phase shifts among the sets which reproduce the cross section. An estimate of a can be obtained only from an at least qualitative theory of the nature of the interaction. It is commonly believed, for instance, that the pion-nucleon scattering proceeds via absorption and re-emission of the pion by the nucleon. If this is the case, a will be of the order of the pion Compton wavelength, i.e., $\hbar/\mu c$. However, it would be quite difficult to tell to what extent $a=2\hbar/\mu c$ is a permissible choice or whether it is necessary to assume $a=3\hbar/\mu c$ or an even larger a, giving less and less stringent forms to (4b). An alternative form of applying the relations of this paper, which might be somewhat more free of this ambiguity, would be to plot R, as calculated from (3b), against the energy and to judge whether any possible deviation of the R obtained this way from a regular R function can be blamed on having assumed a too low value for a.

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Scattering of Polarized Nucleon Beams*

REINHARD OEHME
Institute for Nuclear Studies, University of Chicago, Chicago, Illinois
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The polarization formulas for the general reaction $a+b\rightarrow c+d$ are given in a compact form and specialized for the scattering of polarized nucleon beams at unpolarized targets. These targets may have arbitrary spin, possibly being different before and after collision. An interesting quantity in this problem is the polarization of the scattered nucleon beam. On the basis of general invariance properties of the transition matrix, this polarization is expressed by the polarization of the incoming nucleon beam and the relative momenta before and after collision. The invariant coefficients in this relation are functions of energy and scattering angle only; they are given in terms of the parameters of the transition matrix.

By using these formulas, it is shown that with triple scattering experiments one can obtain two new relations between the parameters of the transition matrix at fixed energy and angle. Quadruple scattering leads to two further relations. These relations represent information in addition to the differential cross section and the polarization resulting from unpolarized beams. The results are specialized for targets of spin zero and spin one-half, where in the latter case also the scattering of identical particles is discussed briefly.

I. INTRODUCTION

A BEAM of identical free particles with equal momentum represents in general a quantum-mechanical mixture, i.e., a classical statistical ensemble of different pure states.[1] We can characterize such an ensemble by the contributing pure states Ψ and their relative abundance $W(\Psi)$, where $\sum_\Psi W(\Psi)=1$. According to the usual rules of probability the expectation value of any operator ω in this beam is given by

$$\langle\omega\rangle_{\text{beam}}=\sum_\Psi W(\Psi)(\Psi,\omega\Psi).$$

Decomposing Ψ with respect to a complete set of orthogonal eigenstates Φ_k, $\Psi=\sum_k a_k(\Psi)\Phi_k$, we can write this expectation value in the form

$$\langle\omega\rangle_{\text{beam}}=\mathrm{Tr}(\rho\omega)=\sum_{ik}\left(\sum_\Psi W(\Psi)a_i(\Psi)a_k^*(\Psi)\right)(\Phi_k,\omega\Phi_i).$$

Here ρ is the density matrix[1] of the beam. If the particles have spin s and differ only by their spin state, then ρ is a Hermitian $2s+1$ by $2s+1$ matrix in spin-space. From the normalization of the state vectors to one follows $\mathrm{Tr}\rho=1$.

It is useful to expand the density matrix in terms of a complete set of $(2s+1)^2$ basic Hermitian matrices ω^μ in spin space,[2] which obey the relation

$$\mathrm{Tr}(\omega^\mu\omega^\nu)=(2s+1)\delta_{\mu\nu}.$$

These ω^μ are related to the irreducible spin tensor moments $T_k^{(q)}$, where q is the rank of the tensor and $k(|k|\leq q\leq 2s)$ indicates its components.[3,4] The tensor moments are not Hermitian operators as the ω^μ, but they transform directly according to the representation $\mathfrak{D}^{(q)}$ of the three dimensional rotation group. Using the Hermitian matrices ω^μ we can express the density matrix by the expectation values of all basic matrices ω^μ in the beam:

$$\rho=\frac{1}{2s+1}\sum_{\mu=1}^{(2s+1)^2}\langle\omega^\mu\rangle_{\text{beam}}\omega^\mu.$$

We say a beam is completely unpolarized if the expectation values of all tensor moments and therefore all operators ω^μ vanish, except the expectation value of the zero rank tensor $T_0^{(0)}=\omega^1=1$.

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[1] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932 and Dover Publications, New York, 1943), p. 174; and H. Weyl, *Theory of Groups and Quantum Mechanics* (Dover Publications, New York, 1931), p. 78.

[2] See, for example, U. Fano, Phys. Rev. **90**, 577 (1953); and F. Coester and J. M. Jauch, Helv. Phys. Acta **26**, 3 (1953); these papers contain further references.

[3] E. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (F. Vieweg, Braunschweig, 1953 and Edwards Brothers, Ann Arbor, 1944), p. 263.

[4] G. Racah, Phys. Rev. **62**, 442 (1942).

II. REACTION $a+b \rightarrow c+d$ WITH POLARIZED BEAMS[5,6]

In the general reaction $a+b \rightarrow c+d$ we can describe the system before collision either by the density matrices $\rho_{\text{in}}^{(a)}$ and $\rho_{\text{in}}^{(b)}$ of the two independent beams of free particles with spin s_a and s_b respectively,[7] or by a density matrix in the combined spin space of the two particle system. In the latter case the expectation value of any operator Ω is

$$\langle\Omega\rangle_{\text{in}} = \mathrm{Tr}(\rho_{\text{in}}\Omega), \tag{1}$$

provided $\mathrm{Tr}\rho_{\text{in}}=1$. Let Ω^μ be a complete set of $(2s_a+1)^2(2s_b+1)^2$ basic Hermitian matrices in this combined spin space with the property

$$\mathrm{Tr}(\Omega^\mu\Omega^\nu) = (2s_a+1)(2s_b+1)\delta_{\mu\nu}. \tag{2}$$

We can expand ρ_{in} in terms of these operators and find with Eqs. (1) and (2):

$$\rho_{\text{in}} = \frac{1}{(2s_a+1)(2s_b+1)} \sum_\mu \langle\Omega^\mu\rangle_{\text{in}}\Omega^\mu. \tag{3}$$

The matrices Ω^μ are the direct products $\Omega^\mu = \omega_a{}^\alpha \times \omega_b{}^\beta$ of the corresponding basic matrices $\omega_a{}^\alpha$ and $\omega_b{}^\beta$ in the spin spaces of the particles a and b respectively. If the expectation values of all basic operators in the incoming beams are known, the density matrix ρ_{in} is completely determined.

The system after collision consists of the two beams of particles c and d, having spin s_c and s_d respectively. These final beams may be completely characterized by a density matrix ρ_{out} in the combined spin space of the particles c and d. If ρ_{in} is known and also the transition matrix $T(a+b \rightarrow c+d)$, which transforms every initial pure state contained in the quantum-mechanical mixture described by ρ_{in} into a corresponding final state in the mixture ρ_{out}, we find for the density matrix of the outgoing beams

$$\rho_{\text{out}} = T\rho_{\text{in}}T^\dagger. \tag{4}$$

The matrix ρ_{out} is not normalized to one, but $\mathrm{Tr}\rho_{\text{out}}$ gives the differential cross section of the reaction. The expectation value of any operator Ω' in the combined spin space of the particles c and d after collision is then

$$\langle\Omega'\rangle_{\text{out}} = \mathrm{Tr}(\rho_{\text{out}}\Omega')/\mathrm{Tr}\rho_{\text{out}}. \tag{5}$$

Using the expansion (3) of ρ_{in} with respect to the operators Ω^μ, we find for the differential cross section

$$Q = \frac{1}{(2s_a+1)(2s_b+1)} \sum_\mu \langle\Omega^\mu\rangle_{\text{in}} \,\mathrm{Tr}(T\Omega^\mu T^\dagger); \tag{6}$$

and with Eq. (5) the expectation value of an operator Ω' becomes:

$$\langle\Omega'\rangle_{\text{out}} = \frac{\sum_\mu \langle\Omega^\mu\rangle_{\text{in}} \,\mathrm{Tr}(T\Omega^\mu T^\dagger\Omega')}{\sum_\nu \langle\Omega^\nu\rangle_{\text{in}} \,\mathrm{Tr}(T\Omega^\nu T^\dagger)}. \tag{7}$$

Equation (7) expresses the expectation value of any operator Ω' in the outgoing beams by the expectation values of the complete set of basic matrices Ω^μ in the incoming beams. The relations (6) and (7) give a complete description of all physically measurable quantities for the general reaction $a+b \rightarrow c+d$ in terms of the transition matrix; they hold for particles of arbitrary spin and beams with any polarization.

III. NUCLEONS SCATTERED AT UNPOLARIZED TARGETS†

Let us now consider the special case of particles with spin one-half scattered at unpolarized targets with spin I before and spin I' after collision. The basic matrices $\Omega^\mu = \omega^\alpha(\frac{1}{2}) \times \omega^\beta(I)$ in the combined spin space of the incoming particles are then of the form $1 \times \omega^\beta(I)$ and $\boldsymbol{\sigma} \times \omega^\beta(I)$. Since the targets are initially unpolarized, only the matrices 1×1 and $\boldsymbol{\sigma}\times 1$ have nonzero expectation values before collision. If T is the transition matrix, we obtain the cross section for the process $a(\frac{1}{2})+b(I, \text{unpolarized}) \rightarrow c(\frac{1}{2})+d(I')$ by specialization of Eq. (6). It is of the form

$$Q^{pu} = \frac{1}{2(2I+1)}\{\mathrm{Tr}(TT^\dagger) + \mathrm{Tr}(T\boldsymbol{\sigma}T^\dagger)\cdot\mathbf{P}^{\text{in}}\}, \tag{8}$$

where we have written $\boldsymbol{\sigma}$ for $\boldsymbol{\sigma}\times 1$ and $\mathbf{P}^{\text{in}}$ for the expectation value $\langle\boldsymbol{\sigma}\times 1\rangle_{\text{in}}$. The expectation value of $\boldsymbol{\sigma}$ in the out-going nucleon beam becomes according to Eq. (7):

$$\mathbf{P}^{pu} \equiv \langle\boldsymbol{\sigma}\times 1\rangle_{\text{out}} = \frac{\mathrm{Tr}(T^\dagger\boldsymbol{\sigma}T) + \mathrm{Tr}(\boldsymbol{\sigma}T\boldsymbol{\sigma}\cdot\mathbf{P}^{\text{in}}T^\dagger)}{\mathrm{Tr}(TT^\dagger) + \mathrm{Tr}(T\boldsymbol{\sigma}T^\dagger)\cdot\mathbf{P}^{\text{in}}}. \tag{9}$$

In the special case of an unpolarized incident nucleon beam the quantities Q^{pu} and $\mathbf{P}^{pu}$ reduce to

$$Q^{uu} = \frac{1}{2(2I+1)}\mathrm{Tr}(TT^\dagger),$$

and

$$\mathbf{P}^{uu} = \mathrm{Tr}(T^\dagger\boldsymbol{\sigma}T)/\mathrm{Tr}(TT^\dagger). \tag{10}$$

In Eqs. (8) and (10) the traces containing $\boldsymbol{\sigma}$ are *a priori* different. But Wolfenstein and Ashkin[6] have shown that these traces must be equal as a consequence of the invariance of the transition matrix with respect to inversion of motion as well as to the usual invariance

[5] See A. Simon and T. A. Welton, Phys. Rev. **90**, 1036 (1953), and A. Simon, Phys. Rev. **92**, 1050 (1953) for a treatment in terms of Fano X-functions.

[6] L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 847 (1952); these authors discuss the special case of particles with spin one-half.

[7] We assume always that incident particle and target particle are incoherent.

† *Note added in proof.*—After having submitted this paper the author was informed about the independent work of L. Wolfenstein [Phys. Rev. **96**, 1654 (1954)], which contains discussions similar to those given in Secs. III, V, and VI. The author is indebted to Professor Wolfenstein for sending him a copy of this paper and for bringing to his attention a surplus term originally contained in Eq. (19) of the present paper.

properties. The consequences derived from inversion of motion are applicable only for pure elastic scattering. Using this we may write Eq. (8) in the form

$$Q^{pu}=Q^{uu}(1+\mathbf{P}^{uu}\cdot\mathbf{P}^{\text{in}})=Q^{uu}(1+A), \tag{11}$$

where $A\equiv\mathbf{P}^{uu}\cdot\mathbf{P}^{\text{in}}(|A|\leq 1)$ is the asymmetry of the scattering. Furthermore, the polarization $\mathbf{P}^{pu}$ can be written:

$$\mathbf{P}^{pu}=\mathbf{P}^{uu}+\mathfrak{T}\mathbf{P}^{\text{in}}/(1+\mathbf{P}^{uu}\cdot\mathbf{P}^{\text{in}}). \tag{12}$$

The components of the real second rank tensor $\mathfrak{T}$ are

$$\mathfrak{T}_{ik}=\text{Tr}(\sigma_i T\sigma_k T^\dagger)/\text{Tr}(TT^\dagger). \tag{13}$$

Invariance of the transition matrix under inversion of motion etc., does not imply the symmetry of this tensor.

We also may ask for the expectation values of spin and higher tensor moments in the beam of the recoiling target particles. From Eq. (7) we find

$$\langle\omega^\beta(I')\rangle_{\text{out}}=\frac{\text{Tr}(T^\dagger\omega^\beta(I')T)+\text{Tr}(\omega^\beta(I')T\boldsymbol{\sigma}\cdot\mathbf{P}^{\text{in}}T^\dagger)}{\text{Tr}(TT^\dagger)+\text{Tr}(T\boldsymbol{\sigma}T^\dagger)\cdot\mathbf{P}^{\text{in}}}, \tag{14}$$

but let us confine our attention to the outgoing Fermion beam.

The polarization $\mathbf{P}^{uu}$ resulting from an unpolarized incident nucleon beam can only be a function of the relative momenta $\mathbf{k}_i$ and $\mathbf{k}_f$ of the particles before and after collision. Because $\mathbf{P}^{uu}$ is an expectation value of a spin operator, it must transform like a pseudovector under rotations and inversions of the observer's coordinate system. Therefore, since the cross product of $\mathbf{k}_i$ and $\mathbf{k}_f$ is the only pseudovector one can form from these vectors, we have

$$\mathbf{P}^{uu}=P^{uu}\mathbf{n},$$

with

$$\mathbf{n}=(\mathbf{k}_i\times\mathbf{k}_f)/|\mathbf{k}_i\times\mathbf{k}_f|. \tag{15}$$

The invariant quantity P^{uu} is a function of the energy and the scalar product $(\mathbf{k}_i\cdot\mathbf{k}_f)$. Equation (15) expresses the well-known fact that the polarization resulting from the scattering of an unpolarized beam by an unpolarized target is always orthogonal to the plane of scattering.[5,8,9] As a consequence of this, measurements of the asymmetry $A=P^{uu}(\mathbf{n}\cdot P^{\text{in}})$ in the scattering of a polarized beam of nucleons from an unpolarized target can only give information about the components of the polarization $\mathbf{P}^{\text{in}}$ orthogonal to the direction of propagation of that beam.

IV. GENERAL FORM OF $\mathbf{P}^{pu}$

In the scattering problem considered here the transition matrix depends on $\mathbf{k}_i$ and $\mathbf{k}_f$, the matrix $\boldsymbol{\sigma}$ and the components of the spin matrices $\mathbf{S}(I)$ and $\mathbf{S}(I')$ of the target particles before and after collision. The vectors $\mathbf{k}_i$ and $\mathbf{k}_f$ are taken as unit vectors in the direction of the corresponding relative momenta in the center-of-mass system. The dependence on the energy in the center-of-mass system shall not be indicated explicitly.

[8] L. Wolfenstein, Phys. Rev. **75**, 1664 (1949).
[9] R. J. Blin-Stoyle, Proc. Phys. Soc. (London) A64, 700 (1951).

TABLE I. Transformation properties of the matrix coefficients in the transition matrix.

	a	b	c	d
Space inversion	Scalar	Scalar	Pseudo-scalar	Pseudo-scalar
Inversion of motion	+	+	−	+

Because T must be invariant under rotations, it can be written in the form

$$T(\mathbf{k}_i,\mathbf{k}_f;\boldsymbol{\sigma},S_i(I),S_k(I'))=a+\boldsymbol{\sigma}\cdot\mathbf{B}, \tag{16}$$

where a and $\mathbf{B}$ are nonquadratic matrices for $I\neq I'$. They depend on $\mathbf{k}_i$, $\mathbf{k}_f$, $S_i(I)$, and $S_k(I')$ and have matrix elements connecting spin space I and spin space I'. We can decompose $\mathbf{B}$ with respect to the three orthogonal unit vectors

$$\mathbf{n}=\frac{\mathbf{k}_i\times\mathbf{k}_f}{\sin\vartheta},\quad \mathbf{m}=\frac{\mathbf{k}_i-\mathbf{k}_f}{2\sin(\vartheta/2)},\quad \mathbf{l}=\frac{\mathbf{k}_i+\mathbf{k}_f}{2\cos(\vartheta/2)},$$

where $\cos\vartheta=(\mathbf{k}_i\cdot\mathbf{k}_f)$ and have

$$\mathbf{B}=b\mathbf{n}+c\mathbf{m}+d\mathbf{l}. \tag{17}$$

The quantities a, b, c, and d are rotation invariant matrices. Considering furthermore the invariance of T under space reflections and with respect to inversion of motion, we find from the known transformation properties of $\boldsymbol{\sigma}$, $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{l}$, that these matrices must transform as indicated in Table I.[10] Note that their behavior under inversion of motion refers only to the case of pure elastic scattering.

If we now express Q^{pu} and $\mathbf{P}^{pu}$ in terms of the matrices a, b, c, and d, there appear only traces of products of two of them. Because these traces are rotation invariant functions of $\mathbf{k}_i$ and $\mathbf{k}_f$, they depend only on $(\mathbf{k}_i\cdot\mathbf{k}_f)$ and therefore must transform as $(S,+)$, i.e., as scalars not changing sign under inversion of motion. As a consequence of this all traces of those matrix products, which do not transform as $(S,+)$, must vanish. The only real coefficients which may appear in observable quantities will be the traces of products of the matrices with their own Hermitian conjugates and the expressions $\text{Tr}(ab^\dagger+ba^\dagger)$ and $i\,\text{Tr}(ab^\dagger-ba^\dagger)$. We find from Eqs. (10) and (13)

$$\begin{aligned} Q^{uu}&=\frac{1}{2(2I+1)}\text{Tr}(aa^\dagger+bb^\dagger+cc^\dagger+dd^\dagger),\\ Q^{uu}\mathbf{P}^{uu}&=\frac{1}{2(2I+1)}\text{Tr}(ab^\dagger+ba^\dagger)\mathbf{n},\\ Q^{uu}(\mathfrak{T}\mathbf{P}^{\text{in}})&=\frac{1}{2(2I+1)}\{\text{Tr}(aa^\dagger-bb^\dagger-cc^\dagger-dd^\dagger)\mathbf{P}^{\text{in}}\\ &\quad+2\,\text{Tr}(bb^\dagger)(\mathbf{n}\cdot\mathbf{P}^{\text{in}})\mathbf{n}+2\,\text{Tr}(cc^\dagger)(\mathbf{m}\cdot\mathbf{P}^{\text{in}})\mathbf{m}\\ &\quad+2\,\text{Tr}(dd^\dagger)(\mathbf{l}\cdot\mathbf{P}^{\text{in}})\mathbf{l}+i\,\text{Tr}(ba^\dagger-ab^\dagger)[\mathbf{n}\times\mathbf{P}^{\text{in}}]\}. \end{aligned} \tag{18}$$

[10] See appendix to reference 6 for similar considerations.

By the use of Eqs. (11), (12), and (18) the quantities Q^{pu} and $\mathbf{P}^{pu}$ can be expressed in terms of the coefficients of the transition matrix.

The last expression in Eq. (18) may be rewritten in the form

$$\mathfrak{T}\mathbf{P}^{\text{in}}=V(\mathbf{n}\cdot\mathbf{P}^{\text{in}})\mathbf{n}+X\{(\mathbf{k}_i\cdot\mathbf{P}^{\text{in}})\mathbf{k}_i+(\mathbf{k}_f\cdot\mathbf{P}^{\text{in}})\mathbf{k}_f\}+Y(\mathbf{k}_f\cdot\mathbf{P}^{\text{in}})\mathbf{k}_i+Z(\mathbf{k}_i\cdot\mathbf{P}^{\text{in}})\mathbf{k}_f. \quad (19)$$

The coefficients in this relation are functions of the energy and of $(\mathbf{k}_i\cdot\mathbf{k}_f)$ and can be easily expressed by the traces mentioned before Eqs. (18). In Eq. (19) the only consequence of the invariance of the transition matrix under inversion of motion is the equality of the coefficients of $(\mathbf{k}_i\cdot\mathbf{P}^{\text{in}})\mathbf{k}_i$ and $(\mathbf{k}_f\cdot\mathbf{P}^{\text{in}})\mathbf{k}_f$. The azimuthal dependence of $\mathbf{P}^{uu}$ and $\mathbf{P}^{pu}$ is exhibited completely in Eqs. (18) and (19).

In principle it is possible to measure the four coefficients V, X, Y, and Z at any given energy and angle. This would give us, aside from Q^{uu} and P^{uu}, four further relations between the parameters of the transition matrix at fixed energy and angle. In order to see to what extent we may measure these four coefficients, we will discuss in the following section special cases of Eq. (19), which can be realized by performing triple and possibly quadruple scattering experiments with nucleons.

V. TRIPLE AND QUADRUPLE SCATTERING

The information obtainable from triple scattering is reduced by the fact that with the scattering of a primary unpolarized beam at unpolarized targets we can only produce a polarization orthogonal to the direction of motion of the scattered beam. Therefore, in order to study the influence of a longitudinal component of $\mathbf{P}^{\text{in}}$ on the polarization $\mathbf{P}^{pu}$ in the outgoing nucleon beam, we must have two previous scatterings to produce such a component. A third scattering then can lead to an additional term in the polarization $\mathbf{P}_3^{pu}$, depending on the longitudinal component of $\mathbf{P}^{\text{in}}=\mathbf{P}_2^{pu}$. A fourth scattering is needed to analyze $\mathbf{P}_3^{pu}$ by measuring the asymmetry $A_4=\mathbf{P}_3^{pu}\cdot\mathbf{P}_4^{uu}$. The production of a nucleon beam with a known longitudinal component of polarization is discussed in Sec. VI. A further restriction in triple scattering experiments is that by measuring the asymmetry in a third scattering we can determine only those components of the polarization $\mathbf{P}_2^{pu}$ which are orthogonal to the direction of motion of the scattered beam in the laboratory system.

Let us now assume that the incident nucleon beam is polarized orthogonal to the plane of scattering: $\mathbf{P}^{\text{in}}=P_n^{\text{in}}\mathbf{n}$. One sees from Eqs. (15) and (19) that $\mathbf{P}^{pu}$ must be also proportional to the normal vector $\mathbf{n}$. By use of Eq. (12), the polarization $\mathbf{P}^{pu}$ becomes

$$\mathbf{P}^{pu}=P_n^{pu}\mathbf{n}=\frac{P^{uu}+VP_n^{\text{in}}}{1+P^{uu}P_n^{\text{in}}}\mathbf{n}. \quad (20)$$

For an incoming nucleon beam which is completely polarized in the direction $\mathbf{n}$, we obtain for the depolarization, which is defined by $D_n\equiv P_n^{pu}-P_n^{\text{in}}$:

$$D_n=\frac{V-1}{1+P^{uu}}.$$

In this expression we have assumed that $P_n^{\text{in}}=+1$. In terms of the parameters of the transition matrix [Eqs. (16) and (17)] the depolarization can be expressed as

$$D_n=-\frac{2}{1+\text{Tr}\{(a+b)(a+b)^\dagger\}/\text{Tr}(cc^\dagger+dd^\dagger)}. \quad (21)$$

D_n is a function of energy and angle; it varies from zero to minus two, which is guaranteed by the positive definiteness of the traces appearing in Eq. (21). Even if the depolarization D_n vanishes in a special case (e.g., if the target has spin zero; Sec. VI), we get a change of polarization for a partially polarized beam, which according to Eq. (20) becomes

$$P_n^{pu}(D_n=0)=\frac{P^{uu}+P_n^{\text{in}}}{1+P^{uu}P_n^{\text{in}}}.$$

The polarization is increased or decreased if P_n^{in} and P^{uu} have the same or the opposite sign respectively.

If, at a certain energy and angle, P^{uu} is known from a double scattering experiment and P_n^{pu} is measured by triple scattering with all three scatterings in the same plane, then we can determine $V(\vartheta)$ and with this the depolarization D_n. Thus we obtain a new relation between the parameters of the transition matrix at this energy and angle. Obviously we do not obtain any new information if the analyzing scattering occurs in a plane orthogonal to the common plane of first and second scattering. In this case $\mathbf{P}_3^{uu}$ is orthogonal to $\mathbf{P}_2^{pu}$, and the symmetry A_3 vanishes.

If we now assume that the incoming nucleon beam is polarized in the plane of scattering, we see from Eq. (19) that also $\mathfrak{T}\mathbf{P}^{\text{in}}$ is a vector in that plane. In this case, according to Eq. (12), the component of $\mathbf{P}^{pu}$ orthogonal to the plane of scattering is independent of the incoming polarization and given by $\mathbf{P}^{uu}$. The asymmetry $(\mathbf{P}^{uu}\cdot\mathbf{P}^{\text{in}})$ occurring in the denominator of Eq. (12) vanishes because $(\mathbf{n}\cdot\mathbf{P}^{\text{in}})$ is zero. Therefore the triple scattering experiment provides no more information than a double scattering experiment, if the first scattering occurs in a plane which is orthogonal to the common plane of second and third scattering. For this arrangement the asymmetry $A_3=\mathbf{P}_2^{pu}\cdot\mathbf{P}_3^{uu}$ becomes equal to $\mathbf{P}_2^{uu}\cdot\mathbf{P}_3^{uu}$.

Of more interest is a triple scattering experiment with producing and analyzing scattering occurring in the same plane, which itself is orthogonal to the plane of the second scattering. Then $\mathbf{P}^{\text{in}}$ is orthogonal to $\mathbf{n}$ and $\mathbf{k}_i$ (the indices referring to the second scattering are always omitted) and for positive P_3^{uu} the vector $\mathbf{P}_3^{uu}$

is parallel to $\mathbf{n}_3=[\mathbf{k}_f(\text{lab})\times\mathbf{n}]/|[\cdot\cdot]|$. $\mathbf{n}_3$ is a unit vector orthogonal to $\mathbf{k}_f(\text{lab})$ and in the plane of the second scattering. Thus we find from Eq. (19) for the special case considered here:

$$\mathbf{P}^{pu}=P^{uu}\mathbf{n}+(\mathbf{k}_f\cdot\mathbf{P}^{\text{in}})\{X\mathbf{k}_f+Y\mathbf{k}_i\}.$$

The asymmetry in a third analyzing scattering becomes

$$A_3=\mathbf{P}^{pu}\cdot\mathbf{P}_3{}^{uu}=P^{\text{in}}P_3{}^{uu}\Lambda,$$

where

$$\Lambda=\sin\vartheta\{X(\mathbf{k}_f\cdot\mathbf{n}_3)+Y(\mathbf{k}_i\cdot\mathbf{n}_3)\} \tag{22}$$

and $\mathbf{P}^{\text{in}}$ is defined by $\mathbf{P}^{\text{in}}=P^{\text{in}}[\mathbf{n}\times\mathbf{k}_i]$. In terms of the coefficients of the transition matrix this is

$$\Lambda=\frac{-1}{2(2I+1)Q^{uu}}\Big\{\text{Tr}(aa^\dagger-bb^\dagger)\cos(\vartheta/2+\chi)+i\,\text{Tr}(ba^\dagger-ab^\dagger)\sin(\vartheta/2+\chi)+\text{Tr}(cc^\dagger-dd^\dagger)\cos(\vartheta/2-x). \tag{22a}$$

Here the angle χ is defined by $\cos\chi\equiv\mathbf{n}_3\cdot\mathbf{m}$ and we have $\chi=0$ if the masses of the incident particles and the target particles are equal. This experiment gives further information about the transition matrix. Note that Λ can be different from zero even if D_n vanishes, i.e., if $\text{Tr}(cc^\dagger)$ and $\text{Tr}(dd^\dagger)$ are both zero.

Because $\mathbf{P}^{pu}(1+A)=\mathbf{P}^{uu}+\mathfrak{T}\mathbf{P}^{\text{in}}$ is a linear vector function of $\mathbf{P}^{\text{in}}$, one can easily see that D_n and Λ are the only independent new relations between the parameters of the transition matrix, which may be obtained by triple scattering experiments with nucleons.

We have yet to discuss the influence of a longitudinal component of the polarization $\mathbf{P}^{\text{in}}$ in the incoming nucleon beam on the transverse component of $\mathbf{P}^{pu}$. As mentioned in the beginning of this section, this influence may be determined, at least theoretically, from quadruple scattering experiments. If $\mathbf{P}^{\text{in}}$ is proportional to $\mathbf{k}_i$, $\mathbf{P}^{\text{in}}=P^{\text{in}}\mathbf{k}_i$, we find from Eq. (19)

$$\mathbf{P}^{pu}=P^{uu}\mathbf{n}+P^{\text{in}}\{(X+Y\cos\vartheta)\mathbf{k}_i+(X\cos\vartheta+Z)\mathbf{k}_f\}.$$

The asymmetry of a fourth scattering occurring in a plane orthogonal to the plane of the third scattering becomes

$$\frac{A_4}{P^{\text{in}}P_4{}^{uu}}=\frac{1}{2(2I+1)Q^{uu}}\Big\{\text{Tr}(aa^\dagger-bb^\dagger)\sin(\vartheta/2+\chi)-i\,\text{Tr}(ba^\dagger-ab^\dagger)\cos(\vartheta/2+\chi)+\text{Tr}(cc^\dagger-dd^\dagger)\sin(\vartheta/2-\chi)\Big\}. \tag{23}$$

Summing up we see that with single, double, and triple scattering we can obtain the four relations Q^{uu}, P^{uu}, D_n, and Λ between the parameters of the transition matrix at fixed energy and angle. Including quadrupole scattering we gain two more independent relations. We could then determine all four coefficients in Eq. (19). Further information would of course be obtainable from the scattering of polarized nucleon beams at polarized targets.

VI. TARGETS WITH SPIN ZERO AND SPIN ONE-HALF

For the scattering of nucleons at spin zero targets,[11–13] the rotation invariant matrices a, b, c, and d in Eqs. (16) and (17) become complex functions of energy and angle. Therefore the pseudoscalars c and d must vanish, and the transition matrix $T=a+b(\boldsymbol{\sigma}\cdot\mathbf{n})$ at fixed energy and angle is determined by the amounts $|a|=a_0$ and $|b|=b_0$ and the relative phase $\psi=\arg(a^*b)$, aside from an absolute phase factor. If one includes triple scattering experiments, the physically measurable quantities are, in terms of these parameters, in the laboratory system:

$$Q^{uu}=a_0{}^2+b_0{}^2,\quad D_n=0,$$
$$P^{uu}=\frac{2a_0b_0\cos\psi}{a_0{}^2+b_0{}^2},\quad \Lambda=\left\{\frac{a_0{}^2-b_0{}^2}{a_0{}^2+b_0{}^2}\cos\vartheta-\frac{2a_0b_0\sin\psi}{a_0{}^2+b_0{}^2}\sin\vartheta\right\}. \tag{24}$$

Measurements of Q^{uu}, P^{uu}, and Λ at given energy and angle are sufficient to determine the transition matrix for this energy and angle aside from the arbitrary absolute phase. If we have once determined T for a certain process, we can compute $\mathbf{P}^{pu}$ for an incident nucleon beam with any polarization $\mathbf{P}^{\text{in}}$. This fact may possibly be used in order to produce a beam with known longitudinal component of polarization. Choosing the polarization $\mathbf{P}^{\text{in}}$ orthogonal to $\mathbf{k}_i$ and $\mathbf{n}$ we find by the use of Eq. (18) or (19) for the polarization $\mathbf{P}^{pu}$ in the beam scattered from a spin zero target

$$\mathbf{P}^{pu}=\mathbf{P}^{uu}\mathbf{n}+P^{\text{in}}\left\{\frac{a_0{}^2-b_0{}^2}{a_0{}^2+b_0{}^2}[\mathbf{n}\times\mathbf{k}_i]+\frac{2a_0b_0\sin\psi}{a_0{}^2+b_0{}^2}\mathbf{k}_i\right\}.$$

The longitudinal component of $\mathbf{P}^{pu}$ becomes in this case

$$(\mathbf{P}^{pu}\cdot\mathbf{k}_f)=P^{\text{in}}\left\{\frac{a_0{}^2-b_0{}^2}{a_0{}^2+b_0{}^2}\sin\vartheta+\frac{2a_0b_0\sin\psi}{a_0{}^2+b_0{}^2}\cos\vartheta\right\}.$$

If, in the special case of targets with spin zero, the incoming nucleon beam is completely polarized in any direction, the scattered beam must also be completely polarized, because the quantum-mechanical mixtures before and after collision reduce to pure states. For any pure state of free fermions the magnitude of the expectation value of $\boldsymbol{\sigma}$ becomes one. The angle between initial and final polarization in this case is given by

$$\cos\Theta=(\mathbf{P}^{pu}\cdot\mathbf{P}^{\text{in}})=\frac{a_0{}^2-b_0{}^2+2(\mathbf{n}\cdot\mathbf{P}^{\text{in}})(a_0\cos\psi+1)b_0}{(a_0{}^2+b_0{}^2)\{1+2(\mathbf{n}\cdot\mathbf{P}^{\text{in}})a_0b_0\cos\psi\}},$$

[11] J. Schwinger, Phys. Rev. **69**, 681 (1946).
[12] L. Wolfenstein, Phys. Rev. **75**, 1664 (1949).
[13] J. V. Lepore, Phys. Rev. **79**, 137 (1950).

which becomes

$$\cos\Theta=\begin{cases}1 \text{ for } (\mathbf{n}\cdot\mathbf{P}^{\text{in}})=1\\ \dfrac{a_0^2-b_0^2}{a_0^2+b_0^2} \text{ for } (\mathbf{n}\cdot\mathbf{P}^{\text{in}})=0.\end{cases}$$

For $|\mathbf{P}^{\text{in}}|\leq 1$ we obtain for the amount of polarization in the scattered beam

$$|\mathbf{P}^{pu}|^2=1-\frac{1-|\mathbf{P}^{uu}|^2}{(1+\mathbf{P}^{uu}\cdot\mathbf{P}^{\text{in}})^2}(1-|\mathbf{P}^{\text{in}}|^2),$$

which reduces to one for $|\mathbf{P}^{\text{in}}|=1$.

Finally let us consider the case of targets with spin one-half. For the matrix-coefficients in the transition matrix we find from their transformation properties listed in Table I:

$$a=\alpha+\gamma(\boldsymbol{\sigma}_2\cdot\mathbf{n}),\quad c=\delta(\boldsymbol{\sigma}_2\cdot\mathbf{m}),$$
$$b=\gamma'+\beta(\boldsymbol{\sigma}_2\cdot\mathbf{n}),\quad d=\epsilon(\boldsymbol{\sigma}_2\cdot\mathbf{l}).$$

The quantities α, etc., are complex functions of energy and scattering angle in the center-of-mass system. Indicating the spin matrices referring to incoming nucleons and target particles by $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ respectively, we obtain the transition matrix in the form

$$T=\alpha+\beta(\boldsymbol{\sigma}_1\cdot\mathbf{n})(\boldsymbol{\sigma}_2\cdot\mathbf{n})+(\gamma'\boldsymbol{\sigma}_1+\gamma\boldsymbol{\sigma}_2\cdot\mathbf{n})$$
$$+\delta(\boldsymbol{\sigma}_1\cdot\mathbf{m})(\boldsymbol{\sigma}_2\cdot\mathbf{m})+\epsilon(\boldsymbol{\sigma}_1\cdot\mathbf{l})(\boldsymbol{\sigma}_2\cdot\mathbf{l}).$$ [6,14]

The experimentally measurable quantities in single, double, and triple scattering become in terms of these parameters

$$Q^{uu}=|\alpha|^2+|\beta|^2+|\gamma|^2+|\gamma'|^2+|\delta|^2+|\epsilon|^2,$$
$$Q^{uu}P^{uu}=2\,\mathrm{Re}(\alpha^*\gamma'+\beta^*\gamma),$$
$$D_n=-\frac{2}{1+(|\alpha+\gamma'|^2+|\beta+\gamma|^2)/(|\delta|^2+|\epsilon|^2)},\tag{25}$$
$$Q^{uu}\Lambda=(-|\alpha|^2+|\beta|^2-|\gamma|^2+|\gamma'|^2-|\delta|^2+|\epsilon|^2)\cos(\vartheta/2)+2\,\mathrm{Im}(\alpha^*\gamma'-\beta^*\gamma)\sin(\vartheta/2).$$

For identical particles the transition matrix must be symmetric under interchange of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. Therefore γ must be equal to γ' in this case. If we neglect electromagnetic interactions, the same holds for neutron proton scattering because of charge independence. Thus the transition matrix for nucleon-nucleon scattering depends on five complex functions, i.e., excluding the absolute phase factor, on nine real parameters at every angle and energy. Including triple scattering we can measure only the four relations (25) between these parameters. This lack of information available at present makes it preferable at lower energies to separate the angular dependence of these coefficients by making a phase shift analysis. This is usually done in a representation of T where the total spin operator S^2 of the two nucleon system and its z-component S_z are diagonal. Using a nomenclature similar to that of Swanson[15] we can then write T in the form:

$$(T_{S'm_s',Sm_s}(\lambda,\vartheta,\phi))$$
$$=\frac{\lambda}{4i}\begin{pmatrix}A & -\sqrt{2}Be^{-i\phi} & Ce^{-2i\phi} & 0\\ \sqrt{2}De^{i\phi} & 2E & -\sqrt{2}De^{-i\phi} & 0\\ Ce^{2i\phi} & \sqrt{2}Be^{i\phi} & A & 0\\ 0 & 0 & 0 & S\end{pmatrix},\tag{26}$$

with the additional condition

$$A-C-2E=2\cot\vartheta(D-B).$$

This relation is a consequence of the invariance of T under inversion of motion. The complex functions A, B, $\cdots$, and S in Eq. (26) can be easily expressed in terms of the functions α, β, $\cdots$, and ϵ. The elements S, A, and E are series in $P_l^0(\vartheta)$, B and D in $P_l^1(\vartheta)$, and C is a series in $P_l^2(\vartheta)$. In all of these series the coefficients depend on the corresponding phase shifts. In terms of the matrix elements in Eq. (26) we get for P^{uu} and D_n

$$Q^{uu}P^{uu}=-\frac{\lambda^2}{16}\tfrac{1}{2}\,\mathrm{Im}\{(A-C+2E)(B^*+D^*)\},$$ [16–19]

$$Q^{uu}[1+D_n(1+P^{uu})]=Q^{uu}V$$
$$=\frac{\lambda^2}{16}\tfrac{1}{2}\,\mathrm{Re}\{(A+C)S^*+(A-C)2E^*+4BD^*\}.$$

Also the cross section Q^{pu} and the quantity Λ defined in Eq. (22) may be easily expressed by the elements of the transition matrix (26). The polarization P^{uu} depends, in a direct way, only on triplet elements of the transition matrix, whereas D_n and also Λ contain singlet-triplet interference terms aside from pure triplet contributions. Of course there is a dependence on singlet terms through the cross section Q^{uu} for unpolarized beams, but this quantity can be measured independently.

For identical particles the singlet part of T must be symmetrized and the triplet matrix elements correspondingly antisymmetrized. We can write the transition matrix $\bar{T}$ for identical particles in the same form as T in Eq. (26), with the quantities A, B, $\cdots$, and S replaced by the corresponding elements $\bar{A}$, $\bar{B}$, $\cdots$, and $\bar{S}$, where $\bar{A}=A(\vartheta)-A(\tau-\vartheta)$, $\bar{B}=B(\vartheta)+B(\tau-\vartheta)$

[14] R. H. Dalitz, Proc. Phys. Soc. (London) **A65**, 175 (1952).

[15] D. R. Swanson, Phys. Rev. **89**, 740 (1953); this paper contains further references.
[16] L. Wolfenstein, Phys. Rev. **76**, 973 (1949).
[17] L. J. B. Goldfarb and D. Feldman, Phys. Rev. **88**, 1099 (1952).
[18] D. R. Swanson, Phys. Rev. **89**, 749 (1953).
[19] G. Breit and J. B. Ehrman, Phys. Rev. **96**, 805 (1954).

$S(\vartheta)+S(\pi-\vartheta)$, etc. In practice this simply means summing only over odd l for triplet elements and even l for the singlet element and multiplying the results by a factor of two. We note that $\bar{A}$ and $\bar{E}$ are proportional to $\cos\vartheta$ and that $\bar{C}$ is proportional to $\sin^2\vartheta\cos\vartheta$. Whereas these elements become zero at ninety degrees in the center-of-mass system, the quantities $\bar{B}$, $\bar{D}$ and $\bar{S}$ do not vanish identically at this angle. The functions $\bar{B}$ and $\bar{D}$ are proportional to $\sin\vartheta$. Using these properties of the matrix elements we find for *identical* particles that $P_n{}^{pu}(\vartheta=\pi/2)$ becomes

$$P_n{}^{pu}(\pi/2)=P_n{}^{\text{in}}+D_n(\pi/2)P_n{}^{\text{in}}$$
$$=\frac{\lambda^2 P_n{}^{\text{in}}}{8Q^{uu}(\pi/2)}\operatorname{Re}\left\{\bar{B}(\pi/2)\bar{D}^*(\pi/2)\right\}.$$

At this angle the depolarization depends only on triplet elements and thus provides a new relation between triplet phase shifts.

If the polarization of the incoming beam is orthogonal to $\mathbf{n}$ and $\mathbf{k}_i$, the polarization $\mathbf{P}^{pu}(\vartheta=\pi/2)$ in the beam scattered at ninety degrees becomes

$$\mathbf{P}^{pu}(\pi/2)=\frac{\lambda^2 P^{\text{in}}}{16Q^{uu}(\pi/2)}[-\operatorname{Re}\{\bar{S}(\pi/2)\bar{D}^*(\pi/2)\}\mathbf{k}_i].$$

In this relation P^{in} is defined by

$$\mathbf{P}^{\text{in}}=P^{\text{in}}[\mathbf{n}\times\mathbf{k}_i].$$

According to Eq. (22), the quantity $\Lambda(\vartheta=\pi/2)$ is then

$$\Lambda(\pi/2)=\frac{-\lambda^2}{\sqrt{2}16Q^{uu}(\pi/2)}\operatorname{Re}\{\bar{S}(\pi/2)\bar{D}^*(\pi/2)\}$$

and depends only on singlet triplet interference terms.

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Mesonic Decay of an Ejected Triton

HERMAN YAGODA
National Institutes of Health, Bethesda, Maryland
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A heavy fragment is ejected from a star of type $21+5p$ which decays at rest into three charged particles. Gap counts indicate that the ejected fragment is singly charged and heavier than a proton. The mass of the particle, as estimated by constant sagitta scattering along its range of 1330 microns, is 2.93 ± 1.36 proton masses. All three secondary particles terminate their range, and one of them can be identified as a negative pi meson of 26.6 ± 0.9 Mev kinetic energy. Two short recoil tracks, if assumed to be protons, have kinetic energies of 1.43 ± 0.04 and 2.31 ± 0.15 Mev, respectively. Momentum balance applied to several decay schemes suggest that the event probably represents $H^{3*}\rightarrow H^1+H^1+n^0+\pi^-+Q=31.5\pm1$ Mev. The binding energy of the excited triton is found to be 5.4 ± 1 Mev, less than that of the normal triton (8.48 Mev). The time of flight of the excited triton is 4.2×10^{-11} sec.

IN the examination of a unit of Ilford G5 emulsions flown in the stratosphere[1] a star of type $21+5p$ was observed in which two of the gray tracks could be identified as negative pi mesons of 13.5 and 16.6 Mev. In the detailed study of this event it was noted that one of the heavy prongs (F of Fig. 1) had a 3-prong star at the end of its range. As indicated in the photomicrograph of this detail (Fig. 2) track F appears to have come to rest, and the tracks of the disintegration products seem to reside in a single plane approximately perpendicular to the field of view. The recoil tracks A and B terminated in the same emulsion sheet (1500 microns thick) and the third track C could be followed into an adjacent pellicle where it terminated with the formation of a 4-prong sigma star. This identification indicated a mesonic-decay process, occurring under exceptionally favorable circumstances, such that the momenta of the decay products could be evaluated accurately from the measured ranges of the tracks. A study was therefore made on the identification of the several members and the evaluation of the most probable mode of decay of the heavy fragment.

IDENTIFICATION OF THE EXCITED FRAGMENT

The range of track F is 1330 microns and it makes a small angle with the original emulsion plane ($\tan\beta=0.344\pm0.017$). This permits the determination of the charge of the particle by gap counting.[2] These measurements are summarized in Fig. 3 where the solid curves

[1] For details of exposure see H. Yagoda, this issue [Phys. Rev. 98, 103 (1955)].

[2] Our gap-counting procedure yields essentially a gap-length measurement. The gap of minimum discernibility of about 0.4 micron is given a single count. Larger gaps are weighted visually in proportion to their apparent length with the aid of a reticule in the eyepiece scale. This technique is rapid and eliminates the need for a special stage constructed with a micro screw feed.