq. (5) shows that very little progress toward a Maxwellian velocity distribution is made by the fast protons in their time of travel from the sun to the earth and the stream will still be in its first phase when it reaches the earth in such a case. Although the actual current in the stream may be much greater than the critical current, it does not follow that the exponential shrinkage towards the axis given by Eq. (16) will occur. Such a shrinkage can not in fact occur for these non-Maxwellian distributions because of its approximate conservation of angular momentum of each individual fast proton, about the axis of the stream. The shrinkage which can occur in this early part of the first phase of the stream is determined by the distribution in angular momentum of the fast protons and not alone by the kind of considerations which led to Eq. (16).

Although the surviving fast protons leaving the jet at the sun may, and probably do, leave in a stream with an oddly shaped cross section far from cylindrical symmetry, the rapid mixing due to the spread in rates of precession about the axis will very quickly bring such a stream to cylindrical symmetry, but of course, the distributions in transverse momenta remain far from Maxwellian.

This theory of low-density streams is not adequate to explain the experiments of Cousins and Ware, and of Bostick, Levine, and Coombes,[2,3] both because large electromotive forces were applied to the stream and also because the gas densities used were too large. Thoneman, Cowhig, and Davenport[4] used densities which were too large for this theory to apply in its entirety, but it is interesting to note that they produced a large direct current using only an alternating emf. This seems to illustrate the strong tendency of a stream magnetically to throw out all subdistributions moving the wrong way as seen by a central observer[16] and so to increase the direct current in the stream. Thus, large densities and small mean particle velocity in the direction of the stream results in rapid effects of collisions with a rapid increase in mean kinetic energy due to transverse momenta, with the attendant rapid driving of subdistributions beyond the criterion of Eq. (11) so that there is a rapid loss of particles at the walls.

[16] W. H. Bennett, Phys. Rev. **90**, 387(A) (1953).

In conclusion the writer wishes to express his great appreciation for the opportunity to discuss with L. H. Thomas some of the mathematical methods used in this development.

MATHEMATICAL APPENDIX

A. Virial of Clausius in Cylindrical Coordinates

The probability that a charged particle of the κth kind is within a velocity range $d\dot{r}$ at $\dot{r}$, $d\dot{\phi}$ at $\dot{\phi}$ and $d\dot{z}$ at $\dot{z}$ at a position r, ϕ and z will be written

$$f_\kappa(r,\phi,z;\,\dot{r},\dot{\phi},\dot{z})d\dot{r}rd\dot{\phi}d\dot{z},$$

where $\dot{r}$ is the velocity in the radial direction (direction of r); $\dot{\phi}$ is the angular velocity about the axis (direction of ϕ); and $\dot{z}$ is the velocity component parallel with the axis (direction of z). When this function is non-Maxwellian, it may be a function of the position coordinates. The function is normalized so that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_\kappa d\dot{r}rd\dot{\phi}d\dot{z}=1.$$

The density ρ_κ of the κth kind of particle is a function of the position coordinates, only,—r, ϕ, and z. The velocity of mass motion of the κth kind of particle has components given by

$$\dot{r}_{\kappa0}=\int\int\int \dot{r} f_\kappa d\dot{r}rd\dot{\phi}d\dot{z},$$

$$r\dot{\phi}_{\kappa0}=\int\int\int r\dot{\phi} f_\kappa d\dot{r}rd\dot{\phi}d\dot{z},$$

$$\dot{z}_{\kappa0}=\int\int\int \dot{z} f_\kappa d\dot{r}rd\dot{\phi}d\dot{z}.$$

The velocity of any of the κth kind of particle may be considered to consist of the velocity of mass motion at the position of that particle plus the velocity of that particle relative to the velocity of mass motion, and the latter has components ξ, $r\eta$, and ζ where

$$\dot{r}=\dot{r}_0+\xi,\quad r\dot{\phi}=r\dot{\phi}_0+r\eta,\quad \dot{z}=\dot{z}_0+\zeta.$$

In the streams considered here, radial symmetry has been assumed so $\dot{\phi}_0=0$; and longitudinal uniformity has been assumed so $\partial\dot{z}_0/\partial z=0$. The mean square velocity of a particle of the κth kind relative to the velocity of mass motion has parts in the r and ϕ directions

$$\langle\xi_\kappa^2\rangle_{\text{Av}}=\int\int\int \dot{r}^2 f_\kappa d\dot{r}rd\dot{\phi}d\dot{z}-\dot{r}_{\kappa0}^2,$$

$$\langle\eta_\kappa^2\rangle_{\text{Av}}=\int\int\int \dot{\phi}^2 f_\kappa d\dot{r}rd\dot{\phi}d\dot{z},$$

$$\langle\zeta_\kappa^2\rangle_{\text{Av}}=\int\int\int \dot{z} f_\kappa d\dot{r}rd\dot{\phi}d\dot{z}-\dot{z}_{\kappa0}^2.$$

If we write the mass of a particle of the κth kind as m_κ, the moment of inertia of that particle about the axis is $m_\kappa r^2$, and

$$\frac{1}{2}\frac{d^2}{dt^2}(m_\kappa r^2)=\frac{1}{2}m_\kappa \dot{r}^2+\frac{r}{2}m_\kappa \ddot{r},$$

where $m_\kappa \ddot{r}$ is related to the force F_κ exerted on that particle (in the radial direction in the streams assumed here) by $m_\kappa \ddot{r}=F_\kappa+m_\kappa r\dot{\phi}^2$. Substituting for F_κ from Eq. (2), integrating and summing over all particles in unit length of stream, and writing the moment of inertia of all particles in unit length of stream as I:

$$\frac{1}{4}\frac{d^2I}{dt^2}=\sum_\alpha\{\tfrac{1}{2}m_\alpha \dot{r}_{\alpha 0}^2+\tfrac{1}{2}m_\alpha\langle\xi_\alpha^2\rangle_{\text{Av}}+\tfrac{1}{2}m_\alpha r^2\langle\eta_\alpha^2\rangle_{\text{Av}}\}+\sum_\alpha\sum_\beta e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\int_0^\infty\int_0^r \rho_\beta(s)2\pi s ds\rho_\alpha 2\pi r dr.$$

Reversing the order of integration in the last term and summing twice

$$\sum_\alpha\sum_\beta\int_0^\infty\int_0^r \rho_\beta(s)2\pi s ds\rho_\alpha 2\pi r dr=\sum_\beta\sum_\alpha\int_0^\infty\int_r^\infty \rho_\alpha(s)2\pi s ds\rho_\beta 2\pi r dr=\sum_\alpha\sum_\beta\int_0^\infty\int_r^\infty \rho_\beta 2\pi s ds\rho_\alpha 2\pi r dr.$$

Adding the first and last integral gives

$$\sum_\alpha\sum_\beta\int_0^\infty\int_0^\infty \rho_\beta 2\pi s ds\rho_\alpha 2\pi r dr=\sum_\alpha\sum_\beta N_\alpha N_\beta=N^2,$$

where N_α is the total number of particles of the αth kind per unit length of stream, and N is the total number of all particles per unit length of stream; so

$$\sum_\alpha\sum_\beta\int_0^\infty\int_0^r \rho_\beta 2\pi s ds\rho_\alpha 2\pi r dr=\tfrac{1}{2}\sum\sum N_\alpha N_\beta.$$

Writing ψ for the mean kinetic energy of particles due to transverse components of velocity,

$$\tfrac{1}{4}d^2I/dt^2=N\psi+\tfrac{1}{2}\{q^2-i^2/c^2\},$$

where $q=\sum_\alpha e_\alpha N_\alpha$ is the charge per unit length of stream and $i=\sum_\alpha e_\alpha u_\alpha N_\alpha$ is the electric current in the stream. From this, the stream will concentrate towards the axis, that is, it is self-focusing if $d^2I/dt^2<0$; that is,

$$i>c(2N\psi+q^2)^{\frac{1}{2}}.$$

B. Boltzmann's Equation in Cylindrical Coordinates

Using arguments analogous to those used by Jeans[17] but in cylindrical coordinates, Boltzmann's equation can be written for the particles of the κth subdistribution as

$$\frac{d(\rho_\kappa f_\kappa)}{dt}+\left(\frac{F_\kappa}{m_\kappa}+r\dot{\phi}^2\right)\frac{\partial(\rho_\kappa f_\kappa)}{\partial\dot{r}}+\frac{1}{r}\left(\frac{\Phi_\kappa}{m_\kappa}-2\dot{r}\dot{\phi}\right)\frac{\partial(\rho_\kappa f_\kappa)}{\partial\dot{\phi}}+\frac{Z_\kappa}{m_\kappa}\frac{\partial(\rho_\kappa f_\kappa)}{\partial\dot{z}}+\dot{r}\frac{\partial(\rho_\kappa f_\kappa)}{\partial r}+\dot{\phi}\frac{\partial(\rho_\kappa f_\kappa)}{\partial\phi}+\dot{z}\frac{\delta(\rho_\kappa f_\kappa)}{\partial z}=0,$$

where Φ_κ and Z_κ are the components of force on a κth kind of particle in the tangential and axial direction respectively and the radially symmetric and longitudinally uniform streams assumed here, are each equal to zero. Multiplying through by $\dot{r}/\rho_\kappa$ and integrating over the velocity ranges:

$$\frac{1}{\rho_\kappa}\frac{d(\rho_\kappa\dot{r}_{\kappa 0})}{dt}-\frac{F_\kappa}{m_\kappa}-r\langle\eta_\kappa^2\rangle_{\text{Av}}+\frac{2\dot{r}_{\kappa 0}^2}{r}+\frac{2\langle\xi_\kappa^2\rangle_{\text{Av}}}{r}+\frac{r}{\rho_\kappa}\frac{\partial}{\partial r}\left(\frac{\rho_\kappa\dot{r}_{\kappa 0}^2}{r}\right)+\frac{r}{\rho_\kappa}\frac{\partial}{\partial r}\left(\frac{\rho_\kappa\langle\xi_\kappa^2\rangle_{\text{Av}}}{r}\right)+\frac{r}{\rho_\kappa}\frac{\partial}{\partial\phi}\left(\frac{\rho_\kappa\langle\xi_\kappa\eta_\kappa\rangle_{\text{Av}}}{r}\right)+\frac{r}{\rho_\kappa}\frac{\partial}{\partial z}\left(\frac{\rho_\kappa\dot{r}_{\kappa 0}\dot{z}_{\kappa 0}}{r}\right)+\frac{r}{\rho_\kappa}\frac{\partial}{\partial z}\left(\frac{\rho_\kappa\langle\xi_\kappa\zeta_\kappa\rangle_{\text{Av}}}{r}\right)=0,$$

where

$$\langle\xi_\kappa\eta_\kappa\rangle_{\text{Av}}=\int\int\int(\dot{r}-\dot{r}_0)\dot{\phi}f_\kappa d\dot{r}rd\dot{\phi}d\dot{z}=0,$$

$$\langle\xi_\kappa\zeta_\kappa\rangle_{\text{Av}}=\int\int\int(\dot{r}-\dot{r}_{\kappa 0})(\dot{z}-\dot{z}_{\kappa 0})f_\kappa d\dot{r}rd\dot{\phi}d\dot{z}=\dot{r}_{\kappa 0}\dot{z}_{\kappa 0},$$

and $\dot{z}_{\kappa 0}$ does not vary with r. In the above equation,

$$\frac{1}{\rho_\kappa}\frac{d(\rho_\kappa\dot{r}_{\kappa 0})}{dt}=\frac{d\dot{r}_{\kappa 0}}{dt}+\frac{\dot{r}_{\kappa 0}}{\rho_\kappa}\frac{d\rho_\kappa}{dt}.$$

$d\rho_\kappa/dt$ can be eliminated by using the equation of continuity:

$$\frac{d\rho_\kappa}{dt}+\frac{1}{r}\left\{\frac{\partial(r\rho_\kappa\dot{r}_{\kappa 0})}{\partial r}+\frac{\partial(r\rho_\kappa\dot{\phi}_{\kappa 0})}{\partial\phi}+\frac{\partial(r\rho_\kappa\dot{z}_{\kappa 0})}{\partial z}\right\}=0,$$

and in a radially symmetric and longitudinally uniform stream:

$$\frac{d\dot{r}_{\kappa 0}}{dt}+\dot{r}_{\kappa 0}\frac{\partial\dot{r}_{\kappa 0}}{\partial r}-\frac{F_\kappa}{m_\kappa}-r\langle\eta_\kappa^2\rangle_{\text{Av}}+\frac{1}{r\rho_\kappa}\frac{\partial}{\partial r}(r\rho_\kappa\langle\xi_\kappa^2\rangle_{\text{Av}})=0. \quad (19)$$

[17] J. H. Jeans, *Dynamical Theory of Gases* (Cambridge University Press, Cambridge, 1925), fourth edition, Chap. VIII.

In the steady state obtained by mixing, $\dot{r}_{\kappa 0}=0$. Multiplying through by $m_\kappa r/2$, writing the mean kinetic energy of the κth kind of particle due to radial motion as $\frac{1}{2}m_\kappa\langle\xi_\kappa^2\rangle_{\text{Av}}=\phi_\kappa$ and the mean kinetic energy due to tangential motion as $\frac{1}{2}m_\kappa r^2\langle\eta_\kappa^2\rangle_{\text{Av}}=\theta_\kappa$, and substituting for F_κ from Eq. (2):

$$e_\kappa \sum_\alpha e_\alpha\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\int_0^r \rho_\alpha 2\pi s ds + \theta_\kappa - \phi_\kappa - \frac{r}{\rho_\kappa}\frac{d(\rho_\kappa\phi_\kappa)}{dr}=0. \quad (20)$$

C. Relativistic Transformation of Variables

The transformation of velocities and of electric charge densities is explained in a book by McCrea.[18] Velocities are transformed by the well-known expression

$$u=\frac{u'+v}{1+u'v/c^2},$$

and the numerical densities are transformed by the seemingly less well known expression

$$\rho=\left(\frac{1+u'v/c^2}{(1-v^2/c^2)^{\frac{1}{2}}}\right)\rho'.$$

The charge on each charged particle is relativistically invariant but the numerical densities are not. It is for this reason that the charge density must be transformed.

The term "temperature" with all its usual implications cannot strictly speaking be applied to a subdistribution except as that subdistribution is viewed by an observer travelling with the velocity of that subdistribution. In the longitudinally uniform streams considered here, the effects of the thermal distribution in velocity component parallel with the axis cancel and only the distributions in velocity components transverse to the axis need be considered. The term temperature has been invoked here as a quantity proportional to θ_κ and to ϕ_κ which are each respectively proportional to the mass and to the square of the corresponding component of velocity. Thus, the temperature T' in a system of coordinates moving at a velocity v and used in this restricted sense is related to the T in the rest system of coordinates by

$$T'=T(m'/m)(u^2/u'^2) \quad \text{and} \quad T=T'(m/m')(u'^2/u^2).$$

Substituting for the ratios of mass and velocities squared[18]

$$T=\left(\frac{1-u'^2/c^2}{1-u^2/c^2}\right)^{\frac{1}{2}}\left(\frac{1-v^2/c^2}{(1+u'v/c^2)^2}\right)T'=\left(\frac{(1-v^2/c^2)^{\frac{1}{2}}}{1+u'v/c^2}\right)T'.$$

[18] W. H. McCrea, *Relativity Physics* (John Wiley and Sons, Inc., New York, 1954), fourth edition, Chaps 3 and 4 for velocities and Chap. 6 for electric charge. Note 8 on p. 61 is of particular interest in this connection.

D. Potential Energy per Unit Length of Stream

The force exerted on a charge e_κ with velocity u_κ parallel to the z-axis and at a distance r from a linear charge density λ_α parallel to the z-axis consisting of charges moving at a velocity u_α parallel to the z-axis is

$$\frac{2e_\kappa\lambda_\alpha}{r}\left(1-\frac{u_\kappa u_\alpha}{c^2}\right),$$

and the potential energy of that charge e_κ is

$$-\int_R^r 2e_\kappa\lambda_\alpha\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\frac{dr}{r} = -2e_\kappa\lambda_\alpha\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\log\left(\frac{r}{R}\right),$$

where the integration is performed from a finite radial distance R in order to avoid logarithmic infinities, where R is by definition the radial distance a charge must be from a linear charge for the potential energy to equal zero. The potential energy of a linear charge density λ_κ parallel to the z-axis (and to λ_α) at a distance r from λ_α is

$$-2\lambda_\kappa\lambda_\alpha\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\log\left(\frac{r}{R}\right).$$

Considering now an infinitely long thin shell of radius r, azimuthally uniform and longitudinally uniform, containing various kinds of charges of various axial velocities but mixed in uniform proportions everywhere in the shell. Among the charges in the shell, the αth kind of charge has N_α particles per unit length of shell, each with charge e_α and axial velocity u_α. The force exerted on a narrow ribbon of the κth kind of particle within an azimuthal angle of the shell $d\theta_\kappa$, by a narrow ribbon of the αth kind of particle within an angle $d\theta$ and located in the shell at an angle θ from $d\theta_\kappa$, is

$$\frac{2(N_\kappa/2\pi)d\theta_\kappa(N_\alpha/2\pi)d\theta}{2r\sin(\theta/2)}\left(1-\frac{u_\kappa u_\alpha}{c^2}\right),$$

and the potential energy equals

$$-\frac{e_\kappa e_\alpha N_\kappa N_\alpha}{2\pi^2}d\theta_\kappa d\theta\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\log\left(\frac{2r\sin(\theta/2)}{R}\right).$$

The potential energy of the κth ribbon due to all of the αth kind of charges is

$$-\frac{e_\kappa e_\alpha N_\kappa N_\alpha}{2\pi^2}d\theta_\kappa\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)2\int_0^\pi \log\left(\frac{2r\sin(\theta/2)}{R}\right)d\theta = -\frac{e_\kappa e_\alpha N_\kappa N_\alpha}{\pi}\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\log\left(\frac{r}{R}\right)d\theta_\kappa,$$

and that due to all kinds of charge in the shell is

$$-\sum_{\alpha}\frac{e_\kappa e_\alpha N_\kappa N_\alpha}{\pi}\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\log\left(\frac{r}{R}\right)d\theta_\kappa.$$

The potential energy of all the κth kind of particle in the shell is

$$-\sum_{\alpha}\frac{e_\kappa e_\alpha N_\kappa N_\alpha}{\pi}\left(1-\frac{u_\kappa u_\alpha}{c^2}\right)\log\left(\frac{r}{R}\right)\int_0^{2\pi}d\theta_\kappa,$$

and potential energy of all kinds of charges in the stream, remembering that in summing twice, each charge is counted twice, is

$$-\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta N_\alpha N_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\log\left(\frac{r}{R}\right).$$

Suppose that the shell has an initial radius R so that the initial potential energy is zero, and that the shell is then shrunken to a small radius S so that the potential energy becomes

$$-\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta N_\alpha N_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\log\left(\frac{S}{R}\right),$$

after which the charges will be redeployed radially to density distributions $\rho(r)$ at radial distance r from the axis. The numbers of charges per unit length of shell are related to the density distributions after redeployment by

$$N_\alpha=\int_0^\infty\rho_\alpha(r)2\pi r dr.$$

The potential energy of the shrunken shell can be written

$$-\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\log\left(\frac{S}{R}\right)\int_0^\infty\int_0^\infty\rho_\beta(\xi)2\pi\xi d\xi$$
$$\times\rho_\alpha(r)2\pi r dr=-2\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\log\left(\frac{S}{R}\right)$$
$$\times\int_0^\infty\int_0^r\rho_\beta(\xi)2\pi\xi d\xi\rho_\alpha(r)2\pi r dr.$$

If in redeploying the charges, the charges are moved out to successive elementary shells beginning with the outermost, then in moving an element of the κth kind of charge $\rho_\kappa(r)2\pi r dr$ to radius r, the charges previously moved out to greater radial distances do not exert a force on these charges because of their symmetry as may be shown using Gauss' theorem, and only the charges left behind in the shrunken shell,

$$e_\alpha N_\alpha-\int_r^\infty e_\alpha\rho_\alpha 2\pi\xi d\xi=e_\alpha\int_0^r\rho_\alpha 2\pi\xi d\xi,$$

exert forces. The energy derived from moving all kinds of charges out to dr at r is

$$2\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\log\left(\frac{r}{S}\right)\int_0^r\rho_\beta(\xi)$$
$$\times 2\pi\xi d\xi\rho_\alpha(r)2\pi r dr,$$

and the energy derived from moving all kinds of charges to all radii greater than S is

$$2\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\int_S^\infty\log\left(\frac{r}{S}\right)\int_0^r\rho_\beta(\xi)$$
$$\times 2\pi\xi d\xi\rho_\alpha(r)2\pi r dr\cdot$$

The total potential energy gained in shrinking the zero-energy shell to radius S and then expanding to density distributions $\rho_\alpha(r)$ is

$$-2\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\int_0^\infty\log\left(\frac{S}{R}\right)$$
$$\times\int_0^r\rho_\beta 2\pi\xi d\xi\rho_\alpha 2\pi r dr,$$
$$-2\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\int_0^\infty\log\left(\frac{r}{S}\right)$$
$$\times\int_0^r\rho_\beta 2\pi\xi d\xi\rho_\alpha 2\pi r dr,$$
$$+2\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\int_0^S\log\left(\frac{r}{S}\right)$$
$$\times\int_0^r\rho_\beta 2\pi\xi d\xi\rho_\alpha 2\pi r dr.$$

As S is allowed to approach zero, the last term vanishes if the densities $\rho(r)$ are finite everywhere, which they would have to be in streams of the kind considered here, in order to avoid the infinitely rapid effects of collisions which would otherwise result from infinite densities. The potential energy per unit length of stream is obtained by adding the first two terms and is

$$-2\sum_{\alpha}\sum_{\beta}e_\alpha e_\beta\left(1-\frac{u_\alpha u_\beta}{c^2}\right)\int_0^\infty\int_0^r\rho_\beta(\xi)2\pi\xi d\xi$$
$$\times\log(r/R)\rho_\alpha(r)2\pi r dr.$$