Multiple Photon Production in Quantum Electrodynamics

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Multiple production of photons in high-energy processes in quantum electrodynamics is investigated, and the electron-positron annihilation is specially discussed in detail. It is found that at very high energies in cosmic rays, multiple production of up to four or five observable photons can easily take place. But, the probability, according to quantum electrodynamics, for the multiple production of a larger number of observable photons is quite small.

1. INTRODUCTION

BECAUSE of the small value of the fine structure constant, multiple processes in quantum electrodynamics have not received much attention. In fact, it is usually believed[1,2] that the cross section for a process involving the multiple production of photons is always much smaller than the cross section for a similar process involving the production of a lesser number of photons. However, in view of the recent discovery of a narrow shower of about 20 high-energy photons in cosmic rays by Schein and co-workers,[3] it seems to be of interest to carry out a proper investigation of multiple photon production by charged particles at high energies.

We shall, therefore, first investigate in some detail the multiple production of photons in the annihilation of a pair of electron and positron, and then we shall consider the multiple photon production in any arbitrary process in quantum electrodynamics. We shall see that at very high energies, which are available in cosmic rays, multiple production of several photons can easily take place. For instance, in cosmic rays it should be possible to observe multiple production of up to four or five photons in electron-positron annihilation, and also in some other processes. However, the probability for the multiple production of 15 to 20 photons is so small that Schein's photon shower cannot be explained solely by multiple photon production within the framework of quantum electrodynamics. This seems to suggest that the Schein shower is probably partly due to multiple photon production and partly due to the usual cascade process. Another possibility seems to be that this shower was produced by some process, which involves mesonic as well as electromagnetic interactions. Therefore, the production of a photon shower by the decay of a π^0 meson[4] and also by some other processes is at present under investigation.

Some preliminary results on multiple photon production, based on rough calculations, have been published earlier by the author.[5] However, some of the conclusions, mentioned there, are unjustified in the light of more accurate calculations, described in the present paper.[6]

2. MATRIX ELEMENT FOR THE PRODUCTION OF THREE PHOTONS IN ELECTRON-POSITRON ANNIHILATION

We shall first consider the annihilation of a pair of electron and positron with the production of three photons. Following Dyson's treatment,[7] we can write the S matrix element for this process as

$$S_3=(e^3/c^3\hbar^3)\int dx\int dx'\int dx'' A_\mu(x)A_\nu(x')A_\lambda(x'')\times\bar{\psi}(x)\gamma_\mu S_F(x-x')\gamma_\nu S_F(x'-x'')\gamma_\lambda\psi(x''), \quad (1)$$

where

$$S_F(x-x')=\lim_{\epsilon\to+0}\frac{1}{(2\pi)^4}\int dp\, e^{ip(x-x')}\frac{ip\gamma-\kappa}{p^2+\kappa^2-i\epsilon}. \quad (2)$$

We now put

$$\psi(x'')=V^{-\frac{1}{2}}a_r(\mathbf{k})u_r(\mathbf{k})e^{ikx''},\quad \bar{\psi}(x)=V^{-\frac{1}{2}}b_s(\mathbf{k}')\bar{v}_s(\mathbf{k}')e^{ik'x}, \quad (3)$$

where k and k' are the propagation four-vectors for the electron and the positron respectively, $a_r(\mathbf{k})$ and $b_s(\mathbf{k}')$ are absorption operators for these particles, $u_r(\mathbf{k})$ and $\bar{v}_s(\mathbf{k}')$ are the spinor amplitudes, and the indices r and s can take the values 1 or 2 depending on the spin states of the particles. We can also express the transverse parts of $A_\mu(x)$, $A_\nu(x')$, and $A_\lambda(x'')$ as

$$\mathbf{A}(x)=\sum_{\mathbf{q}',\mathbf{e}'}V^{-\frac{1}{2}}\left(\frac{c\hbar}{2q_0'}\right)^{\frac{1}{2}}\mathbf{e}'a_{\mathbf{e}'}^*(\mathbf{q}')e^{-iq'x},$$

$$\mathbf{A}(x')=\sum_{\mathbf{q}'',\mathbf{e}''}V^{-\frac{1}{2}}\left(\frac{c\hbar}{2q_0''}\right)^{\frac{1}{2}}\mathbf{e}''a_{\mathbf{e}''}^*(\mathbf{q}'')e^{-iq''x'}, \quad (4)$$

$$\mathbf{A}(x'')=\sum_{\mathbf{q},\mathbf{e}}V^{-\frac{1}{2}}\left(\frac{c\hbar}{2q_0}\right)^{\frac{1}{2}}\mathbf{e}a_{\mathbf{e}}^*(\mathbf{q})e^{-iqx''},$$

[1] W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954).
[2] R. E. Marshak, *Meson Physics* (McGraw-Hill Book Company, Inc., New York, 1952).
[3] Schein, Haskin, and Glasser, Phys. Rev. **95**, 855 (1954). Several cases of Schein's photon shower have been observed more recently by DeBenedetti, Garelli, Tallone, Vigone, and Wataghin, Nuovo cimento **12**, 954 (1954).
[4] The results of the present paper and some possible interpretations of the Schein shower were described by the author in an invited paper at the Chicago meeting of the American Physical Society, November, 1954.
[5] S. N. Gupta, Phys. Rev. **96**, 1453 (1954).
[6] In fact, the cross section σ_n, given in reference 5, must be divided by a factor $(n-2)!$. This was first pointed out to me in a private communication by Professor F. J. Dyson and Dr. R. H. Dalitz, who made use of semiclassical and statistical arguments.
[7] F. J. Dyson, Phys. Rev. **75**, 486, 1736 (1949).

where $a_e^*(\mathbf{q})$, $a_{e'}^*(\mathbf{q}')$, and $a_{e''}^*(\mathbf{q}'')$ are emission operators for the photons, whose propagation four-vectors are q, q', and q'' and whose directions of polarization are given by the unit vectors $\mathbf{e}$, $\mathbf{e}'$, and $\mathbf{e}''$ respectively. Substituting (3) and (4) in (1), we obtain

$$S_3 = V^{-5/2} \sum_{\mathbf{q},e} \sum_{\mathbf{q}',e'} \sum_{\mathbf{q}'',e''} \int dx e^{ix(k+k'-q-q'-q'')} a_r(\mathbf{k}) \times b_s(\mathbf{k}') a_e^*(\mathbf{q}) a_{e'}^*(\mathbf{q}') a_{e''}^*(\mathbf{q}'') \left(\frac{e^2}{2c\hbar}\right)^{\frac{3}{2}} (q_0 q_0' q_0'')^{-\frac{1}{2}} \times \bar{v}_s(\mathbf{k}') \left[(\gamma\cdot\mathbf{e}') \frac{-i(k'-q')\gamma-\kappa}{(k'-q')^2+\kappa^2} (\gamma\cdot\mathbf{e}'') \times \frac{i(k-q)\gamma-\kappa}{(k-q)^2+\kappa^2} (\gamma\cdot\mathbf{e}) \right] u_r(\mathbf{k}). \quad (5)$$

We shall now carry out our calculations in the center-of-mass system, so that

$$\mathbf{k}' = -\mathbf{k}, \quad k_0' = k_0. \quad (6)$$

We shall also assume that the energies of the initial electron and positron are very large compared with their rest energies, i.e.,

$$k_0^2 \gg \kappa^2. \quad (7)$$

We shall denote the angle made by the vector $\mathbf{q}$ with $\mathbf{k}$ as θ, while the angles made by $\mathbf{q}'$ and $\mathbf{q}''$ with $\mathbf{k}'$ will be denoted as θ' and θ'' respectively.

We note that

$$\frac{1}{(k-q)^2+\kappa^2} = \frac{1}{2q_0(k_0-|\mathbf{k}|\cos\theta)}, \quad (8)$$

which, in view of (7), can be written as

$$\frac{1}{(k-q)^2+\kappa^2} = \frac{k_0}{q_0[\kappa^2+2k_0^2(1-\cos\theta)]}. \quad (9)$$

The quantity (9) has a sharp maximum at $\theta=0$. This shows that practically the entire contribution to the matrix element (5) arises from those values of $\mathbf{q}$, which lie within a narrow cone around the vector $\mathbf{k}$. Similarly, the quantity

$$\frac{1}{(k'-q')^2+\kappa^2} = \frac{k_0}{q_0'[\kappa^2+2k_0^2(1-\cos\theta')]} \quad (10)$$

has a sharp maximum at $\theta'=0$, so that practically the entire contribution to the matrix element (5) arises from those values of $\mathbf{q}'$, which lie within a narrow cone around the vector $\mathbf{k}'$. For small values of θ and θ', (9) and (10) reduce to

$$\frac{1}{(k-q)^2+\kappa^2} = \frac{k_0}{q_0[\kappa^2+k_0^2\theta^2]}, \quad (11)$$

$$\frac{1}{(k'-q')^2+\kappa^2} = \frac{k_0}{q_0'[\kappa^2+k_0^2\theta'^2]}. \quad (12)$$

We have seen that the photon $\mathbf{q}$ is contained within a narrow cone around $\mathbf{k}$, and $\mathbf{q}'$ is contained within a narrow cone around $\mathbf{k}'$. It follows from the conservation of momentum that if q_0'' is comparable to q_0 or q_0', the photon $\mathbf{q}''$ must also make a small angle with $\mathbf{k}$ or $\mathbf{k}'$, while if q_0'' is small compared with q_0 or q_0', the photon $\mathbf{q}''$ can make any angle with $\mathbf{k}$ or $\mathbf{k}'$. However, for simplicity, we shall assume that $\mathbf{q}''$ also is always confined within a narrow cone around $\mathbf{k}$ or $\mathbf{k}'$. We then have to consider two cases: (*a*) The photon $\mathbf{q}$ is confined within a narrow cone around $\mathbf{k}$, while the photons $\mathbf{q}'$ and $\mathbf{q}''$ are confined within a narrow cone around $\mathbf{k}'$. (b) The photons $\mathbf{q}$ and $\mathbf{q}''$ are confined within a narrow cone around $\mathbf{k}$, while the photon $\mathbf{q}'$ is confined within a narrow cone around $\mathbf{k}'$. The case *b* can evidently be obtained from the case *a* by interchanging the roles of the electron and the positron. Hence, the cross sections for these two cases are the same, and we need calculate only the cross section for the case *a*.

We can write the matrix element (5) for the case *a* as

$$S_3 = V^{-5/2} \sum_{\mathbf{q}}^{\Omega} \sum_{\mathbf{q}',\mathbf{q}''}^{\Omega'} \sum_{e,e',e''} \int dx e^{ix(k+k'-q-q'-q'')} a_r(\mathbf{k}) b_s(\mathbf{k}') a_e^*(\mathbf{q}) a_{e'}^*(\mathbf{q}') a_{e''}^*(\mathbf{q}'') \times \left(\frac{e^2}{2c\hbar}\right)^{\frac{3}{2}} (q_0 q_0' q_0'')^{-\frac{1}{2}} \frac{\bar{v}_s(\mathbf{k}')(\gamma\cdot\mathbf{e}')[-i(k'-q')\gamma-\kappa](\gamma\cdot\mathbf{e}'')[i(k-q)\gamma-\kappa](\gamma\cdot\mathbf{e})u_r(\mathbf{k})}{[(k'-q')^2+\kappa^2][(k-q)^2+\kappa^2]}, \quad (13)$$

where $\sum_{\mathbf{q}}^{\Omega}$ denotes summation over all values of $\mathbf{q}$ within a small solid angle Ω around $\mathbf{k}$, and $\sum_{\mathbf{q}',\mathbf{q}''}^{\Omega'}$ denotes summation over all values of $\mathbf{q}'$ and $\mathbf{q}''$ within a small solid angle Ω' around $\mathbf{k}'$.

Further, we can interchange the roles of the photons $\mathbf{q}_1$ and $\mathbf{q}_2$ without changing any given physical state. Hence, we can express (13) as

$$S_3 = V^{-5/2} \sum_{\mathbf{q}}^{\Omega} {\sum_{\mathbf{q}',\mathbf{q}''}^{\Omega'}}' \sum_{e,e',e''} \int dx e^{ix(k+k'-q-q'-q'')} \times a_r(\mathbf{k}) b_s(\mathbf{k}') a_e^*(\mathbf{q}) a_{e'}^*(\mathbf{q}') a_{e''}^*(\mathbf{q}'') \times \left(\frac{e^2}{2c\hbar}\right)^{\frac{3}{2}} (q_0 q_0' q_0'')^{-\frac{1}{2}} \frac{1}{[(k-q)^2+\kappa^2]} \times \left\{ \frac{A}{[(k'-q')^2+\kappa^2]} + \frac{B}{[(k'-q'')^2+\kappa^2]} \right\}, \quad (14)$$

with

$$A=\bar{v}_s(\mathbf{k}')(\gamma\cdot\mathbf{e}')[-i(k'-q')\gamma-\kappa](\gamma\cdot\mathbf{e}'')\times[i(k-q)\gamma-\kappa](\gamma\cdot\mathbf{e})u_r(\mathbf{k}), \quad (15)$$

$$B=\bar{v}_s(\mathbf{k}')(\gamma\cdot\mathbf{e}'')[-i(k'-q'')\gamma-\kappa](\gamma\cdot\mathbf{e}')\times[i(k-q)\gamma-\kappa](\gamma\cdot\mathbf{e})u_r(\mathbf{k}), \quad (16)$$

where $\sum_{\mathbf{q}',\mathbf{q}''}^{\Omega'}{}'$ denotes summation over all values of $\mathbf{q}'$ and $\mathbf{q}''$ within the solid angle Ω' such that each physically different state occurs only once, and the second term within the curly brackets in (14) is obtained from the first one by interchanging the roles of the photons $\mathbf{q}'$ and $\mathbf{q}''$.

3. CROSS SECTION FOR THE PRODUCTION OF THREE PHOTONS IN ELECTRON-POSITRON ANNIHILATION

In order to simplify our calculations for the process under consideration, we shall make two types of approximations, which are justified for high energies of the initial electron and the positron. Firstly, we shall neglect κ^2/k_0^2 compared with 1. Secondly, since the photons are confined within narrow cones around $\mathbf{k}$ or $\mathbf{k}'$, we shall also neglect θ^2, θ'^2, and θ''^2 compared with 1. We shall not, however, neglect any of the quantities κ^2/k_0^2, θ^2, θ'^2, and θ''^2 as compared to each other.

Using (6) and the relations

$$(ik\gamma+\kappa)u_r(\mathbf{k})=0,\quad \bar{v}_s(\mathbf{k}')(ik'\gamma-\kappa)=0, \quad (17)$$

we can express A as

$$A=\bar{v}_s(-\mathbf{k})[(q'\gamma)(\gamma\cdot\mathbf{e}')-2(\mathbf{k}\cdot\mathbf{e}')](\gamma\cdot\mathbf{e}'')\times[(\gamma\cdot\mathbf{e})(q\gamma)+2(\mathbf{k}\cdot\mathbf{e})]u_r(\mathbf{k}). \quad (18)$$

We can also choose our x_3 axis along $\mathbf{k}$, and denote the azimuthal angles of $\mathbf{q}$, $\mathbf{q}'$, and $\mathbf{q}''$ around the x_3 axis as ϕ, ϕ', and ϕ'' respectively. Then the components of $\mathbf{q}$, $\mathbf{q}'$, and $\mathbf{q}''$ are

$$\begin{aligned}\mathbf{q}&=(q_0\sin\theta\cos\phi,\ q_0\sin\theta\sin\phi,\ q_0\cos\theta),\\ \mathbf{q}'&=(q_0'\sin\theta'\cos\phi',\ q_0'\sin\theta'\sin\phi',\ -q_0'\cos\theta'),\\ \mathbf{q}''&=(q_0''\sin\theta''\cos\phi'',\ q_0''\sin\theta''\sin\phi'',\ -q_0''\cos\theta'').\end{aligned} \quad (19)$$

Using (17) and (19), and making the approximations mentioned above, we get

$$\begin{aligned}(q\gamma)u_r(\mathbf{k})&=q_0(i\kappa/k_0+\theta\alpha)u_r(\mathbf{k}),\\ \bar{v}_s(-\mathbf{k})(q'\gamma)&=\bar{v}_s(-\mathbf{k})q_0'(-i\kappa/k_0+\theta'\alpha')\end{aligned} \quad (20)$$

with

$$\alpha=\gamma_1\cos\phi+\gamma_2\sin\phi,\quad \alpha'=\gamma_1\cos\phi'+\gamma_2\sin\phi', \quad (21)$$

which enables us to write (18) as

$$A=q_0q_0'\bar{v}_s(-\mathbf{k})[(-i\kappa/k_0+\theta'\alpha')(\gamma\cdot\mathbf{e}')-2(\mathbf{k}\cdot\mathbf{e}')/q_0']\times(\gamma\cdot\mathbf{e}'')[(\gamma\cdot\mathbf{e})(i\kappa/k_0+\theta\alpha)+2(\mathbf{k}\cdot\mathbf{e})/q_0]u_r(\mathbf{k}). \quad (22)$$

In the extreme relativistic case, we also have

$$\begin{aligned}\tfrac{1}{2}(1+i\gamma_4\gamma_3+\kappa\gamma_4/k_0)u_r(\mathbf{k})&=u_r(\mathbf{k}),\\ \tfrac{1}{2}(1+i\gamma_4\gamma_3+\kappa\gamma_4/k_0)v_r(-\mathbf{k})&=0,\\ \tfrac{1}{2}(1-i\gamma_4\gamma_3-\kappa\gamma_4/k_0)v_r(-\mathbf{k})&=v_r(-\mathbf{k}),\\ \tfrac{1}{2}(1-i\gamma_4\gamma_3-\kappa\gamma_4/k_0)u_r(\mathbf{k})&=0.\end{aligned} \quad (23)$$

Therefore, averaging the quantity AA^* in the usual way over the spin states of the electron and the positron in the initial state, we get

$$\begin{aligned}\langle AA^*\rangle_{\text{Av}}=\tfrac{1}{16}q_0^2q_0'^2\,\mathrm{tr}\{&[(\gamma\cdot\mathbf{e})(i\kappa/k_0+\theta\alpha)+2(\mathbf{k}\cdot\mathbf{e})/q_0]\\ &\times(1+i\gamma_4\gamma_3+\kappa\gamma_4/k_0)[(-i\kappa/k_0+\theta\alpha)(\gamma\cdot\mathbf{e})\\ &+2(\mathbf{k}\cdot\mathbf{e})/q_0](\gamma\cdot\mathbf{e}'')[(\gamma\cdot\mathbf{e}')(i\kappa/k_0+\theta'\alpha')\\ &-2(\mathbf{k}\cdot\mathbf{e}')/q_0'](1+i\gamma_4\gamma_3-\kappa\gamma_4/k_0)\\ &\times[(-i\kappa/k_0+\theta'\alpha')(\gamma\cdot\mathbf{e}')\\ &-2(\mathbf{k}\cdot\mathbf{e}')/q_0'](\gamma\cdot\mathbf{e}'')\}.\end{aligned} \quad (24)$$

Further, summing over the states of polarization of the photons $\mathbf{q}$, $\mathbf{q}'$, and $\mathbf{q}''$ in succession, we find

$$\sum_{\mathbf{e},\mathbf{e}',\mathbf{e}''}\langle AA^*\rangle_{\text{Av}}=4q_0^2q_0'^2[\kappa^2/k_0^2+\theta^2(1-2k_0/q_0+2k_0^2/q_0^2)]\times[\kappa^2/k_0^2+\theta'^2(1-2k_0/q_0'+2k_0^2/q_0'^2)]. \quad (25)$$

According to (15) and (16), B can be obtained from A by interchanging the roles of the photons $\mathbf{q}'$ and $\mathbf{q}''$. Hence, we obtain from (22)

$$B=q_0q_0''\bar{v}_s(-\mathbf{k})[(-i\kappa/k_0+\theta''\alpha'')(\gamma\cdot\mathbf{e}'')-2(\mathbf{k}\cdot\mathbf{e}'')/q_0'']\times(\gamma\cdot\mathbf{e}')[(\gamma\cdot\mathbf{e})(i\kappa/k_0+\theta\alpha)+2(\mathbf{k}\cdot\mathbf{e})/q_0]u_r(\mathbf{k}), \quad (26)$$

where

$$\alpha''=\gamma_1\cos\phi''+\gamma_2\sin\phi''. \quad (27)$$

Using (22) and (26), and averaging the quantity AB^* over the spin states of the electron and the positron, we get

$$\begin{aligned}\langle AB^*\rangle_{\text{Av}}=\tfrac{1}{16}q_0^2q_0'q_0''\,\mathrm{tr}\{&[(\gamma\cdot\mathbf{e})(i\kappa/k_0+\theta\alpha)\\ &+2(\mathbf{k}\cdot\mathbf{e})/q_0](1+i\gamma_4\gamma_3+\kappa\gamma_4/k_0)\\ &\times[(-i\kappa/k_0+\theta\alpha)(\gamma\cdot\mathbf{e})+2(\mathbf{k}\cdot\mathbf{e})/q_0](\gamma\cdot\mathbf{e}')\\ &\times[(\gamma\cdot\mathbf{e}'')(i\kappa/k_0+\theta''\alpha'')-2(\mathbf{k}\cdot\mathbf{e}'')/q_0'']\\ &\times(1+i\gamma_4\gamma_3-\kappa\gamma_4/k_0)[(-i\kappa/k_0+\theta'\alpha')\\ &\times(\gamma\cdot\mathbf{e}')-2(\mathbf{k}\cdot\mathbf{e}')/q_0'](\gamma\cdot\mathbf{e}'')\}.\end{aligned} \quad (28)$$

Then, summing over the states of polarization of the photons, we find

$$\sum_{\mathbf{e},\mathbf{e}',\mathbf{e}''}\langle AB^*\rangle_{\text{Av}}=4\theta'\theta''k_0^2q_0^2\cos(\phi'-\phi'')\times[\kappa^2/k_0^2+\theta^2(1-2k_0/q_0+2k_0^2/q_0^2)]. \quad (29)$$

We can now write (14) as

$$S_3=V^{-5/2}\sum_{\mathbf{q}}^{\Omega}\sum_{\mathbf{q}',\mathbf{q}''}^{\Omega'}{}'\sum_{\mathbf{e},\mathbf{e}',\mathbf{e}''}\int dx e^{ix(k+k'-q-q'-q'')}\times a_r(\mathbf{k})b_s(\mathbf{k}')a_{\mathbf{e}}^*(\mathbf{q})a_{\mathbf{e}'}^*(\mathbf{q}')a_{\mathbf{e}''}^*(\mathbf{q}'')K, \quad (30)$$

where

$$K=(e^2/2c\hbar)^{\frac{3}{2}}(q_0q_0'q_0'')^{-\frac{1}{2}}\frac{1}{[(k-q)^2+\kappa^2]}\times\left\{\frac{A}{[(k'-q')^2+\kappa^2]}+\frac{B}{[(k'-q'')^2+\kappa^2]}\right\}. \quad (31)$$

Simplifying the denominators in (31) for small values of the angles θ, θ', and θ'', we obtain

$$K=(e^2/2c\hbar)^{\frac{3}{2}}(q_0q_0'q_0'')^{-\frac{1}{2}}\frac{k_0^2}{q_0[\kappa^2+k_0^2\theta^2]}\times\left\{\frac{A}{q_0'[\kappa^2+k_0^2\theta'^2]}+\frac{B}{q_0''[\kappa^2+k_0^2\theta''^2]}\right\}, \quad (32)$$

so that

$$KK^*=(e^2/2c\hbar)^3\frac{k_0^4}{q_0^3q_0'q_0''[\kappa^2+k_0^2\theta^2]^2}\times\left\{\frac{AA^*}{q_0'^2[\kappa^2+k_0^2\theta'^2]^2}+\frac{BB^*}{q_0''^2[\kappa^2+k_0^2\theta''^2]^2}+\frac{AB^*+BA^*}{q_0'q_0''[\kappa^2+k_0^2\theta'^2][\kappa^2+k_0^2\theta''^2]}\right\}. \quad (33)$$

The cross section for the process under consideration is related to the quantity (33) as

$$\sigma_3=\frac{1}{2V}\sum_{\mathbf{q}'}^{\Omega'}\frac{1}{2(2\pi)^2}\int_\Omega d\omega q_0^2\left[\frac{dq_0}{d(q_0+q_0'')}\right]_{q_0'=\text{constant}}\times\sum_{e,e',e''}\langle KK^*\rangle_{\text{Av}}, \quad (34)$$

where $d\omega$ denotes an element of the solid angle in the direction of $\mathbf{q}$, and $\langle\ \rangle_{\text{Av}}$ denotes an average over the spin states of the electron and the positron in the initial state. But, in the present case, we have approximately

$$q_0'+q_0''=q_0=k_0, \quad (35)$$

which gives

$$\left[\frac{dq_0}{d(q_0+q_0'')}\right]_{q_0'=\text{constant}}=\tfrac{1}{2}. \quad (36)$$

We can also put

$$\frac{1}{V}\sum_{\mathbf{q}'}^{\Omega'}=(2\pi)^{-3}\int_{\Omega'}d\mathbf{q}'=(2\pi)^{-3}\int dq_0'\int_{\Omega'}q_0'^2d\omega'. \quad (37)$$

Hence, we can express (34) as

$$\sigma_3=\tfrac{1}{8}(2\pi)^{-5}\int dq_0'\int_{\Omega'}d\omega'\int_\Omega d\omega q_0'^2q_0^2\sum_{e,e',e''}\langle KK^*\rangle_{\text{Av}}, \quad (38)$$

or

$$\sigma_3=\tfrac{1}{8}(2\pi)^{-3}\int dq_0'\int_0^\delta\theta'd\theta'\int_0^\delta\theta d\theta q_0'^2q_0^2\sum_{e,e',e''}\langle KK^*\rangle_{\text{Av}}, \quad (39)$$

where δ is the angle of the narrow cones within which $\mathbf{q}$ and $\mathbf{q}'$ have been confined.

It is evident that in order to obtain the cross section σ_3, we can interchange the roles of the photons $\mathbf{q}'$ and $\mathbf{q}''$ in any term of the quantity KK^*. Therefore, (33) is equivalent to

$$KK^*=\left(\frac{e^2}{2c\hbar}\right)^3\frac{k_0^4}{q_0^3q_0'q_0''[\kappa^2+k_0^2\theta^2]^2}\left\{\frac{2AA^*}{q_0'^2[\kappa^2+k_0^2\theta'^2]^2}+\frac{2AB^*}{q_0'q_0''[\kappa^2+k_0^2\theta'^2][\kappa^2+k_0^2\theta''^2]}\right\}. \quad (40)$$

Using (25), (29), and (35), we obtain

$$\sum_{e,e',e''}\langle KK^*\rangle_{\text{Av}}=8\left(\frac{e^2}{2c\hbar}\right)^3\frac{1}{k_0[\kappa^2+k_0^2\theta^2][\kappa^2+k_0^2\theta'^2]}\times\left\{\frac{1}{q_0'(k_0-q_0')}+\frac{2\theta'^2k_0^3}{q_0'^3[\kappa^2+k_0^2\theta'^2]}+\frac{\theta'\theta''k_0^4\cos(\phi'-\phi'')}{q_0'^2(k_0-q_0')^2[\kappa^2+k_0^2\theta''^2]}\right\}, \quad (41)$$

and, substituting (41) in (39), we get

$$\sigma_3=\left(\frac{e^2}{4\pi c\hbar}\right)^3\int dq_0'\int_0^\delta\theta'd\theta'\int_0^\delta\theta d\theta\times\frac{k_0^2}{[\kappa^2+k_0^2\theta^2][\kappa^2+k_0^2\theta'^2]}\left\{\frac{q_0'}{(k_0-q_0')k_0}+\frac{2}{q_0'}-\frac{2\kappa^2}{q_0'[\kappa^2+k_0^2\theta'^2]}+\frac{\theta'\theta''k_0^3\cos(\phi'-\phi'')}{(k_0-q_0')^2[\kappa^2+k_0^2\theta''^2]}\right\}. \quad (42)$$

Carrying out the integrations in (42), we find

$$\sigma_3=3(e^2/4\pi c\hbar)^3k_0^{-2}\log(k_0/\lambda)[\log(\delta k_0/\kappa)]^2, \quad (43)$$

where we have denoted the lower limit to the values of q_0' and q_0'' as λ, and we have neglected terms involving lower powers of $\log(k_0/\lambda)$ or $\log(\delta k_0/\kappa)$.

We now put

$$k_0=E/c\hbar,\quad \kappa=\mu/c\hbar,\quad \lambda=\epsilon/c\hbar, \quad (44)$$

where μ is the rest energy of the electron, E is the energy of the electron or the positron in the center-of-mass system, and ϵ is the lower limit to the energy of the emitted photons in the center-of-mass system. We can then express the cross section (43) as

$$\sigma_3=3\alpha^3(c^2\hbar^2/E^2)\log(E/\epsilon)[\log(\delta E/\mu)]^2, \quad (45)$$

where

$$\alpha=e^2/4\pi c\hbar \quad (46)$$

denotes the fine structure constant.

It should further be noted that the expression (45) represents the cross sections for the process, in which two of the three photons are emitted within a narrow cone around the direction of motion of the positron. As explained in Sec. 2, the cross section for the process, in which two of the three photons are emitted within a narrow cone around the direction of motion of the electron, will be the same. Hence, the total cross section for the production of three photons in electron-positron annihilation is

$$\sigma_{3,\,\mathrm{total}}=2\sigma_3=6\alpha^3(c^2\hbar^2/E^2)\log(E/\epsilon)[\log(\delta E/\mu)]^2. \quad (47)$$

4. MULTIPLE PRODUCTION OF PHOTONS IN ELECTRON-POSITRON ANNIHILATION

We shall now calculate the cross section for the production of n photons in electron-positron annihilation. The S matrix element for this process is[8]

$$S_n=(-i)^n(e^2/2c\hbar)^{\frac{1}{2}n}\sum_{\mathbf{q}_1,\mathbf{e}_1}\sum_{\mathbf{q}_2,\mathbf{e}_2}\cdots\sum_{\mathbf{q}_n,\mathbf{e}_n}V^{-(\frac{1}{2}n+1)}\frac{1}{(q_{0,1}q_{0,2}\cdots q_{0,n})^{\frac{1}{2}}}\int dx'e^{i(k+k'-q_1-q_2-\cdots-q_n)x'}$$
$$\times\frac{\bar{v}_s(\mathbf{k}')(\gamma\cdot\mathbf{e}_n)[ip_{n-1}\gamma-\kappa](\gamma\cdot\mathbf{e}_{n-1})\cdots[ip_1\gamma-\kappa](\gamma\cdot\mathbf{e}_1)u_t(\mathbf{k})}{[p_{n-1}^2+\kappa^2]\cdots[p_1^2+\kappa^2]}, \quad (48)$$

where

$$\begin{aligned}&p_1=k-q_1,\quad p_2=k-q_1-q_2,\\&\cdots\cdots\cdots\\&p_{n-2}=k-q_1-\cdots-q_{n-2}=q_n+q_{n-1}-k',\\&p_{n-1}=k-q_1-\cdots-q_{n-1}=q_n-k'.\end{aligned} \quad (49)$$

As before, we shall carry out our calculations in the center-of-mass system so that

$$\mathbf{k}'=-\mathbf{k},\quad k_0'=k_0. \quad (50)$$

Moreover, we shall be interested only in the case when $k_0\gg\kappa$.

In the present case, we shall be largely guided by the calculations of the preceding sections. Thus, we assume that the r photons $\mathbf{q}_1$, $\mathbf{q}_2$, $\cdots$, $\mathbf{q}_r$ are emitted in a narrow cone of angle δ around the vector $\mathbf{k}$, while the remaining $n-r$ photons are emitted in a narrow cone of angle δ around the vector $\mathbf{k}'$. We shall neglect δ^2 compared with 1. Then, from the conservation of momentum and energy, it follows that

$$q_{0,1}+q_{0,2}+\cdots+q_{0,r}=q_{0,r+1}+q_{0,r+2}+\cdots+q_{0,n}=k_0. \quad (51)$$

It will be convenient to express the p's as

$$\begin{aligned}&p_1=k-q_1,\quad\cdots,\quad p_r=k-q_1-\cdots-q_r,\\&p_{n-1}=q_n-k',\quad\cdots,\quad p_{r+1}=q_n+\cdots+q_{r+2}-k',\end{aligned} \quad (52)$$

so that we can express (48) as

$$S_n=(-i)^n(e^2/2c\hbar)^{\frac{1}{2}n}\sum_{\mathbf{q}_1,\mathbf{e}_1}\sum_{\mathbf{q}_2,\mathbf{e}_2}\cdots\sum_{\mathbf{q}_n,\mathbf{e}_n}V^{-(\frac{1}{2}n+1)}(q_{0,1}q_{0,2}\cdots q_{0,n})^{-\frac{1}{2}}\int dx'e^{i(k+k'-q_1-\cdots-q_n)x'}$$
$$\times\frac{\bar{v}_s(\mathbf{k}')(\gamma\cdot\mathbf{e}_n)[i(q_n-k')\gamma-\kappa]\cdots(\gamma\cdot\mathbf{e}_{r+2})[i(q_n+\cdots+q_{r+2}-k')\gamma-\kappa]}{[(q_n-k')^2+\kappa^2]\cdots[(q_n+\cdots+q_{r+2}-k')^2+\kappa^2]}(\gamma\cdot\mathbf{e}_{r+1})$$
$$\times\frac{[i(k-q_1-\cdots-q_r)\gamma-\kappa](\gamma\cdot\mathbf{e}_r)\cdots[i(k-q_1)\gamma-\kappa](\gamma\cdot\mathbf{e}_1)u_t(\mathbf{k})}{[(k-q_1-\cdots-q_r)^2+\kappa^2]\cdots[(k-q_1)^2+\kappa^2]}. \quad (53)$$

In order that S_n may be as large as possible, the q_0's should be as small as possible. But, on account of the relation (51), at least two of the q_0's must have large values. We further note that $q_{0,r}$ and $q_{0,r+1}$ occur the least number of times in the denominators of the propagation functions in (53). Therefore, it is evident that the largest values of S_n correspond to the case, when $q_{0,r}$ and $q_{0,r+1}$ are large while all other q_0's are small. We can expect to obtain a reasonable result by making approximations, which are justified for the above values of the q_0's.

When $q_{0,r}$ and $q_{0,r+1}$ are large and the other q_0's are small, we may simplify the denominators in (53) as

$$[(k-q_1)^2+\kappa^2]\cdots[(k-q_1-\cdots-q_r)^2+\kappa^2]$$
$$=[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]\cdots[(2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1)+\cdots+(2k_0q_{0,r-1}-2\mathbf{k}\cdot\mathbf{q}_{r-1})][2k_0q_{0,r}-2\mathbf{k}\cdot\mathbf{q}_r], \quad (54)$$

$$[(q_n-k')^2+\kappa^2]\cdots[(q_n+\cdots+q_{r+2}-k')^2+\kappa^2]$$
$$=[2k_0q_{0,n}-2\mathbf{k}'\cdot\mathbf{q}_n]\cdots[(2k_0q_{0,n}-2\mathbf{k}'\cdot\mathbf{q}_n)+\cdots+(2k_0q_{0,r+2}-2\mathbf{k}'\cdot\mathbf{q}_{r+2})]. \quad (55)$$

We also have approximately

$$[i(k-q_1)\gamma-\kappa](\gamma\cdot\mathbf{e}_1)u_t(\mathbf{k})=[ik\gamma-\kappa](\gamma\cdot\mathbf{e}_1)u_t(\mathbf{k})=2i(\mathbf{k}\cdot\mathbf{e}_1)u_t(\mathbf{k}), \quad (56)$$

[8] Note that in (48) the quantities q_r and p_r denote four-vectors with the components $(\mathbf{q}_r,iq_{0,r})$ and $(\mathbf{p}_r,ip_{0,r})$ respectively.

and in this way we find

$$[i(k-q_1-\cdots-q_r)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_r)\cdots[i(k-q_1)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_1)u_t(\mathbf{k}) = (2i\mathbf{k}\cdot\mathbf{e}_{r-1})\cdots(2i\mathbf{k}\cdot\mathbf{e}_1)[i(k-q_r)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_r)u_t(\mathbf{k}), \quad (57)$$

$$\bar{v}_s(\mathbf{k}')(\boldsymbol{\gamma}\cdot\mathbf{e}_n)[i(q_n-k')\gamma-\kappa]\cdots(\boldsymbol{\gamma}\cdot\mathbf{e}_{r+2})[i(q_n+\cdots+q_{r+2}-k')\gamma-\kappa]=\bar{v}_s(\mathbf{k}')(-2i\mathbf{k}'\cdot\mathbf{e}_n)\cdots(-2i\mathbf{k}'\cdot\mathbf{e}_{r+2}). \quad (58)$$

Using (54), (55), (57), and (58), we can express (53) as

$$S_n=(-i)^n(e^2/2c\hbar)^{\frac{1}{2}n}\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}V^{-(\frac{1}{2}n+1)}(q_{0,1}q_{0,2}\cdots q_{0,n})^{-\frac{1}{2}}\int dx' e^{i(k+k'-q_1-\cdots-q_n)x'}$$
$$\times\frac{\bar{v}_s(\mathbf{k}')(\boldsymbol{\gamma}\cdot\mathbf{e}_{r+1})[i(k-q_r)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_r)u_t(\mathbf{k})}{[2k_0q_{0,r}-2\mathbf{k}\cdot\mathbf{q}_r]}$$
$$\times\frac{(-2i\mathbf{k}'\cdot\mathbf{e}_n)\cdots(-2i\mathbf{k}'\cdot\mathbf{e}_{r+2})}{[2k_0q_{0,n}-2\mathbf{k}'\cdot\mathbf{q}_n]\cdots[(2k_0q_{0,n}-2\mathbf{k}'\cdot\mathbf{q}_n)+\cdots+(2k_0q_{0,r+2}-2\mathbf{k}'\cdot\mathbf{q}_{r+2})]}$$
$$\times\frac{(2i\mathbf{k}\cdot\mathbf{e}_{r-1})\cdots(2i\mathbf{k}\cdot\mathbf{e}_1)}{[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]\cdots[(2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1)+\cdots+(2k_0q_{0,r-1}-2\mathbf{k}\cdot\mathbf{q}_{r-1})]}. \quad (59)$$

But, we can interchange the roles of the photons $\mathbf{q}_1, \cdots, \mathbf{q}_{r-1}$ without changing any given physical state, and similarly we can interchange the roles of the photons $\mathbf{q}_{r+1}, \cdots, \mathbf{q}_n$. Therefore, we can write S_n as

$$S_n=(-i)^n(e^2/2c\hbar)^{\frac{1}{2}n}\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}\sum_{\mathbf{q}_r,\mathbf{q}_{r+1}}\sum'_{\mathbf{q}_1,\cdots,\mathbf{q}_{r-1}}\sum'_{\mathbf{q}_{r+1},\cdots,\mathbf{q}_n}V^{-(\frac{1}{2}n+1)}(q_{0,1}q_{0,2}\cdots q_{0,n})^{-\frac{1}{2}}$$
$$\times\int dx' e^{i(k+k'-q_1-\cdots-q_n)x'}\frac{\bar{v}_s(\mathbf{k}')(\boldsymbol{\gamma}\cdot\mathbf{e}_{r+1})[i(k-q_r)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_r)u_t(\mathbf{k})}{[2k_0q_{0,r}-2\mathbf{k}\cdot\mathbf{q}_r]}$$
$$\times\left\{\sum^{P}_{\mathbf{q}_n,\cdots,\mathbf{q}_{r+1}}\frac{(-2i\mathbf{k}'\cdot\mathbf{e}_n)\cdots(-2i\mathbf{k}'\cdot\mathbf{e}_{r+2})}{[2k_0q_{0,n}-2\mathbf{k}'\cdot\mathbf{q}_n]\cdots[(2k_0q_{0,n}-2\mathbf{k}'\cdot\mathbf{q}_n)+\cdots+(2k_0q_{0,r+2}-2\mathbf{k}'\cdot\mathbf{q}_{r+2})]}\right\}$$
$$\times\left\{\sum^{P}_{\mathbf{q}_1,\cdots,\mathbf{q}_{r-1}}\frac{(2i\mathbf{k}\cdot\mathbf{e}_{r-1})\cdots(2i\mathbf{k}\cdot\mathbf{e}_1)}{[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]\cdots[(2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1)+\cdots+(2k_0q_{0,r-1}-2\mathbf{k}\cdot\mathbf{q}_{r-1})]}\right\}, \quad (60)$$

where $\sum'_{\mathbf{q}_1,\cdots,\mathbf{q}_{r-1}}$ denotes summation over all values of $\mathbf{q}_1, \cdots, \mathbf{q}_{r-1}$ such that each physically different state occurs only once, and $\sum^{P}_{\mathbf{q}_1,\cdots,\mathbf{q}_{r-1}}$ denotes summation over all possible terms obtained by interchanging the roles of the photons $\mathbf{q}_1, \cdots, \mathbf{q}_{r-1}$. Then, using the identity

$$\sum^{P}_{a_1,\cdots,a_n}\frac{1}{a_1(a_1+a_2)\cdots(a_1+a_2+\cdots+a_n)}=\frac{1}{a_1a_2\cdots a_n}, \quad (61)$$

we can express (61) as

$$S_n=\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}\sum_{\mathbf{q}_r,\mathbf{q}_{r+1}}\sum'_{\mathbf{q}_1,\cdots,\mathbf{q}_{r-1}}\sum'_{\mathbf{q}_{r+1},\cdots,\mathbf{q}_n}V^{-(\frac{1}{2}n+1)}\times\int dx' e^{i(k+k'-q_1-\cdots-q_n)x'}K_n, \quad (62)$$

where

$$K_n=(-i)^n(e^2/2c\hbar)^{\frac{1}{2}n}(q_{0,1}q_{0,2}\cdots q_{0,n})^{-\frac{1}{2}}\times\frac{\bar{v}_s(\mathbf{k}')(\boldsymbol{\gamma}\cdot\mathbf{e}_{r+1})[i(k-q_r)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_r)u_t(\mathbf{k})}{[2k_0q_{0,r}-2\mathbf{k}\cdot\mathbf{q}_r]}$$
$$\times\frac{(-2i\mathbf{k}'\cdot\mathbf{e}_n)\cdots(-2i\mathbf{k}'\cdot\mathbf{e}_{r+2})}{[2k_0q_{0,n}-2\mathbf{k}'\cdot\mathbf{q}_n]\cdots[2k_0q_{0,r+2}-2\mathbf{k}'\cdot\mathbf{q}_{r+2}]}\times\frac{(2i\mathbf{k}\cdot\mathbf{e}_{r-1})\cdots(2i\mathbf{k}\cdot\mathbf{e}_1)}{[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]\cdots[2k_0q_{0,r-1}-2\mathbf{k}\cdot\mathbf{q}_{r-1}]}. \quad (63)$$

According to (63), we have

$$KK^*=(e^2/2c\hbar)^n\frac{2^{2n-4}}{q_{0,1}\cdots q_{0,n}}\frac{CC^*}{[2k_0q_{0,r}-2\mathbf{k}\cdot\mathbf{q}_r]^2}\times\frac{(\mathbf{k}'\cdot\mathbf{e}_n)^2\cdots(\mathbf{k}'\cdot\mathbf{e}_{r+2})^2}{[2k_0q_{0,n}-2\mathbf{k}'\cdot\mathbf{q}_n]^2\cdots[2k_0q_{0,r+2}-2\mathbf{k}'\cdot\mathbf{q}_{r+2}]^2}$$
$$\times\frac{(\mathbf{k}\cdot\mathbf{e}_{r-1})^2\cdots(\mathbf{k}\cdot\mathbf{e}_1)^2}{[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]^2\cdots[2k_0q_{0,r-1}-2\mathbf{k}\cdot\mathbf{q}_{r-1}]^2}, \quad (64)$$

where

$$C=\bar{v}_s(\mathbf{k}')(\gamma\cdot\mathbf{e}_{r+1})[i(k-q_r)\gamma-\kappa](\gamma\cdot\mathbf{e}_r)u_t(\mathbf{k}). \quad (65)$$

We now denote the angles made by $\mathbf{q}_1, \cdots, \mathbf{q}_r$ with $\mathbf{k}$ as $\theta_1, \cdots, \theta_r$ respectively, and the angles made by $\mathbf{q}_{r+1}, \cdots, \mathbf{q}_n$ with $\mathbf{k}'$ as $\theta_{r+1}, \cdots, \theta_n$ respectively. Then, for small values of the θ's, we get

$$[2k_0q_{0,i}-2\mathbf{k}\cdot\mathbf{q}_i]^{-1}=(k_0/q_{0,i})[\kappa^2+k_0^2\theta_i^2]^{-1}, \quad (66)$$

$$[2k_0q_{0,j}-2\mathbf{k}'\cdot\mathbf{q}_j]^{-1}=(k_0/q_{0,j})[\kappa^2+k_0^2\theta_j^2]^{-1}, \quad (67)$$

$$\sum_{\mathbf{e}_i}(\mathbf{k}\cdot\mathbf{e}_i)^2=k_0^2\theta_i^2, \quad \sum_{\mathbf{e}_j}(\mathbf{k}'\cdot\mathbf{e}_j)^2=k_0^2\theta_j^2, \quad (68)$$

where $i\leqslant r$ and $j>r$. Further, averaging over the spin states of the electron and the positron in the initial state, and summing over the directions of polarization of the photons $\mathbf{q}_r$ and $\mathbf{q}_{r+1}$, we find

$$\sum_{\mathbf{e}_r,\mathbf{e}_{r+1}}\langle CC^*\rangle_{\text{Av}}=2(q_{0,r}^2/k_0^2)\times[\kappa^2+k_0^2\theta_r^2(1-2k_0/q_{0,r}+2k_0^2/q_{0,r}^2)]. \quad (69)$$

We substitute (66), (67), (68), and (69) in (64), and, as before, we take $q_{0,r}$ and $q_{0,r+1}$ to be of the order of k_0. We then obtain

$$\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}\langle KK^*\rangle_{\text{Av}}=(e^2/2c\hbar)^n\frac{2^{2n-3}k_0^{2n-4}}{q_{0,1}\cdots q_{0,r-1}q_{0,r+2}\cdots q_{0,n}[\kappa^2+k_0^2\theta_r^2]}\frac{1}{q_{0,1}^2\cdots q_{0,r}^2}\frac{1}{q_{0,r+2}^2\cdots q_{0,n}^2}\times\frac{(k_0^2\theta_1^2)\cdots(k_0^2\theta_{r-1}^2)}{[\kappa^2+k_0^2\theta_1^2]^2\cdots[\kappa^2+k_0^2\theta_{r-1}^2]^2}\frac{(k_0^2\theta_{r+2}^2)\cdots(k_0^2\theta_n^2)}{[\kappa^2+k_0^2\theta_{r+2}^2]^2\cdots[\kappa^2+k_0^2\theta_n^2]^2}. \quad (70)$$

The cross section σ_n for the multiple production of n photons is related to the quantity (70) as

$$\sigma_n=\frac{1}{V^{n-2}}\sum'_{\mathbf{q}_1,\cdots,\mathbf{q}_{r-1}}\sum'_{\mathbf{q}_{r+2},\cdots,\mathbf{q}_n}\frac{1}{2(2\pi)^2}\int d\omega_r q_{0,r}^2\left[\frac{dq_{0,r}}{d(q_{0,r}+q_{0,r+1})}\right]\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}\langle KK^*\rangle_{\text{Av}}, \quad (71)$$

where $d\omega_r$ is an element of solid angle in the direction of $\mathbf{q}_r$, and the derivative within the square brackets in (71) is to be obtained by keeping $q_{0,1}, \cdots, q_{0,r-1}$ and $q_{0,r+2}, \cdots, q_{0,n}$ constant. From (51), we get

$$\left[\frac{dq_{0,r}}{d(q_{0,r}+q_{0,r+1})}\right]=\tfrac{1}{2}, \quad (72)$$

and we can put

$$\frac{1}{V^{r-1}}\sum'_{\mathbf{q}_1,\cdots,\mathbf{q}_{r-1}}=\frac{1}{(r-1)!}\frac{1}{(2\pi)^{3(r-1)}}\int d\mathbf{q}_1\cdots\int d\mathbf{q}_{r-1},$$

$$\frac{1}{V^{n-r-1}}\sum'_{\mathbf{q}_{r+2},\cdots,\mathbf{q}_n}=\frac{1}{(n-r-1)!}\frac{1}{(2\pi)^{3(n-r-1)}}\int d\mathbf{q}_{r+2}\cdots\int d\mathbf{q}_n, \quad (73)$$

where the factorial factors in (73) arise from the fact that $\sum'$ denotes a summation such that each physically different state occurs only once. We further note that

$$\int d\mathbf{q}_i=2\pi\int dq_{0,i}q_{0,i}^2\int_0^\delta\theta_i d\theta_i, \quad (74)$$

where δ is the angle of the cones, within which the photons have been confined. Hence, using (70), (72), (73), and (74), we obtain from (73)

$$\sigma_n=\alpha^n\frac{1}{(r-1)!(n-r-1)!}\frac{2^{n-2}k_0^{2n-4}}{\pi^{n-3}}\int dq_{0,1}\cdots\times\int dq_{0,r-1}\frac{1}{q_{0,1}\cdots q_{0,r-1}}\int dq_{0,r+2}\cdots\times\int dq_{0,n}\frac{1}{q_{0,r+2}\cdots q_{0,n}}\int_0^\delta\theta_1 d\theta_1\cdots\times\int_0^\delta\theta_r d\theta_r\int_0^\delta\theta_{r+2}d\theta_{r+2}\cdots\int_0^\delta\theta_n d\theta_n\times\frac{(k_0^2\theta_1^2)\cdots(k_0^2\theta_{r-1}^2)}{[\kappa^2+k_0^2\theta_1^2]^2\cdots[\kappa^2+k_0^2\theta_{r-1}^2]^2}\frac{1}{[\kappa^2+k_0^2\theta_r^2]}\times\frac{(k_0^2\theta_{r+2}^2)\cdots(k_0^2\theta_n^2)}{[\kappa^2+k_0^2\theta_{r+2}^2]^2\cdots[\kappa^2+k_0^2\theta_n^2]^2}. \quad (75)$$

Carrying out the integrations in (75), and ignoring some lower-order terms, we get

$$\sigma_n=\alpha^n\frac{1}{(r-1)!(n-r-1)!}\frac{2^{n-2}}{\pi^{n-3}}\frac{1}{k_0^2}\times\left(\log\frac{k_0\delta}{\kappa}\right)^{n-1}\left(\log\frac{q_{0,\max}}{\lambda}\right)^{n-2}, \quad (76)$$

where $q_{0,\max}$ and λ are the maximum and the minimum values of $q_{0,1}, \cdots, q_{0,r-1}, q_{0,r+2}, \cdots, q_{0,n}$. Using (44), we can also express (76) as

$$\sigma_n=\alpha^n\frac{1}{(r-1)!(n-r-1)!}\frac{2^{n-2}}{\pi^{n-3}}\frac{c^2\hbar^2}{E^2}\times\left(\log\frac{E\delta}{\mu}\right)^{n-1}\left(\log\frac{\epsilon_{\max}}{\epsilon}\right)^{n-2}, \quad (77)$$

where

$$\epsilon_{\max}=c\hbar q_{0,\max} \quad (78)$$

denotes the upper limit to the energies of the photons $\mathbf{q}_1, \cdots, \mathbf{q}_{r-1}, \mathbf{q}_{r+2}, \cdots, \mathbf{q}_n$.

When $n=3$, $r=1$, $\epsilon_{\max}=E$, (77) becomes

$$2\alpha^3(c^2\hbar^2/E^2)\log(E/\epsilon)[\log(\delta E/\mu)]^2,$$

which differs from the more accurate result (45) only by a factor $\frac{2}{3}$. Following the treatment of Secs. 2–3, we have also investigated more carefully the cross section σ_n for low values of n, and it is found that the result (77) represents a good approximation.

The expression (77) gives the cross section for the production of n photons, out of which r photons are emitted within a narrow cone around the direction of motion of the electron. In order to obtain the total cross section for the production of n photons, we have to sum (77) over all values of r from 1 to $n-1$. Thus, using the relation

$$\sum_{r=1}^{n-1}\frac{1}{(r-1)!}\frac{1}{(n-r-1)!}=\frac{2^{n-2}}{(n-2)!}, \quad (79)$$

we obtain for the total cross section in the center-of-mass system

$$\sigma_{n,\,\text{total}}=\alpha^n\frac{1}{(n-2)!}\frac{4^{n-2}}{\pi^{n-3}}\frac{c^2\hbar^2}{E^2}\left(\log\frac{\delta E}{\mu}\right)^{n-1}\left(\log\frac{\epsilon_{\max}}{\epsilon}\right)^{n-2}. \quad (80)$$

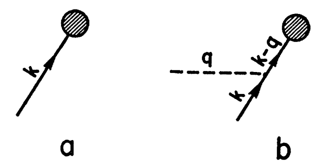

FIG. 1. Feynman diagrams for (a) an arbitrary process with at least one external electron line, and (b) emission of a photon from an external electron line of an arbitrary process.

5. MULTIPLE PHOTON PRODUCTION IN ANY ARBITRARY PROCESS IN QUANTUM ELECTRODYNAMICS

Let us consider any arbitrary process in quantum electrodynamics, whose Feynman diagram contains at least one external electron line,[9] as shown in Fig. 1(a). The S matrix element for this process will be of the form

$$S_0=S_0'(k)u(k), \quad (81)$$

where $u(k)$ is the spinor amplitude of the electron, and $S_0'(k)$ is some quantity, which is a function of k as well as some other variables. If we insert an external photon line in the external electron line, as shown in Fig. 1(b), then according to Dyson's treatment[5] of the S matrix, the contribution of the new process will be

$$S_1=-iV^{-\frac{1}{2}}(e^2/2c\hbar)^{\frac{1}{2}}\sum_{\mathbf{q},\mathbf{e}}q_0^{-\frac{1}{2}}\times\frac{S_0'(k-q)[i(k-q)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e})u(k)}{[(k-q)^2+\kappa^2]}. \quad (82)$$

Similarly, if we introduce n external photon lines in the external electron line of Fig. 1(a), the S matrix element for the resulting process will be

$$S_n=(-i)^nV^{-\frac{1}{2}n}(e^2/2c\hbar)^{\frac{1}{2}n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}(q_{0,1}\cdots q_{0,n})^{-\frac{1}{2}}\times\frac{S_0'(k-q_1-\cdots-q_n)[i(k-q_1-\cdots-q_n)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_n)\cdots[i(k-q_1)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_1)u(k)}{[(k-q_1-\cdots-q_n)^2+\kappa^2]\cdots[(k-q_1)^2+\kappa^2]}. \quad (83)$$

We shall now try to find a relation between the cross section σ_n for the above process and the cross section σ_0 for the process shown in Fig. 1(a), when the external electron line represents an electron of very high energy.

As in the case of the electron-positron annihilation, we may expect that the main contribution to the cross section σ_n arises from low values of the q_0's. Therefore, as in Sec. 4, we may simplify the numerator and the denominator in (83) as

$$[i(k-q_1-\cdots-q_n)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_n)\cdots\times[i(k-q_1)\gamma-\kappa](\boldsymbol{\gamma}\cdot\mathbf{e}_1)u(k)=(2i\mathbf{k}\cdot\mathbf{e}_n)\cdots(2i\mathbf{k}\cdot\mathbf{e}_1)u(k), \quad (84)$$

$$[(k-q_1-\cdots-q_n)^2+\kappa^2]\cdots[(k-q_1)^2+\kappa^2]=[(2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1)+\cdots+(2k_0q_{0,n}-2\mathbf{k}\cdot\mathbf{q}_n)]\cdots\times[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]. \quad (85)$$

If the quantity $S_0'(k)$ is not too sensitive to a small change in k, we may also take

$$S_0'(k-q_1-\cdots-q_n)=S_0'(k). \quad (86)$$

Then, using (84), (85), and (86), we can simplify (83)

[9] For the meaning of an external electron line and other similar terms, see reference 7.

as

$$S_n=(-i)^n V^{-\frac{1}{2}n}(e^2/2c\hbar)^{\frac{1}{2}n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}(q_{0,1}\cdots q_{0,n})^{-\frac{1}{2}}S_0'(k)u(k)$$
$$\times\frac{(2i\mathbf{k}\cdot\mathbf{e}_n)\cdots(2i\mathbf{k}\cdot\mathbf{e}_1)}{[(2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1)+\cdots+(2k_0q_{0,n}-2\mathbf{k}\cdot\mathbf{q}_n)]\cdots[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]},\quad(87)$$

which can be written as

$$S_n=(-i)^n V^{-\frac{1}{2}n}(e^2/2c\hbar)^{\frac{1}{2}n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}{}'\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}(q_{0,1}\cdots q_{0,n})^{-\frac{1}{2}}S_0'(k)u(k)$$
$$\times(2i\mathbf{k}\cdot\mathbf{e}_1)\cdots(2i\mathbf{k}\cdot\mathbf{e}_n)\left\{\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}^{P}\frac{1}{[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]\cdots[(2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1)+\cdots+(2k_0q_{0,n}-2\mathbf{k}\cdot\mathbf{q}_n)]}\right\},\quad(88)$$

where the meaning of $\sum'$ and $\sum^P$ has been explained in Sec. 4. Further, using the identity (61), we can express (88) as

$$S_n=(-i)^n V^{-\frac{1}{2}n}(e^2/2c\hbar)^{\frac{1}{2}n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}{}'\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}(q_{0,1}\cdots q_{0,n})^{-\frac{1}{2}}S_0'(k)u(k)\frac{(2i\mathbf{k}\cdot\mathbf{e}_1)\cdots(2i\mathbf{k}\cdot\mathbf{e}_n)}{[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]\cdots[2k_0q_{0,n}-2\mathbf{k}\cdot\mathbf{q}_n]}\quad(89)$$

or

$$S_n=\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}{}'V^{-\frac{1}{2}n}J_nS_0'(k)u(k),\quad(90)$$

where

$$J_n=(-i)^n(e^2/2c\hbar)^{\frac{1}{2}n}(q_{0,1}\cdots q_{0,n})^{-\frac{1}{2}}\frac{(2i\mathbf{k}\cdot\mathbf{e}_1)\cdots(2i\mathbf{k}\cdot\mathbf{e}_n)}{[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]\cdots[2k_0q_{0,n}-2\mathbf{k}\cdot\mathbf{q}_n]}.\quad(91)$$

The relation (90) shows that σ_n is related to σ_0 as

$$\sigma_n=V^{-n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}{}'\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}J_n^*J_n\sigma_0,\quad(92)$$

so that, substituting (91) in (92), we get

$$\sigma_n=\sigma_0V^{-n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}{}'\sum_{\mathbf{e}_1,\cdots,\mathbf{e}_n}(e^2/2c\hbar)^n\frac{4^n}{q_{0,1}\cdots q_{0,n}}$$
$$\times\frac{(\mathbf{k}\cdot\mathbf{e}_1)^2\cdots(\mathbf{k}\cdot\mathbf{e}_n)^2}{[2k_0q_{0,1}-2\mathbf{k}\cdot\mathbf{q}_1]^2\cdots[2k_0q_{0,n}-2\mathbf{k}\cdot\mathbf{q}_n]^2}.\quad(93)$$

But,

$$\sum_{\mathbf{e}_r}(\mathbf{k}\cdot\mathbf{e}_r)^2=k_0^2\sin^2\theta_r,\quad(94)$$

$$2k_0q_{0,r}-2\mathbf{k}\cdot\mathbf{q}_r=2k_0q_{0;r}[(\kappa^2/2k_0^2)+(1-\cos\theta_r)],\quad(95)$$

where θ_r denotes the angle between $\mathbf{q}$ and $\mathbf{k}$, and we have made use of the fact that $k_0\gg\kappa$. Using (94) and (95), we can express (93) as

$$\sigma_n=\sigma_0V^{-n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}{}'(e^2/2c\hbar)^n(q_{0,1}\cdots q_{0,n})^{-3}$$
$$\times\frac{\sin^2\theta_1}{[(\kappa^2/2k_0^2)+(1-\cos\theta_1)]^2}\cdots$$
$$\times\frac{\sin^2\theta_n}{[(\kappa^2/2k_0^2)+(1-\cos\theta_n)]^2}.\quad(96)$$

Then, putting

$$V^{-n}\sum_{\mathbf{q}_1,\cdots,\mathbf{q}_n}{}'=(1/n!)(2\pi)^{-3n}\int d\mathbf{q}_1\cdots\int d\mathbf{q}_n,\quad(97)$$

$$\int d\mathbf{q}_i=2\pi\int dq_{0,i}q_{0,i}^2\int_0^\pi\sin\theta_i d\theta_i,\quad(98)$$

we obtain

$$\sigma_n=\sigma_0\alpha^n\frac{1}{n!}\frac{1}{(2\pi)^n}\int dq_{0,1}\cdots\int dq_{0,n}\frac{1}{q_{0,1}\cdots q_{0,n}}$$
$$\times\int_0^\pi d\theta_1\cdots\int_0^\pi d\theta_n\frac{\sin^3\theta_1}{[(\kappa^2/2k_0^2)+(1-\cos\theta_1)]^2}\cdots$$
$$\times\frac{\sin^3\theta_n}{[(\kappa^2/2k_0^2)+(1-\cos\theta_n)]^2}.\quad(99)$$

Carrying out the integrations in (99), and ignoring some lower-order terms, we get

$$\sigma_n=\sigma_0\frac{\alpha^n}{n!}\left(\frac{2}{\pi}\right)^n\left(\log\frac{2k_0}{\kappa}\right)^n\left(\log\frac{q_{0,\max}}{\lambda}\right)^n,\quad(100)$$

where $q_{0,\max}$ and λ are the maximum and the minium values of the q_0's. Using (44) and (78), we can also express (100) as

$$\sigma_n=\sigma_0\frac{\alpha^n}{n!}\left(\frac{2}{\pi}\right)^n\left(\log\frac{2E}{\mu}\right)^n\left(\log\frac{\epsilon_{\max}}{\epsilon}\right)^n,\quad(101)$$

where E is the energy of the electron represented by the external electron line.

It should be observed that the above approximate result has been obtained with the assumption that the quantity $S_0'(k)$ of equation (81) is not too sensitive to a small change in k. Therefore, the above general result represents only a rough approximation, and a more accurate determination of σ_n can be made only when we know the quantity $S_0'(k)$.

6. CONCLUSION

We shall now discuss the significance of the results obtained in the preceding sections. It should first be noted that the cross sections (47), (80), and (101) all diverge as the lower limit ϵ to the energy of the emitted photons tends to zero. However, it is known that this "infrared divergence" is harmless, and it is compensated by corresponding divergencies arising in the radiative corrections to cross sections for the production of a lesser number of photons.[10] Therefore, for all practical purposes, ϵ represents the lower limit to the energy of the photons, which can be observed in a given experiment.

As a particular case, let us consider the annihilation of a pair of electron and positron in the center-of-mass system, such that 1 photon is emitted along the direction of motion of the electron while $n-1$ photons are emitted along the direction of motion of the positron. Then, according to (77), the cross section for this process will be

$$\sigma_n=\alpha^n\frac{1}{(n-2)!}\frac{2^{n-2}}{\pi^{n-3}}\frac{c^2\hbar^2}{E^2}\left(\log\frac{\delta E}{\mu}\right)^{n-1}\left(\log\frac{\epsilon_{\max}}{\epsilon}\right)^{n-2}. \quad (102)$$

Since we are interested only in the order of magnitude of σ_n, we can put $\epsilon_{\max}=E$ in (102). Moreover, it follows from (101) that if we include the contributions from all angles instead of a small cone, the factor $[\log(\delta E/\mu)]^{n-1}$ in (102) should be replaced by $[\log(2E/\mu)]^{n-1}$. Therefore, in the present case we may take the cross section as

$$\sigma_n=\alpha^n\frac{1}{(n-2)!}\frac{2^{n-2}}{\pi^{n-3}}\frac{c^2\hbar^2}{E^2}\left(\log\frac{2E}{\mu}\right)^{n-1}\left(\log\frac{E}{\epsilon}\right)^{n-2}. \quad (103)$$

We now pass over from the center-of-mass system to the laboratory system, in which the electron is at rest. We can then put

$$E'=2E^2/\mu,\quad \epsilon'=2\epsilon E/\mu, \quad (104)$$

where E' is the energy of the positron in the laboratory system, and ϵ' is the lower limit to the energy of the photons in this system. Using (104), we can express (103) as

$$\sigma_n=\alpha^n\frac{1}{(n-2)!}\frac{1}{\pi^{n-3}}\frac{c^2\hbar^2}{\mu E'}\left(\log\frac{2E'}{\mu}\right)^{n-1}\left(\log\frac{E'}{\epsilon'}\right)^{n-2}. \quad (105)$$

The lower limit to the energy of photons, which can be observed through pair production in photographic emulsions, will depend on the nature of the emulsion. But, for our general purpose we may take $\epsilon'=50\mu\approx 25$ Mev, which gives us

$$\sigma_n=\alpha^n\frac{1}{(n-2)!}\frac{1}{\pi^{n-3}}\frac{c^2\hbar^2}{\mu E'}\left(\log\frac{2E'}{\mu}\right)^{n-1}\left(\log\frac{E'}{50\mu}\right)^{n-2}. \quad (106)$$

Now, the cross section for the production of two photons in the electron-positron annihilation at high energies is known to be[1]

$$\sigma_2=\pi\alpha^2(c^2\hbar^2/\mu E')\log(2E'/\mu). \quad (107)$$

Therefore, (106) can also be written as

$$\sigma_n=\sigma_2\frac{1}{(n-2)!}\left(\frac{\alpha}{\pi}\log\frac{2E'}{\mu}\log\frac{E'}{50\mu}\right)^{n-2}. \quad (108)$$

When the energy E' of the incident positron is of the order of 10^{14} ev to 10^{16} ev, we find

$$(\alpha/\pi)\log(2E'/\mu)\log(E'/50\mu)\approx 1, \quad (109)$$

so that in the above energy region we have

$$\sigma_n\approx\sigma_2/(n-2)!. \quad (110)$$

According to (110), σ_n is comparable to σ_2 for small values of n, and therefore multiple production of up to four or five photons in electron-positron annihilation can easily take place at very high energies in cosmic rays. However, σ_n for the production of 15 to 20 photons is quite small as compared to σ_2, and therefore we are not likely to observe a shower of 15 to 20 photons due to multiple photon production in electron-positron annihilation even at reasonably high energies.

The general result (101) further seems to show that a similar situation exists with regard to the multiple production of photons in other processes in quantum electrodynamics.

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[10] A general proof of this result has been given recently by J. M. Jauch and F. Rohrlich, Helv. Phys. Acta. 27, 613 (1954).