Fourth-Order Radiative Corrections to Atomic Energy Levels. II*

ROBERT L. MILLS† AND NORMAN M. KROLL
Columbia University, New York, New York
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A rigorous method is described for treating the problem of the hydrogen-like atom in quantum electrodynamics by the separation of integrals into parts which can be evaluated with relativistic and nonrelativistic approximations. The method is based on that developed by Baranger, Bethe, and Feynman for the second-order problem, and is applied here to the problem of the fourth-order radiative energy level displacement. It is shown that the result inferred by Weneser, Bersohn, and Kroll from the study of fourth-order radiative corrections to elastic scattering by a given external potential is correct to order $\alpha^2(\alpha z)^4mc^2$ for a hydrogenic atom. It is also shown that the αZ corrections to this result can be obtained using this method, just as was done in the second-order problem by Baranger, Bethe, and Feynman.

INTRODUCTION

THE possible significance of fourth-order radiative corrections in the interpretation of the experimental results of Lamb,[1] etc., on the $2S_{\frac{1}{2}}-2P_{\frac{1}{2}}$ separation in hydrogen has been discussed by Weneser, Bersohn and Kroll.[2] Their procedure for determining the effect consisted of evaluating the radiative corrections to the elastic scattering (in Born approximation) by a given external potential, inferring a modified potential from the result, and using the modified potential to compute the energy level displacement. While this procedure is certainly a plausible one, it would appear to be more satisfactory to deduce their result from a systematic treatment of an exact expression for the fourth-order self-energy. Such a procedure has the additional advantage that one then sees clearly how to obtain the αZ correction to their result.

Low[3] has discussed the problem of defining this energy level displacement in terms of the experimental methods which might be used in its determination, with particular reference to the line shape problem. He finds that up to, but not including, terms of order $\alpha^2(\alpha Z)^6mc^2$ for a hydrogen atom, the level shift so determined corresponds precisely to that which would be obtained by the application of standard steady-state perturbation theory or by the application of perturbation theory to the modified Dirac equation proposed by Schwinger.[4]

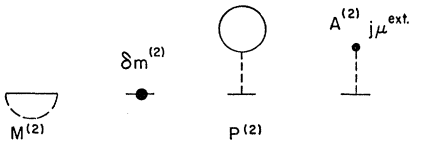

FIG. 1. Second-order contributions to the self-energy, with the appropriate contributions to mass and charge renormalization explicitly displayed. The solid lines represent the propagation function for the electron in the external field, and the dashed lines, the photon propagation function.

Thus, up to the fourth order, one finds

$$\Delta E_a=\Delta E_a{}^{(2)}+\Delta E_a{}^{(4)}+\cdots, \tag{1}$$

where

$$\Delta E_a{}^{(2)}=\mathrm{Re}\,\{iH_{aa}{}^{(2)}(E_a)\},$$

$$\Delta E_a{}^{(4)}=\mathrm{Re}\left\{iH_{aa}{}^{(4)}(E_a)-H_{aa}{}^{(2)}(E_a)\times\left[\frac{\partial}{\partial E}H_{aa}{}^{(2)}(E)\right]_{E=E_a}+\sum_{n\neq a}\frac{H_{an}{}^{(2)}(E_a)H_{na}{}^{(2)}(E_a)}{E_n-E_a}\right\};$$

$$H_{nm}{}^{(p)}(E)=-(2\pi)^3i\hbar c\int\bar{\varphi}_n(\mathbf{p}_2)\Sigma^{(p)}(\mathbf{p}_2,\mathbf{p}_1;E)\times\varphi_m(\mathbf{p}_1)(d\mathbf{p})^2. \tag{2}$$

In the aforementioned expressions, $\Delta E_a{}^{(2)}$ and $\Delta E_a{}^{(4)}$ refer, respectively, to the second- and fourth-order parts of the level shift for the ath state, E_n refers to the unshifted level of the nth state, while $\varphi_n(\mathbf{p})$ is the corresponding steady state solution of the Dirac equation in the given external field.[5] The functions $\Sigma^{(p)}(\mathbf{p}_2,\mathbf{p}_1;E)$, which we shall also write simply as $\Sigma^{(p)}(p_2,p_1)$ with $p_{20}=p_{10}=p_0=E/(\hbar c)$, are for convenience taken to include both what one might call mass operator and vacuum polarization contributions to the self-energy. Accordingly, we write,

$$\Sigma^{(2)}(p_2,p_1)=M^{(2)}+P^{(2)},\quad[p_{20}=p_{10}=p_0],$$

with

$$M^{(2)}=-\tfrac{1}{2}i(2\pi)^5\alpha\int\gamma_\mu S_F{}^e(p_2-k,\,p_1-k)\gamma_\mu D_F(k)d_4k-\delta\kappa^{(2)}\delta_3(\mathbf{p}_2-\mathbf{p}_1), \tag{3}$$

$$P^{(2)}=\tfrac{1}{2}i(2\pi)^5\alpha D_F(p_2-p_1)\{\mathrm{Tr}[\gamma_\mu S_F{}^e(p_2-k,\,p_1-k)]d_4k-[2/(iec)]A^{(2)}j_\mu{}^{\mathrm{ext}}(p_2-p_1)\}.$$

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† Now at Brookhaven National Laboratory, Upton, New York.

[1] Triebwasser, Dayhoff, and Lamb, Phys. Rev. **89**, 98 (1953).

[2] Weneser, Bersohn, and Kroll, Phys. Rev. **91**, 1257 (1953), hereafter referred to as WBK.

[3] Francis Low, Phys. Rev. **88**, 53 (1952).

[4] Julian Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).

[5] The "momenta" p, q, and k in this paper, like the "mass" $\kappa=m_ec/\hbar$, have the dimensions of inverse length. The momentum space wave functions $\varphi_n(\mathbf{p})$ have the normalization $\int|\varphi_n(\mathbf{p})|^2d\mathbf{p}=(2\pi)^{-3}$. If a and b are two four-vectors, we take the convention $a\cdot b=a_\mu b_\mu=a_1b_1+a_2b_2+a_3b_3+a_4b_4=\mathbf{a}\cdot\mathbf{b}-a_0b_0$. Heaviside units are used.

Furthermore,

$$D_F(k) = -2i(2\pi)^{-4}k^{-2} \tag{4}$$

is the Feynman Green's function for the electromagnetic field, while the Feynman Green's function for the Dirac equation, with the specified external potential is

$$S_F{}^e(p_2,p_1) = \frac{i\hbar c}{\pi}\sum_n \frac{\varphi_n(\mathbf{p}_2)\bar{\varphi}_n(\mathbf{p}_1)}{(1-i\epsilon)E_n - \hbar c p_0}. \tag{5}$$

The appropriate mass and charge renormalizations are subtracted out explicitly. The corresponding Feynman diagrams appear in Fig. 1.

Similarly we write

$$\Sigma_4(p_2,p_1) = M_1{}^{(4)} + M_2{}^{(4)} + (MP)_1 + (MP)_2 + P_1{}^{(4)} + P_2{}^{(4)},$$

with

$$\begin{aligned}
M_1{}^{(4)} &= -2i(2\pi)^{-4}[\tfrac{1}{4}(2\pi)^9\alpha]^2 \\
&\quad\times\int \gamma_\mu S_F{}^e(p_2-k_2,\, p_4-k_2)\gamma_\nu \\
&\quad\times S_F{}^e(p_4-k_1-k_2,\, p_3-k_1-k_2)\gamma_\mu \\
&\quad\times S_F{}^e(p_3-k_1,\, p_1-k_1)\gamma_\nu D_F(k_1) \\
&\quad\times D_F(k_2)(d_4k)^2 d\mathbf{p}_3 d\mathbf{p}_4, \\
M_2{}^{(4)} &= \tfrac{1}{4}(2\pi)^9\alpha\int \gamma_\mu S_F{}^e(p_2-k,\, p_4-k) \\
&\quad\times M^{(2)}(p_4-k,\, p_3-k)S_F{}^e(p_3-k,\, p_1-k)\gamma_\mu \\
&\quad\times D_F(k)d_4k d\mathbf{p}_3 d\mathbf{p}_4, \\
(MP)_1 &= -\tfrac{1}{2}i(2\pi)^5\alpha\int \gamma_\mu S_F{}^e(p_2-k_2,\, p_1-k_1)\gamma_\nu \\
&\quad\times[D_F{}^{e(2)}(k_2,k_1)]_{\mu\nu}(d_4k)^2, \\
[D_F{}^{e(2)}(k_2,k_1)]_{\mu\nu} &= -\tfrac{1}{4}(2\pi)^9\alpha D_F(k_2)D_F(k_1) \\
&\quad\times\int \mathrm{Tr}\,[\gamma_\mu S_F{}^e(p_4,p_3)\gamma_\nu S_F{}^e(p_3-k_1,\, p_4-k_2)](d_4p)^2 \\
&\quad - C^{(2)}D_F(k_1)\delta_4(k_2-k_1)\delta_{\mu\nu}.
\end{aligned} \tag{6}$$

We will have no occasion to discuss the remaining terms explicitly and so we content ourselves with illustrating them, as well as the above defined terms, by means of the Feynman diagrams appearing in Fig. 2. These expressions contain fourth-order mass and charge renormalization terms which will be recognized and ignored

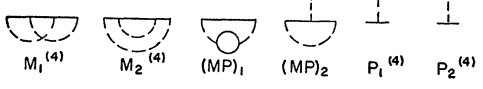

FIG. 2. Fourth-order contributions to the self-energy. Mass and charge renormalization contributions are not explicitly displayed.

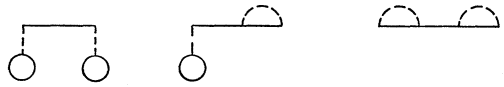

FIG. 3. Improper self-energy diagrams, corresponding to the second and third terms in $\Delta E^{(4)}$ [Eq. (2)].

at a later stage. The improper self-energy diagrams illustrated in Fig. 3 correspond crudely to the second and third terms in the expression for $\Delta E^{(4)}$.

Our task now consists of the separation of $\Delta E^{(4)}$ into four kinds of terms: (1) A set which is recognizable as mass and charge renormalization. (2) A set which corresponds precisely to the set of terms evaluated by WBK, and which gives rise to an energy level displacement of the S state of order $\alpha^2(\alpha Z)^4$. (3) A set which is of the same order of magnitude as the WBK terms, which must be separately discussed, and will ultimately be shown to vanish. (4) The remainder, consisting of terms individually at least of order αZ smaller than the WBK terms.

In the following sections our attention will be devoted chiefly to effecting the previously described separation and to the demonstration that terms in class (3) above do in fact add up to zero. The number of terms which must be put in class (4) is very large and we shall in general assign terms to this class without proof. There are, however, important guiding principles which should be kept in mind. We note that the Fourier transform of the Coulomb potential is given by

$$\frac{ie}{\hbar c}A_\rho{}^e(\mathbf{q}) = \frac{\alpha Z}{2\pi^2\mathbf{q}^2}\delta_{\rho 4}.$$

It generally appears multiplied by a three-dimensional volume element in $\mathbf{q}$-space. Thus the appearance of an extra factor $eA_\mu{}^e$ in an expression will as a rule reduce the value by a factor αZ or $(\alpha Z)^2$, accordingly as the significant values of $\mathbf{q}$ are of order unity or αZ. Furthermore, the momenta $\mathbf{p}$ associated with a bound state may usually be taken as being of order αZ. The binding energy, $\kappa - p_0$, is of order $(\alpha Z)^2$. One of these factors will always be involved when approximations "valid to order $\alpha^2(\alpha Z)^4$" are made or when terms are consigned to class (4). The problem at hand would be trivial if there were not important exceptions to the criteria given above, and it is these exceptions which will require our attention.

First of all, it can be readily shown that the terms $(MP)_2$, $P_2{}^{(4)}$, and $\sum_n H_{an}H_{na}(E_n-E_a)^{-1}$ belong in class (4), that is, they do not contribute to the order in which we are interested, nor did any scattering corrections which one might associate with these terms occur in the WBK calculation. The contribution of $P_1{}^{(4)}$ has been computed by Baranger, Dyson, and Salpeter,[6] and will not be discussed here. We therefore confine our attention to the terms $M_1{}^{(4)}$, $M_2{}^{(4)}$, $(MP)_1$,

[6] Baranger, Dyson, and Salpeter, Phys. Rev. 88, 680 (1952).

and $H^{(2)}\partial H^{(2)}/\partial E$. The terms of the WBK calculation (which will hereafter be referred to as the scattering approximation) to be associated with each of these terms can be readily guessed and will appear explicitly later. These are illustrated in Fig. 4. It is worth noting that in the case of $(MP)_1$, the scattering approximation yields a result which is finite in the infrared. This is not the case for the remaining terms; one gets a finite result only when the contributions from the three terms are added. Associated with this situation is the fact that the scattering approximation is valid for $(MP)_1$, considered separately. This fact makes the treatment of $(MP)_1$ essentially trivial, and we shall therefore discuss it first. On the other hand the scattering approximation is not valid for the other terms considered separately; one finds, indeed, $\alpha^2(\alpha Z)^4$ terms with infrared divergent coefficients. The fact that the infrared difficulties disappear in the scattering approximation after adding the terms together implies that the same will happen to the corrections, but does not obviously imply that the corrections add up to zero. In order to demonstrate that they do we have found it necessary to compute these corrections explicitly.

FIG. 4. Contributions to the scattering approximation calculated by WBK. The crosses represent interactions with the external Coulomb field.

VALIDITY OF THE SCATTERING APPROXIMATION FOR THE TERM $(MP)_1$

The most straightforward way of separating the scattering approximation from the expressions we have written down is simply to replace $S_F^e(p_2,p_1)$ by

$$\begin{aligned}S_F^e(p_2,p_1)&=S_F(p_2)\delta_3(\mathbf{p}_2-\mathbf{p}_1)-\tfrac{1}{2}i(2\pi)^4[ie/(\hbar c)]S_F(p_2)\\&\times\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)S_F(p_1)-\tfrac{1}{4}(2\pi)^8[ie/(\hbar c)]^2S_F(p_2)\\&\times\Big[\int\gamma\cdot A^e(\mathbf{q}_2)S_F^e(p_2-q_2,\,p_1+q_1)\\&\qquad\times\gamma\cdot A^e(\mathbf{q}_1)(d\mathbf{q})^2\Big]S_F(p_1),\quad(7)\\&[q_{10}=q_{20}=0],\end{aligned}$$

obtained by iterating the integral equations satisfied by $S_F^e(p_2,p_1)$:

$$\begin{aligned}S_F^e(p_2,p_1)&=S_F(p_2)\delta_3(\mathbf{p}_2-\mathbf{p}_1)-\tfrac{1}{2}i(2\pi)^4[ie/(\hbar c)]\\&\times S_F(p_2)\int\gamma\cdot A^e(\mathbf{q})S_F^e(p_2-q,\,p_1)d\mathbf{q}\\&=S_F(p_1)\delta_3(\mathbf{p}_2-\mathbf{p}_1)-\tfrac{1}{2}i(2\pi)^4[ie/(\hbar c)]\\&\times\Big[\int S_F^e(p_2,\,p_1+q)\gamma\cdot A^e(\mathbf{q})d\mathbf{q}\Big]S_F(p_1),\end{aligned}\quad(8)$$

where

$$S_F(p)=-\frac{2i}{(2\pi)^4}\frac{i\gamma\cdot p-\kappa}{p^2+\kappa^2}.\quad(9)$$

Rather than consider the consequences of such a substitution in full detail we shall simply assert that the S_F^e functions appearing in the closed loop can be replaced by free-particle propagators. The expression for $(MP)_1$ can then be written[7]

$$(MP)_1=-\tfrac{1}{2}i(2\pi)^4\alpha^2\int\gamma_\mu S_F^e(p_2-k,\,p_1-k)\times\gamma_\mu\bar{D}_F^{(2)}(k)d_4k,$$

with

$$\bar{D}_F^{(2)}(k)=-\frac{i}{(2\pi)^4}\int_0^1 dv\frac{v^2(1-\frac{1}{3}v^2)}{\kappa^2+\frac{1}{4}k^2(1-v^2)}.$$

Furthermore,

$$\begin{aligned}(MP)_1=-\tfrac{1}{2}i(2\pi)^4\alpha^2\Big\{&\gamma_\mu S_F(p_2-k)\gamma_\mu\bar{D}_F^{(2)}(k)d_4k\\&\times\delta_3(\mathbf{p}_2-\mathbf{p}_1)-\tfrac{1}{2}i(2\pi)^4[ie/(\hbar c)]\\&\times\int\gamma_\mu S_F(p_2-k)\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\\&\times S_F(p_1-k)\gamma_\mu\bar{D}_F^{(2)}(k)d_4k\\&-\tfrac{1}{4}(2\pi)^8[ie/(\hbar c)]^2\int\gamma_\mu S_F(p_2-k)\gamma\cdot A^e(\mathbf{q}_2)\\&\times S_F^e(p_2-q_2-k,\,p_1+q_1-k)\gamma\cdot A^e(\mathbf{q}_1)\\&\times S_F(p_1-k)\gamma_\mu\bar{D}_F^{(2)}(k)d_4k(d\mathbf{q})^2\Big\}.\end{aligned}\quad(10)$$

It is clear that $(MP)_1$ bears a close resemblance to $M^{(2)}$, the only difference being the replacement of the D_F function by the $\bar{D}_F^{(2)}$ function. This difference is in fact very significant, as $\bar{D}_F^{(2)}(0)$ is finite. As a consequence, the infrared difficulties associated with the evaluation of $M^{(2)}$ do not manifest themselves here. To order $\alpha^2(\alpha Z)^4$, one can, in fact, simply replace the S_F^e appearing in the third term by the free-particle propgation function. Using familiar manipulations (see, for example, Kroll and Pollock[8]) one can extract from

$$\langle(MP)_1\rangle=(2\pi)^3\int\varphi_a(\mathbf{p}_1)(MP)_1\varphi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2,$$

to order $\alpha^2(\alpha Z)^4$, four kinds of terms. The first term of

[7] R. Karplus and N. M. Kroll, Phys. Rev. **77**, 536 (1950), Eq. (13).

[8] N. M. Kroll and F. Pollock, Phys. Rev. **86**, 876 (1952).

(10) yields expressions of the form

$$(1)\quad \int \bar{\varphi}(\mathbf{p})\varphi(\mathbf{p})d\mathbf{p},$$

$$(2)\quad [ie/(\hbar c)]\int \bar{\varphi}(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\varphi(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2,$$

$$(3)\quad [ie/(\hbar c)]^2\int \bar{\varphi}(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_3)\times\gamma\cdot A^e(\mathbf{p}_3-\mathbf{p}_1)\varphi(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2d\mathbf{p}_3.$$

The second term of (10) yields terms like (2) and (3), plus a term proportional to:

$$(4)\quad [ie/(\hbar c)]\int \bar{\varphi}(\mathbf{p}_2)(\mathbf{p}_2-\mathbf{p}_1)^2\times\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\varphi(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2.$$

This last term, when one notes the form of the second term in $(MP)_1$ [Eq. (10)], is easily seen to be identical with the contribution of diagram g, Fig. 4, to the scattering approximation. The third term of (10) yields a term like (3). Terms (1) and (2) are obviously mass and charge terms, to be dropped by renormalization. The term (3) corresponds simply to the expectation value of the square of the potential. We now make use of the fact that $\langle(MP)_1\rangle$, which we may write as

$$-\tfrac{1}{2}i(2\pi)^8\alpha\int \bar{\varphi}(\mathbf{p}_2)\gamma_\mu S_F{}^e(p_2-k_2,\, p_1-k_1)\times\gamma_\nu\varphi(\mathbf{p}_1)[D_F{}^{e(2)}(k_2,k_1)]_{\mu\nu}d_4k_1d_4k_2,$$

is a guage invariant quantity, so that the sums of the coefficients of the charge renormalization $(\gamma\cdot A^e)$ and potential squared $[(\gamma\cdot A^e)^2]$ terms must vanish. Consequently, only the $(\mathbf{p}_2-\mathbf{p}_1)^2\gamma\cdot A^e$ term remains and the scattering approximation is seen to be valid for $(MP)_1$.

In view of the fact that the $(\gamma\cdot A^e)^2$ term is not removable by charge renormalization and is of the same order of magnitude as the main level shift term $[(\mathbf{p}_2-\mathbf{p}_1)^2\gamma\cdot A^e]$, previous writers have, in dealing with the second-order level shift, always computed the coefficient to verify explicitly that it vanishes. The full evaluation of the $(\gamma\cdot A^e)^2$ term for the fourth-order level shift would, however, be a task of magnitude comparable to that of evaluating the $(\mathbf{p}_2-\mathbf{p}_1)^2\gamma\cdot A^e$ term. We shall therefore depend completely upon the kind of gauge invariance argument given above. Thus the identification of $\alpha^2(\alpha Z)^4$ contributions which can be attributed to $(\gamma\cdot A^e)^2$ terms will be an essential part of our task, but a term so identified will be subsequently ignored.

CONTRIBUTION OF $M_1{}^{(4)}$

It would be a simple matter to separate the scattering approximation out of $M_1{}^{(4)}$ using a technique identical to that employed in the discussion of $(MP)_1$. After replacing $S_F{}^e(p_2,p_1)$ with the form appearing in Eq. (7), one then finds that the terms in which $\gamma\cdot A^e$ appears once explicitly contain the scattering approximation. On the other hand, examination of the multiple-potential terms, i.e., terms containing $\gamma\cdot A^e$ more than once, shows that these yield contributions at least of order $\alpha^2(\alpha Z)^4$ and possibly larger, not identifiable as $(\gamma\cdot A^e)^2$ terms. It therefore becomes necessary to evaluate the contributions arising from these terms. As pointed out by Baranger, Bethe, and Feynman,[9] the separation described above is not a convenient one in those cases in which the evaluation of the multiple-potential contribution is essential. We therefore extend the BBF procedure to the fourth-order problem.

We wish to separate the scattering approximation in a way which leaves the remainder tractable. We proceed as follows:

BBF gives the general identity, valid for any operator $\mathbf{O}$, and any two vectors p_1 and p_2:

$$\mathbf{O}=B(\mathbf{O};\, p_2,p_1)(i\gamma\cdot p_1+\kappa)-(i\gamma\cdot p_2+\kappa)B(\mathbf{O};\, p_2,p_1), \tag{11}$$

where

$$B(\mathbf{O};\, p_2,p_1)=\frac{(i\gamma\cdot p_2)\mathbf{O}+\mathbf{O}(i\gamma\cdot \mathbf{p}_1)}{p_2{}^2-p_1{}^2}. \tag{12}$$

We shall use a less symmetric, but more convenient, form as follows:

$$\begin{aligned}&\mathbf{O}=F(\mathbf{O};\, p,k)(i\gamma\cdot p+\kappa)-(i\gamma\cdot(p-k)+\kappa)F(\mathbf{O};\, p,k)\\&\quad=(i\gamma\cdot p+\kappa)F'(\mathbf{O};\, p,k)-F'(\mathbf{O};\, p,k)(i\gamma\cdot(p-k)+\kappa);\\&F(\mathbf{O};\, p,k)=B(\mathbf{O};\, p-k,\, p)=\frac{(i\gamma\cdot(p-k))\mathbf{O}+\mathbf{O}(i\gamma\cdot\mathbf{p})}{k^2-2p\cdot k},\\&F'(\mathbf{O};\, p,k)=-B(\mathbf{O};\, p,\, p-k)\\&\qquad=\frac{(i\gamma\cdot p)\mathbf{O}+\mathbf{O}(i\gamma\cdot(p-k))}{k^2-2p\cdot k}.\end{aligned} \tag{13}$$

When $\mathbf{O}=\gamma_\mu$, we get the forms actually used by BBF in the second-order calculation. Now φ_a and $\bar{\varphi}_a$ satisfy the integral equations

$$\begin{aligned}&(i\gamma\cdot p+\kappa)\varphi_a(\mathbf{p})=[ie/(\hbar c)]\int\gamma\cdot A^e(\mathbf{q})\varphi_a(\mathbf{p}-\mathbf{q})d\mathbf{q}\qquad[p_0=E_a/(\hbar c)],\\&\bar{\varphi}_a(\mathbf{p})(i\gamma\cdot p+\kappa)=[ie/(\hbar c)]\int\bar{\varphi}_a(\mathbf{p}+\mathbf{q})\gamma\cdot A^e(\mathbf{q})d\mathbf{q},\end{aligned} \tag{14}$$

while $S_F{}^e$ satisfies (8), which can be rewritten

$$\begin{aligned}&(i\gamma\cdot p_2+\kappa)S_F{}^e(p_2,p_1)\\&\quad=[ie/(\hbar c)]\int\gamma\cdot A^e(\mathbf{q})S_F{}^2(p_2-q,\, p_1)d\mathbf{q}\\&\qquad+2i(2\pi)^{-4}\delta_3(\mathbf{p}_2-\mathbf{p}_1),\end{aligned}$$

[9] Baranger, Bethe, and Feynman, Phys. Rev. **92**, 482 (1953), hereafter referred to as BBF.

$$S_F^e(p_2,p_1)(i\gamma\cdot p_1+\kappa)$$
$$=[ie/(\hbar c)]\int S_F^e(p_2,\,p_1+q)\gamma\cdot A^e(\mathbf{q})d\mathbf{q}$$
$$+2i(2\pi)^{-4}\delta_3(\mathbf{p}_2-\mathbf{p}_1). \quad (15)$$

When we use the identities (13) in the expressions $S_F^e\mathbf{O}\varphi_a$ and $\bar{\varphi}_a\mathbf{O}S_F^e$, we get

$$\begin{aligned}
&\int S_F^e(p_3-k,\,p_1-k)\mathbf{O}(p_1)\varphi_a(\mathbf{p}_1)d\mathbf{p}_1\\
&=[ie/(\hbar c)]\int S_F^e(p_3-k,\,p_1+q-k)\\
&\quad\times\{F[\mathbf{O}(p_1+q);\,p_1+q,\,k]\gamma\cdot A^e(\mathbf{q})\\
&\quad-\gamma\cdot A^e(\mathbf{q})F[\mathbf{O}(p_1);\,p_1,\,k]\}\varphi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{q}\\
&\quad-2i(2\pi)^{-4}F[\mathbf{O}(p_3);\,p_3,k]\varphi_a(\mathbf{p}_3)\\
&=[ie/(\hbar c)]\int S_F^e(p_3-k,\,p_1+q-k)A_\rho^e(\mathbf{q})\\
&\quad\times\mathfrak{F}_\rho(\mathbf{O};\,p_1,q,k)\varphi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{q}\\
&\qquad-2i(2\pi)^{-4}F[\mathbf{O}(p_3);\,p_3,k]\varphi_a(\mathbf{p}_3);\\
&\int\bar{\varphi}_a(\mathbf{p}_2)\mathbf{O}(p_2)S_F^e(p_2-k,\,p_4-k)d\mathbf{p}_2\\
&=[ie/(\hbar c)]\int\bar{\varphi}_a(\mathbf{p}_2)A_\rho^e(\mathbf{q})\mathfrak{F}_\rho'(\mathbf{O};\,p_2,q,k)\\
&\quad\times S_F^e(p_2-q-k,\,p_4-k)d\mathbf{p}_2d\mathbf{q}\\
&\qquad-2i(2\pi)^{-4}\bar{\varphi}_a(\mathbf{p}_4)F'[\mathbf{O}(p_4);\,p_4,k];
\end{aligned} \quad (16)$$

where

$$\begin{aligned}
\mathfrak{F}_\rho(\mathbf{O};\,p,q,k)&=F[\mathbf{O}(p+q);\,p+q,\,k]\gamma_\rho\\
&\quad-\gamma_\rho F[\mathbf{O}(p);\,p,k];\\
\mathfrak{F}_\rho'(\mathbf{O};\,p,q,k)&=\gamma_\rho F'[\mathbf{O}(p-q);\,p-q,\,k]\\
&\quad-F'[\mathbf{O}(p);\,p,k]\gamma_\rho.
\end{aligned} \quad (17)$$

We shall apply this only to operators $\mathbf{O}$ which do not involve the external potential at all, so that, according to (16), $S_F^e\mathbf{O}\varphi_a$ or $\bar{\varphi}_a\mathbf{O}S_F^e$ is split into a term involving A^e once explicitly, as well as through S_F^e, and a term not involving A^e at all, except through the wave function φ_a. In the latter term, if the total integral still involves S_F^e, we can incorporate the F into a new operator $\mathbf{O}$, and apply the identity (13) again. We thus work in from both ends of $\langle M_1^{(4)}\rangle$ until we are left with terms of two types: terms in which A^e appears twice explicitly, and implicitly through S_F^e; and terms with no S_F^e, so that A^e appears implicitly only through φ_a, and explicitly no more than once. These one-potential and no-potential terms, which we call A_0, will be seen to yield terms identifiable as mass renormalizations, charge renormalization, and $(\gamma\cdot A^e)^2$, all of which can be ignored, and terms which correspond to the scattering approximation contributions associated with the diagrams a, b, c in Fig. 4. The only nonignorable corrections to the scattering approximation are of order $(\alpha Z)^2$ smaller. Apart from a demonstration of this last fact, our real task consists of the evaluation of the contribution from the multiple-potential terms A_1.

We now carry through the process described previously. Thus,

$$\langle M_1^{(4)}\rangle=-(i/\pi)[\tfrac{1}{4}(2\pi)^9\alpha]^2A.$$
$$\begin{aligned}
A&=\int\bar{\varphi}_a\gamma_\mu S_F^e\gamma_\nu S_F^e\gamma_\mu S_F^e\gamma_\nu\varphi_aD_FD_F(d_4k)^2(d\mathbf{p})^4\\
&=[ie/(\hbar c)]\int\bar{\varphi}_aA_\rho^e\mathfrak{F}'_{1\mu,\rho}S_F^e\gamma_\nu S_F^e\gamma_\mu S_F^eA_\sigma^e\\
&\quad\times\mathfrak{F}_{1\nu,\sigma}\varphi_aD_FD_F(d_4k)^2(d\mathbf{p})^4(d\mathbf{q})^2\\
&\quad-2i(2\pi)^{-4}[ie/(\hbar c)]\int\bar{\varphi}_aA_\rho^e\mathfrak{F}'_{1\mu,\rho}S_F^e\gamma_\nu\\
&\quad\times S_F^e\gamma_\mu F_{1\nu}\varphi_aD_FD_F(d_4k)^2(d\mathbf{p})^3d\mathbf{q}_2\\
&\quad-2i(2\pi)^{-4}[ie/(\hbar c)]\int\bar{\varphi}_aF'_{1\mu}\gamma_\nu S_F^e\gamma_\mu S_F^eA_\sigma^e\\
&\quad\times\mathfrak{F}_{1\nu,\sigma}\varphi_aD_FD_F(d_4k)^2(d\mathbf{p})^3d\mathbf{q}_1\\
&\quad-4(2\pi)^{-8}\int\bar{\varphi}_aF'_{1\mu}\gamma_\nu S_F^e\gamma_\mu F_{1\nu}\varphi_a\\
&\qquad\times D_FD_F(d_4k)^2(d\mathbf{p})^2,
\end{aligned} \quad (18)$$

where

$$\begin{aligned}
&F'_{1\mu}=F'(\gamma_\mu;\,p_2,k_2);\qquad F_{1\nu}=F(\gamma_\nu;\,p_1,k_1);\\
&\mathfrak{F}'_{1\mu,\rho}=\mathfrak{F}_\rho'(\gamma_\mu;\,p_2,q_2,k_2);\quad \mathfrak{F}_{1\nu,\sigma}=\mathfrak{F}_\sigma(\gamma_\nu;\,p_1,q_1,k_1).
\end{aligned} \quad (19)$$

We continue the process with the terms containing F or F', taking

$$\begin{aligned}
F'_{2\mu\nu}&=F'[-2i(2\pi)^{-4}F'_{1\mu}\gamma_\nu;\,p_2,\,k_1+k_2];\\
F_{2\mu\nu}&=F[-2i(2\pi)^{-4}\gamma_\mu F_{1\nu};\,p_1,\,k_1+k_2];\\
F'_{3\nu}&=F'[-2i(2\pi)^{-4}F'_{2\mu\nu}\gamma_\mu;\,p_2,\,k_1];\\
F_{3\mu}&=F[-2i(2\pi)^{-4}\gamma_\nu F_{2\mu\nu};\,p_1,\,k_2];
\end{aligned} \quad (20)$$

with the $\mathfrak{F}$'s defined correspondingly. Finally,

$$A=A_1+A_0, \quad (21)$$
$$A_1=a_1+a_2+a_2'+a_3+a_3'+a_4, \quad (22)$$

where

$$\begin{aligned}
a_1&=[ie/(\hbar c)]^2\int\bar{\varphi}_a(\mathbf{p}_2)A_\rho^e(\mathbf{q}_2)\mathfrak{F}'_{1\mu,\rho}\\
&\quad\times S_F^e(p_2-q_2-k_2,\,p_4-k_2)\\
&\quad\times\gamma_\nu S_F^e(p_4-k_1-k_2,\,p_3-k_1-k_2)\\
&\quad\times\gamma_\mu S_F^e(p_3-k_1,\,p_1+q_1-k_1)A_\sigma^e(\mathbf{q}_1)\\
&\quad\times\mathfrak{F}_{1\nu,\sigma}\varphi_a(\mathbf{p}_1)D_F(k_1)D_F(k_2)(d_4k)^2(d\mathbf{p})^4(d\mathbf{q})^2;
\end{aligned}$$

$$a_2=[ie/(\hbar c)]^2\int\bar{\varphi}_a(\mathbf{p}_2)A_\rho{}^e(\mathbf{q}_2)\mathfrak{F}'_{2\mu\nu,\rho}$$
$$\times S_F{}^e(p_2-q_2-k_1-k_2,\ p_3-k_1-k_2)$$
$$\times\gamma_\mu S_F{}^e(p_3-k_1,\ p_1+q_1-k_1)A_\sigma{}^e(\mathbf{q}_1)\mathfrak{F}_{1\nu,\sigma}\varphi_a(\mathbf{p}_1)$$
$$\times D_F(k_1)D_F(k_2)(d_4k)^2(d\mathbf{p})^3(d\mathbf{q})^2;$$

$$a_2'=[ie/(\hbar c)]^2\int\bar{\varphi}_a(\mathbf{p}_2)A_\rho{}^e(\mathbf{q}_2)\mathfrak{F}'_{1\mu,\rho}$$
$$\times S_F{}^e(p_2-q_2-k_2,\ p_4-k_2)\gamma_\nu S_F{}^e(p_4-k_1-k_2,$$
$$\times p_1+q_1-k_1-k_2)A_\sigma{}^e(\mathbf{q}_1)\mathfrak{F}_{2\mu\nu,\sigma}\varphi_a(\mathbf{p}_1) \quad (23)$$
$$\times D_F(k_1)D_F(k_2)(d_4k)^2(d\mathbf{p})^3(d\mathbf{q})^2;$$

$$a_3=[ie/(\hbar c)]^2\int\bar{\varphi}_a(\mathbf{p}_2)A_\rho{}^e(\mathbf{q}_2)\mathfrak{F}'_{3\nu,\rho}$$
$$\times S_F{}^e(p_2-q_2-k_1,\ p_1+q_1-k_1)A_\sigma{}^e(\mathbf{q}_1)\mathfrak{F}_{1\nu,\sigma}$$
$$\times\varphi_a(\mathbf{p}_1)D_F(k_1)D_F(k_2)(d_4k)^2(d\mathbf{p})^2(d\mathbf{q})^2;$$

$$a_3'=[ie/(\hbar c)]^2\int\bar{\varphi}_a(\mathbf{p}_2)A_\rho{}^e(\mathbf{q}_2)\mathfrak{F}'_{1\mu,\rho}$$
$$\times S_F{}^e(p_2-q_2-k_2,\ p_1+q_1-k_2)A_\sigma{}^e(\mathbf{q}_1)\mathfrak{F}_{3\mu,\sigma}$$
$$\times\varphi_a(\mathbf{p}_1)D_F(k_1)D_F(k_2)(d_4k)^2(d\mathbf{p})^2(d\mathbf{q})^2;$$

$$a_4=[ie/(\hbar c)]^2\int\bar{\varphi}_a(\mathbf{p}_2)A_\rho{}^e(\mathbf{q}_2)\mathfrak{F}'_{2\mu\nu,\rho}$$
$$\times S_F{}^e(p_2-q_2-k_1-k_2,\ p_1+q_1-k_1-k_2)$$
$$\times A_\sigma{}^e(\mathbf{q}_1)\mathfrak{F}_{2\mu\nu,\sigma}\varphi_a(\mathbf{p}_1)D_F(k_1)D_F(k_2)(d_4k)^2(d\mathbf{p})^2(d\mathbf{q})^2.$$

(Of course, $a_2=a_2'$ and $a_3=a_3'$, but it is convenient to keep them separate in this way.) Furthermore,

$$A_0=-2i(2\pi)^{-4}[ie/(\hbar c)]\int\bar{\varphi}_a(\mathbf{p}_2)A_\rho{}^e(\mathbf{q})$$
$$\times[F'_{3\nu}\mathfrak{F}_{1\nu,\rho}+\mathfrak{F}'_{1\mu,\rho}F_{3\mu}+\mathfrak{F}'_{2\mu\nu,\rho}F_{2\mu\nu}]\varphi_a(\mathbf{p}_1)$$
$$\times D_F(k_1)D_F(k_2)(d_4k)^2(d\mathbf{p})^2d\mathbf{q}$$
$$-4(2\pi)^{-8}\int\bar{\varphi}_a(\mathbf{p})F'_{2\mu\nu}\gamma_\mu F_{1\nu}\varphi_a(\mathbf{p})$$
$$\times D_F(k_1)D_F(k_2)(d_4k)^2d\mathbf{p}. \quad (24)$$

In order actually to carry out the indicated integrations over the momentum space of the photons, it is necessary to "regulate" the photon propagation function both in the ultraviolet and the infrared. This may be easily accomplished by replacing $D_F(k)=-[2i/(2\pi)^4](1/k^2)$ by

$$D_{FR}(k)=-\frac{2i}{(2\pi)^4}\frac{\Lambda^2}{(k^2+\Lambda^2)(k^2+\lambda^2)}. \quad (25)$$

When the calculation is completed one lets $\Lambda\to\infty$ and $\lambda\to 0$. The need for ultraviolet regulation arises from the fact that we do not remove the charge renormalizations associated with the second-order vertex operator. As a consequence, even the physically important parts of $M_1{}^{(4)}$ are divergent in the ultraviolet. Such a procedure is possible because $M^{(2)}$ and $H^{(2)}\partial H^{(2)}/\partial E$ contain similar renormalizations associated both with the second-order vertex operator and with the second-order electron propagation function. It is known that these renormalizations sum to zero, so their explicit removal is unnecessary. The infrared regulation is needed in part for the same reason, and also because the separation into A_0 and A_1 introduces additional infrared divergences arising from the fact that $B(\mathbf{0};p_2,p_1)\to\infty$ as $p_2-p_1\to 0$. In the course of a computation, of course, one always allows Λ and λ to approach their limiting values as early as possible, to minimize the complications they introduce.

Demonstration that A_0 Yields the Scattering Approximation

We now show that, apart from previously discussed ignorable contributions, A_0 yields to order $\alpha^2(\alpha Z)^4$, just the scattering approximation. First of all, the last term of A_0 is of the general form

$$\int\bar{\varphi}_a(\mathbf{p})H(p,k_2,k_1)\varphi_a(\mathbf{p}_1)(d_4k)^2d\mathbf{p}$$
$$=\int\bar{\varphi}_a(\mathbf{p})[h_0(p^2)+(i\gamma\cdot p)h_1(p^2)]\varphi_a(\mathbf{p})d\mathbf{p},$$

which, to order $\alpha^2(\alpha Z)^4$, separates into mass renormalization, charge renormalization, and $(\gamma\cdot A^e)^2$ terms, all of which we ignore.

The terms with A^e appearing once are all of the general form

$$-2i(2\pi)^{-4}[ie/(\hbar c)]\int\bar{\varphi}_aF'A^e\cdot\mathfrak{F}\varphi_a$$
$$=-2i(2\pi)^{-4}[ie/(\hbar c)]\Big\{\int\bar{\varphi}_aF'F\gamma\cdot A^e\varphi_a$$
$$-\int\bar{\varphi}_aF'\gamma\cdot A^eF\varphi_a\Big\}. \quad (26)$$

The first term on the right, after integration over k_1 and k_2, is of the general form

$$\int\bar{\varphi}_a(\mathbf{p}_2)[h_2(p_2{}^2)+(i\gamma\cdot p_2)h_3(p_2{}^2)]$$
$$\times\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\varphi_a(\mathbf{p}_1)(d\mathbf{p})^2,$$

which, to order $\alpha^2(\alpha Z)^4$ yields a charge renormalization term and a $(\gamma\cdot A^e)^2$ term, so that this term may also be ignored. To deal with the remaining term in (26) we

note that F can be rewritten as follows, using (13):

$$-\frac{2i}{(2\pi)^4}F(\mathbf{O};p,k)$$
$$=-\frac{2i}{(2\pi)^4}\left[\frac{i\gamma\cdot(p-k)-\kappa}{k^2-2p\cdot k}\mathbf{O}+\mathbf{O}\frac{i\gamma\cdot p+\kappa}{k^2-2p\cdot k}\right]$$
$$=S_F(p-k)\mathbf{O}+S_F(p-k)\mathbf{O}\frac{p^2+\kappa^2}{k^2-2p\cdot k}$$
$$-\frac{2i}{(2\pi)^4}\mathbf{O}\frac{i\gamma\cdot p+\kappa}{k^2-2p\cdot k}, \quad (27)$$

since

$$-\frac{2i}{(2\pi)^4}\frac{i\gamma\cdot(p-k)-\kappa}{k^2-2p\cdot k}$$
$$=-\frac{2i}{(2\pi)^4}\frac{i\gamma\cdot(p-k)-\kappa}{(p-k)^2+\kappa^2}\left(1+\frac{p^2+\kappa^2}{k^2-2p\cdot k}\right)$$
$$=S_F(p-k)\left(1+\frac{p^2+\kappa^2}{k^2-2p\cdot k}\right). \quad (28)$$

So

$$-2i(2\pi)^{-4}F(\mathbf{O};p,k)\varphi_a(\mathbf{p})$$
$$=S_F(p-k)\mathbf{O}\varphi_a(\mathbf{p})-[ie/(\hbar c)]\int(k^2-2p\cdot k)^{-1}$$
$$\times[S_F(p-k)\mathbf{O}(i\gamma\cdot p-\kappa)+2i(2\pi)^{-4}\mathbf{O}]$$
$$\times\gamma\cdot A^e(\mathbf{p}-\mathbf{p}')\varphi_a(\mathbf{p}')d\mathbf{p}'. \quad (29)$$

The second term on the right will produce expressions which can be broken up into pure charge renormalization terms and $(\gamma\cdot A^e)^2$ terms, which we ignore. We are left only with the first term, and since $\mathbf{O}$ itself is either γ_μ or of the form $\gamma_\mu F$, the process can be repeated until we are left with a product of S_F's and γ's. An analogous procedure for the factor $\bar{\varphi}_a F'$ reduces it also to a product of S_F's and γ's. Thus the surviving term of (26) involves a product of factors γS_F to the left and $S_F\gamma$ to the right of $\gamma\cdot A^e$, and is readily identifiable with a term in the scattering approximation. In carrying out the scattering approximation one always takes $p_1^2+\kappa^2=p_2^2+\kappa^2=0$, and $\bar{\varphi}_a(\mathbf{p}_2)(i\gamma\cdot p_2+\kappa)=(i\gamma\cdot p_1+\kappa)\varphi_a(\mathbf{p}_1)=0$. This approximation, again to order $\alpha^2(\alpha Z)^4$, corresponds to the neglect of $(\gamma\cdot A^e)^2$ terms.

On carrying out the procedure described previously, one finds in a straightforward manner that the three terms linear in A^e appearing in Eq. (24) correspond to the terms of the scattering approximation illustrated in Fig. 4, (c), (b), and (a), respectively.

Identification and Evaluation of $\alpha^2(\alpha Z)^4$ Terms in A_1

There remains the problem of isolating and evaluating the $\alpha^2(\alpha Z)^4$ contributions from A_1. These are of two general types: (1) $(\gamma\cdot A^e)^2$ terms of the type encountered in the discussion of $(MP)_1$. Once such terms have been separated off, they may be evaluated by replacing $S_F{}^e$ by S_F. As noted before, terms of this character will simply be isolated and recognized. (2) Terms which can be expressed as $(\mathbf{p}_2-\mathbf{p}_1)^2\gamma\cdot A^e$ terms; these are clearly identical in form with the scattering approximation terms and will be explicitly evaluated. They arise from terms apparently quadratic in the external potential by virtue of identities like the following:

$$\frac{i\hbar c}{\pi}\left(\frac{ie}{\hbar c}\right)^2\sum_n\int\bar{\varphi}_a(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{q}_2)\mathbf{q}_1\cdot\mathbf{q}_2$$
$$\times\frac{\varphi_n(\mathbf{p}_2-\mathbf{q}_2)\bar{\varphi}_n(\mathbf{p}_1+\mathbf{q}_1)}{E_n-E_a}\gamma\cdot A^e(\mathbf{q}_1)\varphi_a(\mathbf{p}_1)(d\mathbf{p})^2(d\mathbf{q})^2$$
$$=-(i/\pi)[ie/(\hbar c)]\sum\int\varphi_a{}^*(\mathbf{p}_2)\mathbf{p}_2\cdot\mathbf{q}_1\varphi_n(\mathbf{p}_2)$$
$$\times\bar{\varphi}_n(\mathbf{p}_1+\mathbf{q}_1)\gamma\cdot A^e(\mathbf{q}_1)\varphi_a(\mathbf{p}_1)(d\mathbf{p})^2d\mathbf{q}_1 \quad (30)$$
$$=-2i(2\pi)^{-4}[ie/(\hbar c)]\int\bar{\varphi}_a(\mathbf{p}_2)\mathbf{p}_2\cdot(\mathbf{p}_2-\mathbf{p}_1)$$
$$\times\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\varphi_a(\mathbf{p}_1)(d\mathbf{p})^2$$
$$=-i(2\pi)^{-4}[ie/(\hbar c)]\int\bar{\varphi}_a(\mathbf{p}_2)(\mathbf{p}_2-\mathbf{p}_1)^2$$
$$\times\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\varphi_a(\mathbf{p}_1)(d\mathbf{p})^2.$$

The factors E_n-E_a appearing in the denominators come from the $S_F{}^e$ functions; their appearance implies that one cannot, for these terms, expand $S_F{}^e$ in terms of S_F and powers of the potential. One always can, on the other hand, make appropriate nonrelativistic approximations, familiar from the treatments given for the second-order level shift problem. To see how terms of types (1) and (2) may be separated, we write

$$\mathfrak{F}_\sigma(\mathbf{O};p,q,k)=\mathfrak{F}_\sigma(0)+\mathfrak{G}_\sigma,$$
$$\mathfrak{F}_\sigma(0)=\mathfrak{F}_\sigma(\mathbf{O};p,0,k)=[F(\mathbf{O};p,k),\gamma_\sigma],$$
$$\mathfrak{G}_\sigma=\{F[\mathbf{O}(p+q);p+q,k]-F[\mathbf{O}(p);p,k]\}\gamma_\sigma; \quad (31)$$
$$\mathfrak{F}_\rho'(\mathbf{O};p,q,k)=\mathfrak{F}_\rho'(0)+\mathfrak{G}_\rho',$$

with $\mathfrak{F}_\rho'(0)$ and $\mathfrak{G}_\rho'$ defined similarly. A direct examination of the $\mathfrak{F}$ functions shows that the $\mathfrak{F}(0)$ is at least one factor k better behaved as $k\to 0$ than $\mathfrak{G}$. This is the crucial property which makes the rather involved scheme we are using for the $M_1{}^{(4)}$ term superior to the simple expansion of the $S_F{}^e$ used for treating $(MP)_1$. As a consequence of this fact, the terms involving only $\mathfrak{F}(0)$ factors are sufficiently well behaved in the infrared to permit the replacement of $S_F{}^e$ by S_F. These terms are very similar to corresponding terms in $(MP)_1$ and the only $\alpha^2(\alpha Z)^4$ terms which they yield are $(\gamma\cdot A^e)^2$ terms.

The $\mathcal{G}$ factors obviously vanish when $\mathbf{q}$ vanishes, and correspond in a crude way to differentiation of the external potential with respect to the spatial coordinates. It follows that terms involving $\mathcal{G}$ cannot be $(\gamma\cdot A^e)^2$ terms. In the absence of singular infrared behavior which brings into prominence the energy denominators of the S_F^e functions, such terms are at least a factor αZ smaller than the order of interest. The infrared singularities are sufficiently mild, in the case of cross terms between an $\mathcal{F}(0)$ and a $\mathcal{G}$, for such terms to be indeed a factor αZ smaller. On the other hand, the terms containing two $\mathcal{G}$ factors are, in fact, of order $\alpha^2(\alpha Z)^4$ and must be evaluated. For these terms, nonrelativistic approximations make evaluation straightforward.

The entire procedure is best described in terms of an example. We discuss the term a_2 [Eq. (23)]. We note first of all that

$$F_{1\nu}=\frac{2ip_{1\nu}-(i\gamma\cdot k_1)\gamma_\nu}{k_1^2-2p_1\cdot k_1};$$

$$F'_{2\mu\nu}=-\frac{2i}{(2\pi)^4}\frac{1}{(k_1+k_2)^2-2p_2\cdot(k_1+k_2)}\times\left\{\gamma_\mu\gamma_\nu+\frac{2ip_{2\mu}-\gamma_\mu(i\gamma\cdot k_2)}{k_2^2-2p_2\cdot k_2}\times[2i(p_{2\nu}-k_{2\nu})-\gamma_\nu(i\gamma\cdot k_1)\right\}. \quad (32)$$

The terms of lowest order in the k's in both cases evidently commute with γ_ρ, and so give no contribution to $\mathcal{F}_{1\nu,\sigma}(0)$ or $\mathcal{F}'_{2\mu\nu,\rho}(0)$. One may now directly verify that the term arising from $\mathcal{F}_{1\nu,\sigma}(0)$ and $\mathcal{F}'_{2\mu\nu,\rho}(0)$ is finite in the infrared and ultraviolet, using S_F for S_F^e and unregulated D_F functions. One therefore gets a $(\gamma\cdot A^e)^2$ term. One can also verify that the cross terms, while more singular, give contributions no larger than $\alpha^2(\alpha Z)^5$. We now evaluate the contribution from the part involving two $\mathcal{G}$ factors. The important contributions in this case come entirely from $k\sim(\alpha Z)^2$. Consequently one may neglect k as compared to $\mathbf{p}$, $\mathbf{q}$, and p_0. Furthermore, one may assume $|\mathbf{q}|<p_0$, $\mathbf{q}^2<|\mathbf{q}|\kappa$. Taking advantage of these approximations the $\mathcal{G}$'s simplify to

$$\mathcal{G}_{1\nu,\sigma}\sim\frac{1}{\kappa}\left[\frac{iq_{1\nu}}{k_{10}}-\delta_{\nu,4}\frac{\mathbf{q}_1\cdot\mathbf{k}_1}{k_{10}^2}\right]\gamma_\sigma;$$

$$\mathcal{G}'_{2\mu\nu,\rho}\sim-\frac{2i}{(2\pi)^4}\frac{\gamma_\rho}{\kappa}\left[\frac{q_{2\mu}\delta_{\nu4}+q_{2\nu}\delta_{\mu,4}}{k_{20}(k_{10}+k_{20})}-\frac{\delta_{\mu4}\delta_{\nu4}}{k_{20}(k_{10}+k_{20})}\left(\frac{\mathbf{q}_2\cdot\mathbf{k}_2}{k_{20}}+\frac{\mathbf{q}_2\cdot(\mathbf{k}_1+\mathbf{k}_2)}{k_{10}+k_{20}}\right)\right]. \quad (33)$$

The expression for a_2 also contains the factor

$$\int S_F^e(p_2-q_2-k_1-k_2,\ p_3-k_1-k_2)\gamma_\mu S_F^e(p_3-k_1,\ p_1+q_1-k_1)d\mathbf{p}_3$$
$$=-\frac{1}{\pi^2}\sum_{n,m}\int\frac{\varphi_n(\mathbf{p}_2-\mathbf{q}_2-\mathbf{k}_1-\mathbf{k}_2)\bar\varphi_n(\mathbf{p}_3-\mathbf{k}_1-\mathbf{k}_2)\gamma_\mu\varphi_m(\mathbf{p}_3-\mathbf{k}_1)\bar\varphi_m(\mathbf{p}_1+\mathbf{q}_1-\mathbf{k}_1)}{(\delta_n+k_{10}+k_{20})(\delta_m+k_{10})}d\mathbf{p}_3$$
$$\sim-\frac{1}{\pi^2}\sum_{n,m}\int\frac{\varphi_n(\mathbf{p}_2-\mathbf{q}_2)\bar\varphi_n(\mathbf{p}_3)\gamma_\mu\varphi_m(\mathbf{p}_3)\bar\varphi_m(\mathbf{p}_1+\mathbf{q}_1)}{(\delta_n+k_{10}+k_{20})(\delta_m+k_{10})}d\mathbf{p}_3 \quad (34)$$
$$[\delta_n=((1-i\epsilon)E_n-E_a)/(\hbar c)],$$

since, as previously noted, k may be neglected as compared to $\mathbf{p}$ and $\mathbf{q}$. This is simply the familiar dipole approximation, corresponding to the neglect of retardation. This expression may be further simplified by taking into account the fact that the significant contributions come only when n and m are positive energy states of nonrelativistic energy. One therefore takes

$$\int\bar\varphi_n(\mathbf{p}_3)\gamma_\mu\varphi_m(\mathbf{p}_3)d\mathbf{p}_3\sim\delta_{\mu4}\int\varphi_n^*(\mathbf{p}_3)\varphi_m(\mathbf{p}_3)d\mathbf{p}_3$$
$$=(2\pi)^{-3}\delta_{\mu4}\delta_{nm}, \quad (35)$$

so that (34) reduces to

$$-\frac{1}{8\pi^5}\delta_{\mu4}\sum_n\frac{\varphi_n(\mathbf{p}_2-\mathbf{q}_2)\bar\varphi_n(\mathbf{p}_1+\mathbf{q}_1)}{(\delta_n+k_{10}+k_{20})(\delta_n+k_{10})}.$$

With these approximations a_2 becomes

$$a_2=\frac{i}{\pi}\left(\frac{2i}{(2\pi)^4}\right)^4\left(\frac{ie}{\hbar c}\right)^2\frac{1}{\kappa^2}\sum_n\int\bar\varphi_a(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{q}_2)\times\frac{\varphi_n(\mathbf{p}_2-\mathbf{q}_2)\bar\varphi_n(\mathbf{p}_1+\mathbf{q}_1)}{(\delta_n+k_{10}+k_{20})(\delta_n+k_{10})}\gamma\cdot A^e(\mathbf{q}_1)\varphi_a(\mathbf{p}_1)$$
$$\times\frac{1}{k_{10}k_{20}(k_{10}+k_{20})}\left[\mathbf{q}_1\cdot\mathbf{q}_2-\frac{(\mathbf{q}_1\cdot\mathbf{k}_1)(\mathbf{q}_2\cdot\mathbf{k}_2)}{k_{10}k_{20}}-\frac{(\mathbf{q}_1\cdot\mathbf{k}_1)(\mathbf{q}_2\cdot(\mathbf{k}_1+\mathbf{k}_2))}{k_{10}(k_{10}+k_{20})}\right]\frac{d_4k_1}{k_1^2}\frac{d_4k_2}{k_2^2}(d\mathbf{p})^2(d\mathbf{q})^2. \quad (36)$$

The integrations of the timelike components and angular domains of k_1 and k_2 may now be carried out

easily. The k_0 integrations are performed simply by closing the contours on the negative imaginary side of the real axis. The low-energy poles from $k^2-2p\cdot k-i\epsilon$, here approximated by $2\kappa_0 k_0-i\epsilon$, and those from the denominators δ_n+k_0 for positive energy states are thereby avoided. Contributions of the order of interest come only from the poles of the D_F functions. One then finds

$$a_2=\frac{2}{\pi}\left(\frac{2i}{(2\pi)^4}\right)^3\left(\frac{ie}{\hbar c}\right)^2\frac{1}{\kappa^2}\sum_n\int\bar{\varphi}_a(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{q})$$
$$\times\frac{\varphi_n(\mathbf{p}_2-\mathbf{q}_2)\bar{\varphi}_n(\mathbf{p}_1+\mathbf{q}_1)}{(\delta_n+k_1+k_2)(\delta_n+k_1)}\gamma\cdot A^e(\mathbf{q}_1)\varphi_a(\mathbf{p}_1)$$
$$\times\frac{\mathbf{q}_1\cdot\mathbf{q}_2}{k_1k_2(k_1+k_2)}\left(1-\frac{1}{3}\frac{k_1}{k_1+k_2}\right)\frac{k_1^2dk_1}{k_1}$$
$$\times\frac{k_2^2dk_2}{k_2}(d\mathbf{p})^2(d\mathbf{q})^2. \quad (37)$$

(Here $k_1=|\mathbf{k}_1|$, $k_2=|\mathbf{k}_2|$.)

Now, letting $k_1\rightarrow\delta_n k_1$, $k_2\rightarrow\delta_n k_2$, we find

$$a_2=\sum_n C_n\int\frac{dk_1dk_2}{(k_1+k_2)(1+k_1)(1+k_1+k_2)}$$
$$\times\left(1-\frac{1}{3}\frac{k_1}{k_1+k_2}\right), \quad (38)$$

with

$$C_n=(2/\pi)[2i(2\pi)^{-4}]^3[ie/(\hbar c)]^2(1/\kappa^2)$$
$$\times\int\bar{\varphi}_a(\mathbf{p}_2)\gamma\cdot A^e(\mathbf{q}_2)\varphi_n(\mathbf{p}_2-\mathbf{q}_2)(1/\delta_n)$$
$$\times\bar{\varphi}_n(\mathbf{p}_1+\mathbf{q}_1)\gamma\cdot A^e(\mathbf{q}_1)\varphi_n(\mathbf{p}_1)(d\mathbf{p})^2(d\mathbf{q})^2. \quad (39)$$

The expression $\sum C_n$ has been discussed at the beginning of this section and has been shown to be proportional to $\int\bar{\varphi}_a(\mathbf{p}_2)(\mathbf{p}_2-\mathbf{p}_1)^2\gamma\cdot A^e(\mathbf{p}_2-\mathbf{p}_1)\varphi_a(\mathbf{p}_1)d\mathbf{p}_1d\mathbf{p}_2$. One has, therefore, only to evaluate the integrals over k_1 and k_2 to obtain the contribution of this term to the level shift.[10]

The reduction of a_2 is particularly simple because it does not have any infrared divergences. In most of the terms the integral multiplying C_n still involves δ_n through the ratio λ/δ_n. After carrying out the k-space integrations one has, in addition to $\sum C_n$, terms of the form $\sum C_n\ln(\lambda/\delta_n)$ and $\sum C_n\ln^2(\lambda/\delta_n)$. Terms such as these occur in the second-order level shift problem. It is clear, however, that no such terms can remain after summing over all contributions, simply because of the fact that all dependence on λ must vanish.[11] We shall find, in fact, that it is unnecessary to carry out most of the k integrations, many terms vanishing simply as the result of combining integrands.

Carrying out the reduction for the remaining terms in $M_1^{(4)}$, we find, using $k=|\mathbf{k}|$, $\omega=[k^2+(\lambda/\delta_n)^2]^{\frac{1}{2}}$,

$$a_1=0;$$

$$a_2=\sum_n C_n\int\frac{dk_1dk_2}{(k_1+k_2)(1+k_1)(1+k_1+k_2)}$$
$$\times\left(1-\frac{1}{3}\frac{k_1}{k_1+k_2}\right);$$

$$a_2'=\sum_n C_n\int\frac{dk_1dk_2}{(k_1+k_2)(1+k_2)(1+k_1+k_2)}$$
$$\times\left(1-\frac{1}{3}\frac{k_2}{k_1+k_2}\right);$$

$$a_3=\left(\ln\frac{\Lambda}{\kappa}-2\ln\frac{\kappa}{\lambda}+\frac{9}{4}\right)\mathrm{II}_1$$
$$+\sum_n C_n\int\frac{k_1^2dk_1k_2^2dk_2}{\omega_1^2\omega_2^3(\omega_1+\omega_2)(1+\omega_1)}$$
$$\times\left(-1+\frac{1}{3}\frac{k_1^2}{\omega_1(\omega_1+\omega_2)}\right); \quad (40)$$

$$a_3'=\left(\ln\frac{\Lambda}{\kappa}-2\ln\frac{\kappa}{\lambda}+\frac{9}{4}\right)\mathrm{II}_1$$
$$+\sum_n C_n\int\frac{k_1^2dk_1k_2^2dk_2}{\omega_1^3\omega_2^2(\omega_1+\omega_2)(1+\omega_2)}$$
$$\times\left(-1+\frac{1}{3}\frac{k_2^2}{\omega_2(\omega_1+\omega_2)}\right);$$

$$a_4=\sum_n C_n\int\frac{k_1^2dk_1k_2^2dk_2}{\omega_1^2\omega_2^2(\omega_1+\omega_2)^2(1+\omega_1+\omega_2)}$$
$$\times\left(2-\frac{1}{3}\frac{k_1^2}{\omega_1(\omega_1+\omega_2)}-\frac{1}{3}\frac{k_2^2}{\omega_2(\omega_1+\omega_2)}\right.$$
$$\left.-\frac{1}{3}\frac{k_1^2+k_2^2}{(\omega_1+\omega_2)^2}\right);$$

where

$$\mathrm{II}_1=4(2\pi)^{-10}[ie/(\hbar c)]^2\int\bar{\varphi}A^e\cdot\mathcal{F}'_{1\mu}S_F{}^eA^e\cdot\mathcal{F}_{1\mu}\varphi$$
$$\times D_Fd_4k(d\mathbf{p})^2(d\mathbf{q})^2 \quad (41)$$
$$=-\frac{1}{2}\sum_n C_n\int\frac{k^2dk}{\omega^3(1+\omega)}\left(1-\frac{1}{3}\frac{k^2}{\omega^2}\right).$$

[10] It might be noted that the contribution of a_2+a_2' to the $2S$ energy level in hydrogen is about -1.3 megacycles.

[11] This argument is somewhat oversimplified as terms of the form $\ln(\kappa/\lambda)$ also occur. To complete the argument one must identify all $\ln(\kappa/\lambda)$ terms with the second-order charge renormalizations which we neglected to remove.

The expression II_1 is just the multiple-potential contribution to the second-order level shift, apart from a numerical factor. Its appearance as a factor is expected in view of the fact that we have retained charge renormalization terms.

CONTRIBUTION OF $M_2^{(4)}$

While the main outline of the treatment of $M_2^{(4)}$ [Eq. (6)] is identical to that of $M_1^{(4)}$, the presence of the second-order mass term leads to significant complications in its treatment. Furthermore, the fact that the photon momenta appearing in the first and third S_F^e factors are the same makes the infrared problem somewhat more complicated.

We note first of all that $M_2^{(4)}$ can be separated, by successive applications of the identities (13) into a set of terms B_1 involving the potential twice explicitly and S_F^e, and another set, B_0, not involving S_F^e, and involving the potential no more than once explicitly, all in a manner exactly analogous to the treatment of $M_1^{(4)}$. Again the terms B_0 yield the scattering approximation, renormalization terms, and $(\gamma\cdot A^e)^2$ terms. The terms forming B_1, (they are exact analogs of a_1, a_2, a_3, a_4) are, schematically,

$$
\begin{aligned}
b_1=&[ie/(\hbar c)]^2\int\bar\varphi_a A_\rho^e\mathfrak{F}'_{1\mu,\rho}S_F^e\gamma_\nu S_F^e\gamma_\nu S_F^e A_\sigma^e\\
&\times\mathfrak{F}_{1\mu,\sigma}\varphi_a D_F D_F(d_4k)^2(d\mathbf{p})^4(d\mathbf{q})^2-2i(2\pi)^{-5}\alpha^{-1}\\
&\times\delta\kappa^{(2)}[ie/(\hbar c)]\int\bar\varphi_a A_\rho^e\mathfrak{F}'_{1\mu,\rho}S_F^e S_F^e A_\sigma^e\\
&\times\mathfrak{F}_{1\mu,\sigma}\varphi_a D_F d_4k(d\mathbf{p})^3(d\mathbf{q})^2;\\
b_2=&[ie/(\hbar c)]^2\int\bar\varphi_a A_\rho^e\mathfrak{F}'_{2\mu\nu,\rho}S_F^e\gamma_\nu S_F^e A_\sigma^e\\
&\times\mathfrak{F}_{1\mu,\sigma}\varphi_a D_F D_F(d_4k)^2(d\mathbf{p})^3(d\mathbf{q})^2;\\
b_2'=&[ie/(\hbar c)]^2\int\bar\varphi_a A_\rho^e\mathfrak{F}'_{1\mu,\rho}S_F^e\gamma_\nu S_F^e A_\sigma^e\\
&\times\mathfrak{F}_{2\nu\mu,\sigma}\varphi_a D_F D_F(d_4k)^2(d\mathbf{p})^3(d\mathbf{q})^2,\\
b_3=&[ie/(\hbar c)]^2\int\bar\varphi_a A_\rho^e\mathfrak{F}'^R{}_{3\mu,\rho}S_F^e A_\sigma^e\\
&\times\mathfrak{F}_{1\mu,\sigma}\varphi_a D_F d_4k_1(d\mathbf{p})^2(d\mathbf{q})^2;\\
b_3'=&[ie/(\hbar c)]^2\int\bar\varphi_a A_\rho^e\mathfrak{F}'_{1\mu,\rho}S_F^e A_\sigma^e\\
&\times\mathfrak{F}^R{}_{3\mu,\sigma}\varphi_a D_F d_4k_1(d\mathbf{p})^2(d\mathbf{q})^2;\\
b_4=&[ie/(\hbar c)]^2\int\bar\varphi_a A_\rho^e\mathfrak{F}'_{2\mu\nu,\rho}S_F^e A_\sigma^e\\
&\times\mathfrak{F}_{2\nu\mu,\sigma}\varphi_a D_F D_F(d_4k)^2(d\mathbf{p})^2(d\mathbf{q})^2,
\end{aligned}
\tag{42}
$$

where $\mathfrak{F}'^R{}_{3\mu,\rho}$ (and, analogously, $\mathfrak{F}^R{}_{3\mu,\sigma}$) is formed by taking the operator $\mathbf{O}$ in (13) to be

$$
-2i(2\pi)^{-4}\left[\int F'_{2\mu\nu}\gamma_\nu D_F(k_2)d_4k_2-2i(2\pi)^{-5}\alpha^{-1}\delta\kappa^{(2)}F'_{1\mu}\right].
$$

Thus mass renormalization appears explicitly only in b_1, and in the formation of $\mathfrak{F}_3{}^R$ appearing in b_3 and b_3'.

Further reduction of b_1 is necessary to remove the mass renormalization term. The reduction is accomplished by using the identities (13) on one or another of the remaining γ_ν factors. Apart from the mass term,

$$
\begin{aligned}
b_1=&[ie/(\hbar c)]^2\int\bar\varphi_a(\mathbf{p}_2)A_\rho^e(\mathbf{q}_2)\mathfrak{F}'_{1\mu,\rho}\\
&\times S_F^e(p_2-q_2-k_1,p_4-k_1)\gamma_\nu S_F^e(p_4-k_1-k_2,p_3-k_1-k_2)\\
&+\gamma_\nu S_F^e(p_3-k_1,\ p_1+q_1-k_1)A_\sigma^e(\mathbf{q}_1)\mathfrak{F}_{1\mu,\sigma}\\
&\times\varphi_a(\mathbf{p}_1)D_F(k_1)D_F(k_2)(d_4k)^2(d\mathbf{p})^4(d\mathbf{q})^2.
\end{aligned}
\tag{43}
$$

$[\mathfrak{F}'_{1\mu,\rho}=\mathfrak{F}'_\rho(\gamma_\mu;\,p_2,q_2,k_1);\quad \mathfrak{F}_{1\mu,\sigma}=\mathfrak{F}_\sigma(\gamma_\mu;\,p_1,q_1,k_1).]$

Now

$$
\begin{aligned}
&\int S_F^e(p_2-q_2-k_1,\ p_4-k_1)\gamma_\nu\\
&\quad\times S_F^e(p_4-k_1-k_2,\ p_3-k_1-k_2)d\mathbf{p}_4\\
&=[(ie/(\hbar c)]\int S_F^e(p_2-q_2-k_1,\ p_4-k_1)A_\tau^e(\mathbf{q}_3)\\
&\quad\times\mathfrak{F}'_{1\nu,\tau}(p_4-k_1)S_F^e(p_4-q_3-k_1-k_2,\ p_3-k_1-k_2)\\
&\quad\times d\mathbf{p}_4 d\mathbf{q}_3+2i(2\pi)^{-4}F_{1\nu}'(p_2-q_2-k_1)\\
&\quad\times S_F^e(p_2-q_2-k_1-k_2,\ p_3-k_1-k_2)-2i(2\pi)^{-4}\\
&\qquad\times S_F^e(p_2-q_2-k_1,\ p_3-k_1)F_{1\nu}'(p_3-k_1),
\end{aligned}
\tag{44}
$$

where

$$
F_{1\nu}'(p)=F'(\gamma_\nu;\,p,k_2),\quad \mathfrak{F}'_{1\nu,\tau}(p)=\mathfrak{F}_\tau'(\gamma_\nu;\,p,q_3,k_2).
\tag{45}
$$

Therefore this part of b_1 breaks up into three terms:

$$
\begin{aligned}
b_1=&[ie/(\hbar c)]^3\int\bar\varphi_a A^e\cdot\mathfrak{F}_{1\mu}'S_F^e A^e\cdot\mathfrak{F}_{1\nu}'S_F^e\gamma_\nu\\
&\times S_F^e A^e\cdot\mathfrak{F}_{1\mu}\varphi_a D_F D_F(d_4k)^2(d\mathbf{p})^4(d\mathbf{q})^3\\
&+2i(2\pi)^{-4}[ie/(\hbar c)]^2\int\bar\varphi_a A^e\cdot\mathfrak{F}_{1\mu}'F_{1\nu}'S_F^e\gamma_\nu S_F^e\\
&\times A^e\cdot\mathfrak{F}_{1\mu}\varphi_a D_F D_F(d_4k)^2(d\mathbf{p})^3(d\mathbf{q})^2\\
&-2i(2\pi)^{-4}[ie/(\hbar c)]^2\int\bar\varphi_a A^e\cdot\mathfrak{F}_{1\mu}'S_F^e F_{1\nu}'\gamma_\nu\\
&\times S_F^e A^e\cdot\mathfrak{F}_{1\mu}\varphi_a D_F D_F(d_4k)^2(d\mathbf{p})^3(d\mathbf{q})^2\\
=&b_{11}+b_{12}+b_{13}.
\end{aligned}
\tag{46}
$$

In b_{11}, the middle S_F^e factor can be replaced by S_F, so that the k_2 integration can be carried out, yielding

$$
\begin{aligned}
b_{11}=&2i(2\pi)^{-6}[ie/(\hbar c)]^3\left(3\ln\frac{\Lambda}{\kappa}-\frac{9}{4}\right)\int\bar\varphi_a(\mathbf{p}_2)\\
&\times A_\rho^e(\mathbf{q}_2)\mathfrak{F}'_{1\mu,\rho}S_F^e(p_2-q_2-k_1,\ p_3+q_3-k_1)\\
&\times\gamma\cdot A^e(\mathbf{q}_3)S_F^e(p_3-k_1,\ p_1+q_1-k_1)A_\sigma^e(\mathbf{q}_1)\\
&\times\mathfrak{F}_{1\mu,\sigma}\varphi_a(\mathbf{p}_1)D_F(k_1)d_4k_1(d\mathbf{p})^3(d\mathbf{q})^3,
\end{aligned}
\tag{47}
$$

which, on application of Eq. (14) to

$$[ie/(\hbar c)]S_F^e(p_2-q_2-k_1,\ p_3+q_3-k_1)\gamma\cdot A^e(\mathbf{q}_3)\times S_F^e(p_3-k_1,\ p_1+q_1-k_1)_1 d\mathbf{p}_3 d\mathbf{q}_3,$$

combines with b_{13} to yield a multiple of II_1.

The term b_{12} must be further reduced by using the identities (13) on the remaining γ_ν factor, again yielding three terms. One of these involves the potential explicitly three times and may be shown to be too small. The other two can be evaluated using the methods applied in the preceding section. The k_2 integration in b_{13} can be done immediately, yielding a mass term and a term combining with b_{11}, as noted above, to give a multiple of II_1.

Apart from slight modifications in b_2 and b_3, the evaluation of the remaining terms in B_1 is carried out along the same lines followed in evaluating A_1, to yield

$$b_1=-\left(\ln\frac{\Lambda}{\kappa}-2\ln\frac{\kappa}{\lambda}+\frac{9}{4}\right)\mathrm{II}_1+\sum_n C_n\int\frac{k_1^2dk_1k_2^2dk_2}{\omega_1^3\omega_2^3(1+\omega_1+\omega_2)}\left(1-\frac{1}{3}\frac{k_1^2}{\omega_1^2}\right);$$

$$b_2=b_2'=-b_3+\sum_n C_n\int\frac{k_1^2dk_1k_2^2dk_2}{\omega_1^3\omega_2^2(\omega_1+\omega_2)(1+\omega_1+\omega_2)}\times\left(-1+\frac{1}{3}\frac{k_1^2}{\omega_1^2}+\frac{1}{3}\frac{k_1^2}{\omega_1(\omega_1+\omega_2)}\right);$$

$$b_3=b_3'=4\left(\ln\frac{\Lambda}{\kappa}-2\ln\frac{\kappa}{\lambda}\right)\mathrm{II}_1-8i(2\pi)^{-10}[ie/(\hbar c)]^2\times\left(3\ln\frac{\Lambda}{\kappa}-\frac{9}{4}\right)\int\bar\varphi_a(\mathbf{p}_2)\frac{p_{2\mu}}{k_1^2-2p_2\cdot k_1}\gamma\cdot A^e(\mathbf{q}_2)\times S_F^e(p_2-q_2-k_1,\ p_1+q_1-k_1)A_\sigma^e(\mathbf{q}_1)\mathfrak{F}_{1\mu,\sigma}\times\varphi_a(\mathbf{p}_1)D_F(k_1)d_4k_1(d\mathbf{p})^2(d\mathbf{q})^2+\sum_n C_n\int\frac{k_1^2dk_1k_2^2dk_2}{\omega_1^2\omega_2^3(\omega_1+\omega_2)(1+\omega_1)}\times\left(-1+\frac{1}{3}\frac{k_1^2}{\omega_1(\omega_1+\omega_2)}\right);\quad(48)$$

$$b_4=\sum_n C_n\int\frac{k_1^2dk_1k_2^2dk_2}{\omega_1^3\omega_2(\omega_1+\omega_2)^2(1+\omega_1+\omega_2)}\left(2-\frac{1}{3}\frac{k_1^2}{\omega_1^2}-\frac{2}{3}\frac{k_1^2}{\omega_1(\omega_1+\omega_2)}-\frac{1}{3}\frac{k_1^2+k_2^2}{(\omega_1+\omega_2)^2}\right).$$

It is to be noted that b_3 contains a term which cannot readily be evaluated with the techniques we have been using. Fortunately a similar term appearing in b_2 just cancels it.

Inspection of the various contributions to $M_1^{(4)}$ and $M_2^{(4)}$ yields the result that the integrands of the integrals having $\sum C_n$ as a factor add to zero, so that $M_1^{(4)}+M_2^{(4)}$ may be evaluated without carrying out any of the integrations. We find that the contribution to $\Delta E^{(4)}$ is

$$\hbar c[\langle M_1^{(4)}\rangle+\langle M_2^{(4)}\rangle]=-(i/\pi)[\tfrac{1}{4}(2\pi)^9\alpha]^2\hbar c\left(\ln\frac{\Lambda}{\kappa}-2\ln\frac{\kappa}{\lambda}+\frac{9}{4}\right)\mathrm{II}_1.\quad(49)$$

CONTRIBUTION OF $-H^{(2)}\partial H^{(2)}/\partial E$ AND THE DEMONSTRATION OF THE VALIDITY OF THE SCATTERING APPROXIMATION

The evaluation of the remaining significant contribution to $\Delta E^{(4)}$ presents no difficulties. This term, corresponding crudely to diagram 1 of Fig. 3, is pure charge renormalization, and was therefore not calculated by WBK. We have not removed the second-order charge renormalizations from the fourth-order terms, and so must include this as well. $iH_{aa}^{(2)}$ is simply the second-order level shift. After mass renormalization it is of order $\alpha(\alpha Z)^4$. Thus only a crude approximation to $\partial H^{(2)}/\partial E$ is needed. We note that it may be written

$$\partial H_{aa}^{(2)}/\partial E=\tfrac{1}{4}i(2\pi)^{12}\alpha\int\bar\varphi_a\gamma_\mu S_F^e\gamma_4 S_F^e\gamma_\mu\varphi_a\times D_F d_4k(d\mathbf{p})^3.\quad(50)$$

It is sufficient to replace S_F^e by S_F, in which case one recognizes the expression as the second-order vertex charge renormalization constant. $H^{(2)}$ is to be separated into a scattering part, which is to be amalgamated with the scattering approximation terms, and a multiple-potential part, which we have denoted, apart from a factor, by II_1. The contribution of $-H^{(2)}\partial H^{(2)}/\partial E$ to the multiple-potential corrections to the scattering approximation is, indeed, simply

$$(i/\pi)[\tfrac{1}{4}(2\pi)^9\alpha]^2\hbar c\left(\ln\frac{\Lambda}{\kappa}-2\ln\frac{\kappa}{\lambda}+\frac{9}{4}\right)\mathrm{II}_1,$$

which precisely cancels the contribution of $A+B$. The validity of the scattering approximation has therefore been demonstrated.

It is clear that a method of avoiding the rather elaborate calculation described here would be desirable, and a considerable effort in this direction has been expended. The calculation could be avoided if a method could be found for preventing the appearance of infrared divergences in the scattering approximation. The use of the transverse gauge for the electromagnetic field offers some promise in this direction. However, because of the fact that the charge renormalizations appear in a much less recognizable form, and because of the

generally much greater complexity of the transverse gauge, it does not appear that any significant amount of labor could be saved by such an approach even if it were to be successfully carried through.

CORRECTIONS TO THE SCATTERING APPROXIMATION

In the previous sections it has been shown that there are no corrections to the scattering approximation for $\Delta E^{(4)}$ which are of the same order in αZ as the scattering approximation itself. On the other hand, there are corrections one order in αZ smaller. While it appears very unlikely that these are experimentally significant, it is not certain that this is the case. A few brief remarks concerning their evaluation will therefore be made.

In particular we note that the problem here is very similar to that discussed by BBF with reference to the second-order problem. Consequently a procedure very similar to theirs can be followed. Consider, for example, $\langle M_1{}^{(4)}\rangle$, as split into two terms A_0 and A_1. We have shown that, to order $\alpha^2(\alpha Z)^4$, A_0 contributes renormalization terms, $(\gamma\cdot A^e)^2$ terms, and scattering approximation terms. We now assert that, in complete analogy with the BBF problem, there are no corrections to A_0 of order αZ smaller. This comes about as a result of the special form of the denominators of the F and $\mathfrak{F}$ operators. Following the procedure of Kroll and Pollock, one can always express the αZ corrections as the square of the wave function at the origin multiplied by an integral over the momentum space of the potential. Thus, a typical term of A_0 might, after application of the Dirac equation, have the form

$$\int \bar{\varphi}_a(\mathbf{p}_2)A_\rho{}^e(\mathbf{p}_2-\mathbf{p}_3)K_{\rho\sigma}(p_3)A_\sigma{}^e(\mathbf{p}_3-\mathbf{p}_1)\varphi_a(\mathbf{p}_1)(d\mathbf{p})^3$$

$$\approx \int \bar{\varphi}_a(\mathbf{p}_2)d\mathbf{p}_2\int \varphi_a(\mathbf{p}_1)d\mathbf{p}_1\int A_\rho{}^e(-\mathbf{p}_3)$$

$$\times K_{\rho\sigma}(p_3)A_\sigma{}^e(\mathbf{p}_3)d\mathbf{p}_3. \quad (51)$$

For the Coulomb field,

$$\int A_\rho{}^e(-\mathbf{p})K_{\rho\sigma}(p)A_\sigma{}^e(\mathbf{p})d\mathbf{p}\propto \mathcal{P}\int_0^\infty K(p^2)p^{-2}dp.$$

This last integral vanishes since $K(p^2)/p^2$ turns out to be sufficiently well behaved as $p^2\to 0$, and $K(p^2)$ may be written $\int G(p^2;z_1,z_2,\cdots,z_r)dz_1dz_2\cdots dz_r$, where G is an analytic function of p^2 with no singularities except poles on the real axis. (The integral is to be evaluated as a principal part.) Integrals of this sort vanish identically, as long as the integrands are not singular at infinity.

Thus, for $M_1{}^{(4)}$ and $M_2{}^{(4)}$, the αZ corrections come only from A_1 and B_1. These may always be evaluated by replacing $S_F{}^e$ by S_F everywhere, and setting $\mathbf{p}_1$ and $\mathbf{p}_2$, the momenta of the initial and final states, equal to zero everywhere except in $\bar{\varphi}_a(\mathbf{p}_2)$ and $\varphi_a(\mathbf{p}_1)$. One immediately obtains the square of the wave function at the origin multiplied by a definite integral of typical form. Our previous calculation guarantees that all low-momentum divergences will cancel when all terms are combined.

It should, of course, be noted that various contributions, such as $(MP)_2$, which were not considered here because they do not contribute to the scattering approximation, do contribute to the αZ corrections. Methods very similar to those just discussed can again be applied.

The aforementioned remarks should indicate that a method exists for writing down all αZ corrections in terms of the wave function at the origin and a definite integral involving the square of the Fourier transform of the potential, two photon propagation functions, and a finite number of known functions which are identical with or similar to free-electron propagation functions. There are, of course, very many such terms, so that an actual evaluation of the corrections appears to be a far lengthier task than was the evaluation in the case of the scattering approximation.