APPENDIX

The density expansion of $\bar{H}$, for a hard sphere gas of Fermi particles, follows from the expansion of (18), (22), and (23) in powers of $(k_f r_{ij})^2$:

$$l(k_f r) = 1 - (k_f r)^2/10 + (k_f r)^4/280 - \cdots,$$

$$Q_{\alpha_i\alpha_j} = [(k_{\alpha_i} - k_{\alpha_j})^2/6]\int r^2 h(r) d^3r - [(k_{\alpha_i} - k_{\alpha_j})^4/120]\int r^4 h(r) d^3r + \cdots.$$

Inserting these expressions in (21)–(24) and carrying out the sums over k-space, we obtain the following series in n:

$$(2m/h^2)(\bar{T}_c/N) = n\int d^3r_{12} f^2(r_{12})[V^2 f(r_{12})/f(r_{12})] \times[(6\pi^2/5)^{\frac{2}{3}} r_{12}^2 n^{\frac{2}{3}} - (3/175)(6\pi^2)^{4/3} r^4 n^{4/3} + \cdots] \quad (29a)$$

$$+n^2\int d^3r_{12} d^3r_{13} f^2(r_{12})[\nabla^2 f(r_{12})/f(r_{12})]h(r_{13})h(r_{23}) \times[(6\pi^2)^{4/3} n^{4/3}(1/280 - 3/175)(r_{12}^4 + r_{13}^4 + r_{23}^4) + (2/100)(r_{12}^2 r_{13}^2 + r_{12}^2 r_{23}^2 + r_{13}^2 r_{23}^2) + \cdots]. \quad (29b)$$

$$(2m/h^2)(\bar{T}_F/N) = (3/5)(6\pi^2)^{\frac{2}{3}} n^{\frac{2}{3}} \quad (30a)$$

$$+6(6\pi^2)^{4/3}(1/50 - 23/1050) n^{7/3}\int h(r) d^3r + \cdots, \quad (30b)$$

$$(2m/\hbar^2)(\bar{T}_{FC}/N) = (6/5)(6\pi^2)^{\frac{2}{3}} n^{\frac{2}{3}} \times\int r_{12} f(r_{12}) f'(r_{12}) d^3r_{12} + \cdots, \quad (31)$$

$\bar{V}/N$: replace $(\hbar^2/2m)(\nabla^2 f/f)$ by V in (33a). (32)

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Statistical Model for High-Energy Events*

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The relative probabilities for alternate processes initiated by a nucleon-nucleon collision depend on the dynamics involved and on the volume in phase space accessible to each final state. The assignment of relative *a priori* probabilities to the final states proportional to their extension in phase must be consistent with the translational, rotational, and Lorentz-invariant properties of the colliding system. The latter in particular implies a conservation law for the center of energy. Its effect is not only to lower the power of the configurational volume by one dimension but also to severely reduce the contributions from high momenta to the phase space integrals.

The limitations on accessibility arising from the controllable constants of motion are not sufficient to insure well defined probabilities. Some additional restriction on the configurational part of the phase space must be imposed. A cutoff factor for *each* particle is accordingly introduced. The configurational volume accessible to the particle thus decreases with increasing energy, a picture not inconsistent with the uncertainty principle.

THIS note describes a statistical model which differs in some essential respects from the one proposed by Fermi.[1]

Following Fermi we assume that in a high-energy collision a state approximating that of equilibrium is established. The probability of disintegration into various possible modes is then taken proportional to their relative extensions in accessible phase space. The limitations on accessibility arise from the assumed controllable constants of motion.

In this note they are taken to be energy, momentum, center of energy (the relativistic analog of center of mass), and isotopic spin. For simplicity conservation of angular momentum has been neglected.

If it is assumed that the extension in accessible phase space (in the center-of-momentum system) corresponding to particles of masses $M_1, M_2, \cdots$ can be approximated by the classical phase integral divided by $h^{3(n-1)}$,

$$P_n = \frac{S_n T_n}{(2\pi\hbar)^{3(n-1)}}\int \prod_{i=1}^{i=n} d\mathbf{p}_i d\mathbf{x}_i \delta(E - \sum_i E_i) \times\delta(-\sum \mathbf{p}_i)\delta\left(\frac{\sum \mathbf{x}_i E_i}{E}\right), \quad (1)$$

one sees immediately that this integral does not converge and therefore some additional restriction is necessary to give the phase integral a well-defined meaning. This difficulty is overcome in the quantum theory by enclosing the system in a container whose walls aer eventually removed to infinity since (having been

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] E. Fermi, Progr. Theoret. Phys. (Japan) 5, 4, 570 (1950).

cancelled by the normalization factor in the wave functions) its volume does not appear in any physically significant context. This procedure cannot be followed, however, in our case. Imagine a light and a heavy particle with center of mass fixed starting out from the origin and moving according to Newtonian mechanics in opposite directions. When the lighter particle reaches the boundary of the container and is reflected, the heavier one must also turn back, even though it is still in the interior of the region, in order that the center of mass may be conserved. Thus, the conservation of center of mass and a finite volume, the same for light and heavy particles, give rise to unwanted spatial correlations amounting to interactions with the walls of the imaginary container. Since these cannot have any physical basis, this trivial example suggests that one enclose the particles in spheres whose radii decrease in some manner with increasing mass. In a nonrelativistic theory such a restriction may be imposed by inserting into the spatial part of the phase integral factors $f_i(M_i,x_i)$ which are to be chosen in such a way to make the integral well defined. A simple physical motivation can be supplied for such factors by an appeal to the uncertainty principle. One may imagine that at the instant of the collision between two colliding nucleons a virtual cloud of particles of various masses is formed of which that portion due to particles of mass M_i extends to a distance $\hbar/M_ic$. This suggests that one represent f_i by a Kennard packet

$$f_i=\exp(-x_i^2\kappa_i^2\tau_i).$$

κ_i is the Compton wavelength appropriate to the mass M_i, and τ_i is some scaling factor. Of course any other monotonic function with similar properties would do as well. This choice does, however, simplify the calculations and the results should not depend too drastically on the choice of $f_i(M_i,x_i)$.

In a relativistic theory, however, center of energy rather than center of mass is conserved, so if different cutoffs in configurational volume for the various particles are introduced these cutoffs must be energy-dependent. Relativistically, the notion that the functions f_i represent the extension of a virtual cloud of particles with energy E_i implies that they should have a range $\hbar c/E_i$. For simplicity one may choose

$$f_i(E_i,x_i)=\exp\left(-x_i^2\frac{E_i^2}{\hbar^2c^2}\tau_i\right).$$

As before, this choice is arbitrary to the extent that any other monotonic function with similar properties would also be suitable. Nonrelativistically the assumption of absence of fictious correlations (i.e., interactions with the walls of an imaginary container) has the simple effect of multiplying the momentum space part of the phase integral by a factor depending on the ratios of the masses of various particles. Relativistically, however, the energy-dependent range of the virtual cloud introduces a drastic modification in the momentum integral.

No mention has so far been made of any contraction factor such as that which occupies a prominent position in Fermi's statistical model. This indeed is absent here. What one gets instead is something like a uniform shrinkage of the configurational volume with increasing energy due to the energy-dependent cutoffs. The effect of shrinkage of configurational volume with energy is of course more significant for a light particle than for a heavy one. As a result one finds in calculating the energy spectrum of a single emitted meson in a nucleon-nucleon collision that low-meson-energy emissions are favored above the high ones.

A quantitative formulation of the preceding discussion is now given. The expression for the number of states per unit energy interval in the accessible phase space as restricted by conservation of energy, mo-

TABLE I. Values of P_n for a nucleon-nucleon collision when $S_n=T_n=\tau_{(n)}=1$. The last column indicates the number of nucleons and pions which the reaction yields.

E	2.5	6.3	10	Process
P_2	0.030	0.025	0.022	$2N$
P_3	0.012	0.022	0.022	$2N$, 1π
P_4	0.0043	0.006	0.0089	$2N$, 2π
P_4'	0	0.000001	0.00022	$4N$

TABLE II. Values of $P_n/\tau_{(n)}^{\frac{1}{2}(n-1)}$ for a p-p collision at three energies.

	1.5	6.3	10
pp	0.015	0.0125	0.011
pp^0	0.003	0.0055	0.0055
pn^+	0.018	0.033	0.033
pp^{+-}	0.00026	0.0036	0.0053
pp^{00}	0.000043	0.0006	0.00089
pn^{+0}	0.00077	0.0108	0.016
nn^{++}	0.000065	0.0009	0.0013

TABLE III. Values of $P_n/\tau_n^{\frac{1}{2}(n-1)}$ for an n-p collision in a $T=0$ isotopic spin state at three energies.

	1.5	6.3	10
pn	0.030	0.025	0.022
pp^-	0.002	0.0037	0.0037
pn^0	0.004	0.0073	0.0073
nn^+	0.002	0.0037	0.0037
pp^{-0}	0.00072	0.001	0.0015
pn^{00}	0.00072	0.001	0.0015
pn^{+-}	0.00043	0.006	0.0089
nn^{+0}	0.00072	0.001	0.0015

mentum, and center of energy may be written as

$$P_n=\frac{1}{(2\pi)^{3(n-1)}}\frac{(2K)^4}{2c\hbar K}S_n\cdot T_n\prod_{i=1}^{i=n}$$
$$\times\int \mathbf{x}d\mathbf{k}_i\delta(2K-\sum_i(k_i^2+\kappa_i^2)^{\frac{1}{2}})\delta(-\sum\mathbf{k}_i)$$
$$\times\int \mathbf{x}d\mathbf{x}_i\exp[-\tau_i x_i^2(k_i^2+\kappa_i^2)^{\frac{1}{2}}]$$
$$\times\delta(\sum\mathbf{x}_i(k_i^2+\kappa_i^2)^{\frac{1}{2}}). \quad (2)$$

In this formula the index n refers to n particles with Compton wavelengths $\kappa_1, \kappa_2, \cdots\kappa_n$. The letter S_n denotes a constant whose exact value depends on the number of indistinguishable groups and the population of each occurring among the n particles. The isotopic spin weight factors are denoted by T_n. The symbol $\boldsymbol{K}$ stands for the energy per incident nucleon in the center of energy system. A gaussian cutoff with an, as yet, arbitrary parameter τ_i for the configuration space of each particle has been introduced in the integrand. The quantities κ_i, $\boldsymbol{K}$, $\mathbf{k}$ have the dimensions of reciprocal length. Thus the integral has the dimensions of the product of four delta functions, or that of $(2\kappa)^{-4}$. The weight factors S_n and T_n are of course dimensionless. Carrying out the integration over configuration space we obtain

$$P_n=\frac{S_nT_n}{(4\pi)^{\frac{3}{2}(n-1)}}\frac{(2K)^4}{\hbar c(2K)}\cdot\frac{1}{\sum_{i=1}^{n}\frac{1}{\tau_i}}\prod_{i=1}^{n}\frac{1}{\tau_i^{\frac{3}{2}}}$$
$$\times\int\frac{d\mathbf{k}_i}{(k_i^2+\kappa_i^2)^{\frac{3}{2}}}\delta(2K-(k_i^2+\kappa_i^2)^{\frac{1}{2}})\delta(-\sum\mathbf{k}_i), \quad (3)$$

where the τ_i's are constants, independent of energy. Looking back at (3) we see that the size of the "wave packet" of a particle is determined by its energy and the τ_i's play the role of intrinsic scaling factors. To simplify the formula we introduce a reduced scaling factor for the system of n particles

$$\frac{1}{\tau_{(n)}{}^{n-1}}=\frac{1}{n}\sum_{i=1}^{n}\tau_i\Big/\prod_{i=1}^{n}\tau_i, \quad (4a)$$

and define a quantity

$$Q_n=\left(\frac{\tau_{(n)}}{4\pi}\right)^{\frac{3}{2}(n-1)}\frac{1}{n^{\frac{3}{2}}}. \quad (4b)$$

Equation (2) then assumes the form

$$P_n=Q_nS_nT_nJ_n, \quad (5a)$$

where

$$J_n=\frac{(2K)^4}{\hbar c2K}\int\prod_{i=1}^{n}\frac{d\mathbf{k}_i}{[k_i^2+\kappa_i^2]^{\frac{3}{2}}}$$
$$\times\delta(2K-\sum(k_i^2+\kappa_i^2)^{\frac{1}{2}})\delta(-\sum\mathbf{k}_i). \quad (5b)$$

The nonrelativistic limit of expression (5) is not essentially different from that of Fermi's model with the conservation of linear momentum.[2] A short calculation carried out along the lines of reference 2 yields

$$P_n=\tau_n{}^{\frac{3}{2}(n-1)}S_nT_n\frac{1}{2}\frac{1}{mc^2}\frac{1}{n^{\frac{3}{2}}}\frac{1}{(\frac{3}{2}n-5/2)!}\left(\frac{T}{mc^2}\right)^{\frac{3}{2}}, \quad (6)$$

where

$$\frac{1}{m^{n-1}}=\sum_{i=1}^{n}m_i\Big/\prod_{i=1}^{n}m_i,$$

and T is the kinetic energy per colliding nucleon in the center of mass system.

It is readily seen, however, that the factor $[k_i^2+\kappa_i^2]^{-\frac{3}{2}}$ will introduce essential modifications in the high energy domain. These will show up primarily in the shape of the energy spectrum of the emitted mesons. For a fixed multiplicity, this is independent of any uncertainties in the factor Q_n. Figures 1, 2, and 3 show the general

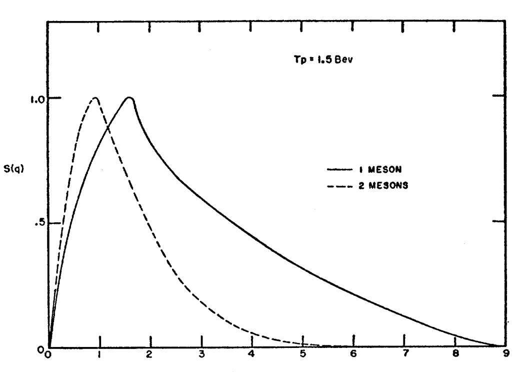

FIG. 1. Momentum spectra of singly and doubly emitted mesons in the laboratory system for incident nucleons with 1.5-Bev kinetic energy.

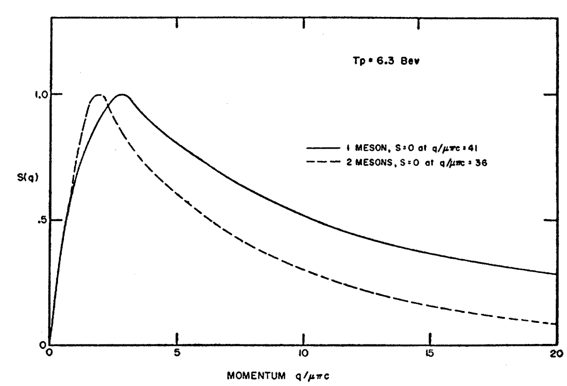

FIG. 2. Momentum spectra of singly and doubly emitted mesons in the laboratory system for incident nucleons with 6.3-Bev kinetic energy.

[2] J. V. Lepore and R. Stuart, Phys. Rev. 94, 1724 (1954).

TABLE IV. Values of $P_n/\tau_{(n)}^{\frac{3}{2}(n-1)}$ for an n-p collision in a $T=1$ isotopic spin state at three energies.

	1.5	6.3	10
pn	0.030	0.025	0.022
pp^-	0.003	0.0055	0.0055
pn^0	0.012	0.022	0.022
nn^+	0.003	0.0055	0.0055
pp^{-0}	0.00017	0.0024	0.0036
pn^{00}	0.00013	0.0018	0.0027
pn^{+-}	0.00077	0.011	0.016
nn^{+0}	0.00017	0.0024	0.0036

tendency for mesons to be emitted with low rather than high energies. This is a general feature of the model resulting from the fact that high-energy mesons have less configurational volume available to them than the low-energy ones. This feature is present in single- as well as multiple-meson production. An experiment carried out under conditions where single-meson production would be expected to dominate would therefore be of some interest.

At higher energies where multiple production is also possible the predictions of the model would depend on the assumed values of the scaling factors τ_π and τ_N that enter into the definition of Q_n. Explicitly

$$Q_2(nn)=(\tau_N/8\pi)^{\frac{3}{2}},$$
$$Q_3(NN\pi)=\left(\frac{\tau_N}{4\pi}\right)^3\bigg/\left(1+2\frac{\tau_N}{\tau_\pi}\right)^{\frac{3}{2}}, \tag{7}$$
$$Q_4(NN2\pi)=\left(\frac{\tau_N}{4\pi}\right)^{9/2}\bigg/\left[2\frac{\tau_N}{\tau_\pi}\left(1+\frac{\tau_N}{\tau_\pi}\right)\right]^{\frac{3}{2}}.$$

It is seen from these expressions that even if $\tau_n=\tau_\pi=\tau$, the assumed value of this single parameter will still effect the predictions of the model in an essential manner.

In view of the general qualitative agreement of the shapes of the spectra with those observed at the Brookhaven Cosmotron, a further attempt might be made to account for the observation in greater detail by fixing one or two parameters ($\tau_\pi=\tau_N=\tau$, or τ_N, τ_π/τ_N)

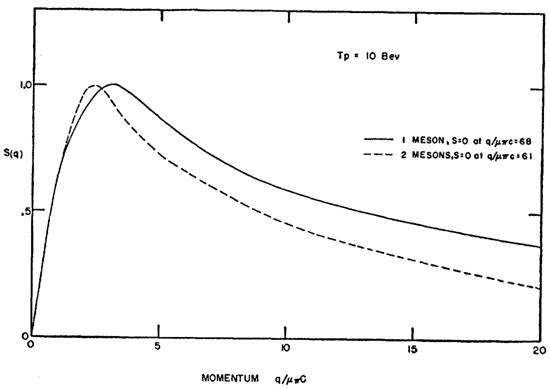

FIG. 3. Momentum spectra of singly and doubly emitted mesons in the laboratory system for incident nucleons with 10-Bev kinetic energy.

empirically on the basis of single-meson production. With a view towards such a program when further data become available we include tables with some computed values of the expressions appearing in Eq. (5). Table I lists the values of P_n when $S_n=T_n=\tau_{(n)}=1$, Table II those of $P_n/(\tau_n)^{\frac{3}{2}(n-1)}$ for a p-p collision, Table III the same quantity for an n-p collision if the colliding particles are in an isotopic spin state $T=0$, and Table IV if they are in a state $T=1$. Columns are labeled by the kinetic energy (in Bev) of the incident nucleon in the laboratory system.

In constructing Table I we reduced J_4 to a double, J_3 to a single integral. J_2 can be expressed analytically in terms of elementary functions. The remaining integration were carried out numerically on the IBM Card Program Calculator at the Livermore site of this laboratory. Tables II, III, and IV were constructed using Table I and the numerical coefficients T_n calculated by Fermi.[3]

It may perhaps be worth-while to point out that the probability for the production of a nucleon pair in a nucleon-nucleon collision is seen from Table I to be exceedingly small even at 10 Bev.

We are grateful to Miss H. Cox and Mrs. M. Harrison for carrying out the computations.

[3] E. Fermi, Phys. Rev. 92, 452 (1953).