Nonrelativistic Interaction Between Two Nucleons*†

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The problem of the derivation of a two-nucleon Schrödinger equation from quantum field theory has been investigated, where only those mesons which are exchanged between the nucleons are taken into account. One expects that the two-particle Schrödinger picture will be useful if the energy (less rest energy) is small ($\lesssim\mu^2/M$), and if the important matrix elements are those which couple states of small momenta ($\lesssim\mu$). The procedure which has been followed has been to go over to Fock space in the manner of Tamm and Dancoff, and then to decouple the two-particle Tamm-Dancoff amplitude from all the others by a series of canonical transformations (the over-all coupling is assumed weak). Unlike the methods developed by Lévy-Klein and Bethe-Salpeter, the characteristic difficulties such as energy-dependent and non-Hermitian potentials are avoided. By way of application, the formalism is used to analyze the nonrelativistic nuclear forces for the neutral scalar and charge-symmetric pseudoscalar theories (with both pseudoscalar and pseudovector coupling). In this approximation, it is shown that there is agreement with the results of Lévy-Klein. In the course of the calculations, it is made evident that the "nonadiabatic velocity-dependent" corrections of Lévy-Klein appear even when the nucleons are taken to be fixed sources. Within the context of the method of canonical transformations, there is no justification for dropping these corrections as has been suggested by Brueckner and Watson. Finally, there is some evidence that the Tamm-Dancoff approximation is not an improvement over weak-coupling perturbation theory when applied to the nuclear-force problem, at least when the coupling constant is small.

I. INTRODUCTION

It appears by now to be reasonably well established that, within the domain of what is commonly referred to as classical nuclear physics (corresponding to energies up to 10 or 20 Mev, say), the behavior of nuclear systems is governed by orthodox nonrelativistic quantum mechanics. This means, in particular, that the state of a system of l-nucleons is characterized by an l-particle wave function $\psi(x_1,x_2,\cdots,x_l)$ where x_i denotes all the coordinates (space, spin, and charge) of the ith particle and where ψ is a properly antisymmetrized solution of an l-particle Schrödinger equation. An important drawback of this type of theory is that the interaction between nucleons needs to be given phenomenologically. It is, of course, an essential task of quantum field theory to give a fundamental basis for this interaction.

Now, the description of a nuclear system as given by quantum field theory is generally a very complicated affair. This is due to the fact that all meson theories must be in accord with the special theory of relativity and, when so constructed, give rise to the possibility of the creation and annihilation of particles. Thus, if we assume that we have to deal with an interaction Hamiltonian which is linear in the meson field variables and bilinear in the nucleon field variables, the state Ψ of our system can be conveniently represented in terms of an infinite, denumerable sequence of configuration-space wave functions $\psi_m{}^n$ in the manner given by Fock,[1] where m denotes the number of mesons and n the number of nucleon-antinucleon pairs which are present in addition to the original l-particles; both m and n can assume the values 0, 1, 2, $\cdots$. The state functional Ψ satisfies a Schrödinger equation from which one can deduce an infinite set of coupled integral equations[2] for the $\psi_m{}^n$.

We accordingly have two descriptions of our nuclear system; the one is field-theoretical and relativistic, the other is phenomenological and nonrelativistic. It is clear that any consistent treatment of the nuclear-force problem must reconcile these two pictures, i.e., from the Schrödinger equation for the state functional Ψ, or, equivalently, from the infinite set of coupled equations for the Fock-space amplitudes $\psi_m{}^n$, there must emerge, at least for low energies, an equally valid description in the form of a Schrödinger equation for an l-particle wave function ψ. It must be emphasized that our standpoint is that, in classical nuclear physics, the hypothesis that the wave function of a system of l-nucleons need only involve the coordinates of these particles has been justified, at least empirically. It is a moot point as to how far one can in fact push this type of description sensibly; we shall return to this question shortly.

We restrict ourselves, in this paper, to the two-nucleon problem which is the simplest mathematically and perhaps the most important physically. There has been a good deal of interest recently in the construction of relativistic two-body equations which are suitable for application to the deuteron, say. One approach, due

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† The essential contents of this paper were presented at the Chicago meeting of the American Physical Society (November 27–28, 1953).
[1] V. Fock, Z. Physik **75**, 622 (1932).

[2] A configuration-space wave function can be represented in either coordinate space or momentum space, and we shall generally distinguish between these two possibilities by using the notations ψ and ϕ, respectively. As a matter of practical convenience, it is simpler to work in momentum space in which case the equations for the Fock-space amplitudes are coupled linear integral equations, whereas they are integro-differential equations in coordinate space. In the detailed considerations of the following sections, we shall take advantage of this simplification. For the purposes of the general discussion of this section only, however, we let ψ denote generically a configuration-space wave function in either representation, and refer to the equations satisfied by the ψ's simply as "integral equations."

to Tamm[3] and Dancoff[4] and extended by Lévy[5] and Klein,[6] proceeds as follows: one assumes that all the Fock-space amplitudes which correspond to having more than a limited number of mesons and nucleon-antinucleon pairs in the field vanish, and proceeds to try to treat the now finite number of coupled equations which are satisfied by the nonzero $\psi_m{}^n$'s as rigorously as possible. Clearly, the Tamm-Dancoff method presupposes, for its validity, certain restrictions on the energy of the system and the strength of the meson-nucleon coupling, but it is equally clear that some higher-order effects beyond those of a straightforward weak-coupling calculation are included.

In the simplest Tamm-Dancoff approximation, we have two coupled equations involving the amplitudes $\psi_0{}^0$ and $\psi_1{}^0$ which, for brevity, we write as ψ_0 and ψ_1. These equations are adequate for a relativistic treatment of both the bound and scattering states of the two-nucleon system within the framework of the Fock-space formalism and subject only to the limitations inherent in the original Tamm-Dancoff approximation. A convenient procedure in the solution of these equations has been to eliminate algebraically the amplitude ψ_1, thus leading to a linear integral equation for ψ_0. This equation has customarily been identified as a genuine Schrödinger equation for the two-nucleon system, with ψ_0 being taken to be the two-particle wave function for the system.[7]

This identification, however, cannot be generally correct. From a physical standpoint, the functions ψ_0 and ψ_1 have a very definite meaning; they represent the probability amplitudes for finding two nucleons and two nucleons plus one meson, respectively, in the field. They correspond to a Fock-space description of the system, and so must be normalized so that, in terms of a symbolic notation, $|\psi_0|^2+|\psi_1|^2=1$;[4] furthermore, in the computation of expectation values, the contribution from ψ_1 cannot be ignored. The construction of a Schrödinger-like equation for ψ_0 by algebraic elimination of ψ_1 may be convenient mathematically, but one does not, in this way, decouple the two-nucleon-plus-one-meson state from the two-nucleon state.

There are also other difficulties. The interaction operator in the ψ_0 equation depends on the energy of the system[3] so that the solutions of this equation are not even orthogonal to one another. Furthermore, as we shall see, when an iteration procedure due to Lévy and Klein is applied so as to remove this explicit energy-dependence, the resultant interaction operator is not even Hermitian. These results are again clearly manifestations of the fact that the description of the two-nucleon system as given by the Tamm-Dancoff method is inherently a relativistic one, allowing for the creation and annihilation of particles, and that the transition to the nonrelativistic Schrödinger picture has yet to be made.

It is perhaps useful at this point to recall another situation in quantum mechanics which is strikingly analogous to the one under discussion. We have in mind the general question of the reduction of the Dirac equation to its nonrelativistic Pauli form. According to the relativistically covariant Dirac theory, a spin $\frac{1}{2}$ particle is characterized by a four-component wave function which can always be regarded as having been expanded in terms of the complete set of free-particle solutions of the interaction-free Dirac equation. In the interests of clarity, let us assume that we are working in the Foldy-Wouthuysen representation[8] in which the positive- and negative-energy free-particle spinors have no more than two nonvanishing components each, which, for definiteness, we can take to be the upper and lower two, respectively. In this representation, a particular component of an arbitrary Dirac spinor can be directly interpreted as the probability amplitude for finding the spin $\frac{1}{2}$ particle in a state of given spin orientation and sign of energy.

Now, for nonrelativistic energies, it is known, phenomenologically, that a spin $\frac{1}{2}$ particle can be adequately described by a two-component Pauli wave function. One has then the problem of demonstrating that, even in the presence of interaction with, say, an external field, a reduction to the Pauli form can be effected, at least, in the nonrelativistic limit.

One is again faced with the task of reconciling two descriptions of the same physical situation; the one is of relativistic and fundamental theoretical origin, the other is nonrelativistic and phenomenological. One might first ask: Under what circumstances is it, in fact, sensible to try to carry through a reduction to the Pauli form? Here, the answer, as given in the recent comprehensive discussion by Foldy and Wouthuysen, is that the two-component representation of a spin $\frac{1}{2}$ particle is meaningful only if the energies involved are small and if the coupling to the high-momentum components of the external field is weak, i.e., only if one has to deal with basically nonrelativistic problems.

One might next ask: How is one to carry through the transition from the Dirac to the Pauli form within the domain of applicability, i.e., for nonrelativistic energies? The usual procedure has been to eliminate the small components of the spinor wave functions in favor of the large ones. However, this procedure cannot be justified beyond the lowest-order approximation, since the algebraic elimination of the small components does not

[3] I. Tamm, J. Phys. (U.S.S.R.) **9**, 449 (1945).
[4] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).
[5] M. M. Lévy, Phys. Rev. **88**, 72 (1952); Phys. Rev. **88**, 725 (1952).
[6] A. Klein, Phys. Rev. **90**, 1101 (1953).
[7] In practical applications to the nuclear-force problem (i.e., when more than two amplitudes are included), the interaction operator in the ψ_0 equation has generally been expanded in the form of an infinite series in the coupling constant, with the series being terminated at some convenient point. One must accordingly be careful to distinguish between exact and approximate treatments of the Tamm-Dancoff equations, although, for the purposes of our general discussion, this distinction is immaterial.

[8] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

constitute a physical decoupling of the positive from the negative energy states. The correct procedure, again due to Foldy and Wouthuysen, is to carry through a series of canonical transformations which rigorously separate the positive- and negative-energy states to any desired order in v/c. The Pauli Hamiltonian appears in the form of an infinite series which is useful only if the aforementioned restrictions on the energy of the system and the strength of the coupling to the external field are satisfied.

We may now return to our original problem: the derivation of a two-particle Schrödinger equation from quantum field theory. It is evident that, in a very real sense, we are dealing with the quantum-field theoretical analog of the reduction of the Dirac to the Pauli formalism. It will therefore not be surprising to find that a description of the two-nucleon system in terms of a wave function involving solely nucleon coordinates can, in fact, be justified only if the energies are nonrelativistic and if the coupling to high-momentum states of mesons and heavy particles is weak. The actual separation of the two-particle state in Fock space from states having also a finite number of mesons and/or nucleon-antinucleon pairs can be carried through by an infinite series of canonical transformations.

The detailed content of this paper will be devoted to an exposition of the thesis contained in the preceding paragraph. Starting with the Tamm-Dancoff approximation, we shall investigate, in Sec. II, the problem of how to derive, in a consistent way, a nonrelativistic two-particle Schrödinger equation from meson theory. In Sec. III, we proceed with applications to several cases of interest, *viz.*, the scalar and pseudoscalar theories (the latter with both direct and gradient coupling). For the sake of simplicity, we shall ignore those transitions in which a given meson is emitted and absorbed by the same nucleon. We shall also not find it necessary to consider explicitly the creation and annihilation of virtual nucleon-antinucleon pairs. The problem of the removal of divergences is therefore not discussed in this paper.

In the course of carrying through the detailed considerations of the nuclear-force problem contained in Secs. II and III, we shall also be in a position to discuss two aspects of this problem which are, perhaps, of particular interest. The first has to do with the question as to whether the Tamm-Dancoff approximation is an improvement over weak-coupling perturbation theory. The evidence will be that it is not, at least when one restricts oneself to essentially weak interactions. The second relates to the question as to whether certain "nonadiabatic velocity-dependent" corrections to nuclear forces which have been reported in the literature[5,6] ought to be retained[9] or dropped.[10] In point of fact, these controversial terms are simply the higher-order ladder corrections which are contained in the lowest-order Tamm-Dancoff approximation. Upon carrying through the reduction of the Tamm-Dancoff equations to the two-particle Schrödinger formalism, it will become clear that these terms are, in fact, not at all nonadiabatic or velocity-dependent since they are present even in the fixed-source limit; also, they cannot reasonably be ignored, at least when one deals with weak interactions.

[9] E. M. Henley and M. A. Ruderman, Phys. Rev. **92**, 1036 (1953).

[10] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953).

II. REDUCTION TO A TWO-PARTICLE SCHRÖDINGER EQUATION

Let us suppose that we have two fields—nucleon and meson—whose interaction with one another can be characterized by a Hamiltonian $\mathcal{K}$ which is linear in the meson field and bilinear in the nucleon field. If we denote the free-particle Hamiltonian by $\mathcal{H}$, the stationary states Ψ of the system, where these may be either bound or scattering states, are determined by the Schrödinger equation

$$(E-\mathcal{H})\Psi=\mathcal{K}\Psi. \tag{1}$$

We restrict ourselves immediately to a consideration of the two-nucleon problem. In order to make the transition from Ψ to a set of Fock-space wave functions which are given, in particular, in a momentum-space representation, we introduce the orthonormal subset of vectors $\Psi_m{}^n(P)$, where the latter are those stationary solutions of the interaction-free Schrödinger equation,

$$[E_m{}^n(P)-\mathcal{H}]\Psi_m{}^n(P)=0, \tag{2}$$

which can be coupled by $\mathcal{K}$ to the free-particle two-nucleon states. We use the indices m and n, as before, to specify the number of mesons and nucleon-antinucleon pairs which are present in addition to the original two nucleons. For a given m and n, there exists, of course, an infinite number of free-particle states corresponding to the different possibilities for the momenta, spins, and isotopic spins of the particles; we use the symbol P to distinguish these possibilities from one another.

We now expend Ψ (assumed normalized to unity) in terms of the set $\Psi_m{}^n(P)$, and so write

$$\Psi=\sum \phi_m{}^n(P)\Psi_m{}^n(P). \tag{3}$$

Clearly, $\phi_m{}^n(P)=(\Psi_m{}^n(P),\Psi)$ is the probability amplitude for finding the compound two-nucleon system in the state $\Psi_m{}^n(P)$. The normalization for the Fock-space amplitudes then has to be

$$\sum|\phi_m{}^n(P)|^2=1. \tag{4}$$

It is to be understood that the summations in (3) and (4) are to be taken over all indices, *viz.*, m, n, and P.

Now, while, in detailed calculations, it is convenient to use a momentum-space representation in each configuration subspace, in the development of a general formalism it is simpler to use a somewhat more compact, symbolic notation in which it is the particle aspects alone of the various Fock subspaces which are emphasized—the particular working frame of reference within each subspace being left unspecified. We can introduce such a symbolic notation by simply suppressing the variable P. Equations (3) and (4) can then be rewritten in the forms

$$\Psi=\sum \phi_m{}^n\Psi_m{}^n, \tag{3a}$$

and

$$\sum|\phi_m{}^n|^2=1, \tag{4a}$$

respectively. The summations in (3a) and (4a) are now to be taken over m and n. We have denoted by $\Psi_m{}^n$ the unit vector in Fock space which characterizes the (m,n)th subspace. It is then evident that $\phi_m{}^n$ is a vector which spans the (m,n)th subspace.

We now make the lowest-order Tamm-Dancoff approximation, i.e., we assume that all the $\phi_m{}^n$'s vanish identically with the exception of the two-nucleon amplitude $\phi_0{}^0$ and the two-nucleon-plus-one-meson amplitude $\phi_1{}^0$. If we suppress the upper index, Eqs. (3a) and (4a) become

$$\Psi=\phi_0\Psi_0+\phi_1\Psi_1 \tag{5}$$

and

$$|\phi_0|^2+|\phi_1|^2=1, \tag{6}$$

respectively.

Let us write the unit vectors Ψ_0 and Ψ_1 as two-component column matrices, i.e.,

$$\Psi_0=\begin{pmatrix}1\\0\end{pmatrix},\quad \Psi_1=\begin{pmatrix}0\\1\end{pmatrix}; \tag{7}$$

we then have, for the state vector Ψ,

$$\Psi=\begin{pmatrix}\phi_0\\\phi_1\end{pmatrix}. \tag{8}$$

In terms of this notation, the Hamiltonian operators $\mathcal{H}$ and $\mathcal{K}$ which enter in Eq. (1) will appear as two-by-two matrices, with the former having only nonvanishing diagonal elements, and the latter only nonvanishing off-diagonal elements; they can accordingly be written in the following way:

$$\mathcal{H}=\begin{pmatrix}W_0 & 0\\ 0 & W_1\end{pmatrix},\quad \mathcal{K}=\begin{pmatrix}0 & K_{01}\\ K_{10} & 0\end{pmatrix}. \tag{9}$$

It is to be noticed that the matrix elements in terms of which $\mathcal{H}$ and $\mathcal{K}$ are here expressed are not c-numbers but are, in turn, also operators. Thus, the diagonal elements W_0 and W_1 are the appropriate free-particle Hamiltonian operators for the zero-meson and one-meson configuration subspaces, respectively. The off-diagonal elements K_{01} and K_{10} serve to couple the two subspaces. Denoting the Hermitian adjoint of an operator by a dagger, we observe also that the Hermitian character of $\mathcal{H}$ and $\mathcal{K}$ implies $(W_0)^\dagger=W_0$, $(W_1)^\dagger=W_1$, and $(K_{01})^\dagger=K_{10}$.

Finally, the Schrödinger equation (1), in lowest-order Tamm-Dancoff approximation, will appear as a two-component matrix equation, *viz.*,

$$\begin{pmatrix}E-W_0 & 0\\ 0 & E-W_1\end{pmatrix}\begin{pmatrix}\phi_0\\\phi_1\end{pmatrix}=\begin{pmatrix}0 & K_{01}\\ K_{10} & 0\end{pmatrix}\begin{pmatrix}\phi_0\\\phi_1\end{pmatrix}; \tag{10}$$

upon carrying through the indicated matrix multiplication, we are led, in turn, to the following pair of linear equations:

$$(E-W_0)\phi_0=K_{01}\phi_1, \tag{11a}$$

$$(E-W_1)\phi_1=K_{10}\phi_0. \tag{11b}$$

Now, Eqs. (11a) and (11b) can be used as the basis for a discussion of the stationary states of the two-nucleon system subject only to our assumption that no more than one meson is present in the field at any one time and that no nucleon-antinucleon pairs are produced. As has already been indicated in Part I of this paper, the usual procedure[3–6] in the solution of these equations is to solve (11b) for ϕ_1 in terms of ϕ_0,

$$\phi_1=(E-W_1)^{-1}K_{10}\phi_0,$$

and then to substitute this expression for ϕ_1 into (11a), thus leading to a Schrödinger-like equation for the configuration-space two-particle amplitude ϕ_0,

$$(E-W_0)\phi_0=K_{01}(E-W_1)^{-1}K_{10}\phi_0. \tag{12}$$

The essential objections to the use of Eq. (12) as a starting point for a discussion of the nonrelativistic nuclear-force problem have been listed in Part I and need not be repeated here. We want simply to add the remark that precisely the same criticisms can be directed against the use of the Bethe-Salpeter equation[11] for the same purpose. Indeed, the equal-times Bethe-Salpeter wave function for a system of two nucleons is directly related to a two-particle Tamm-Dancoff amplitude,[12] so that it is evident that, in both formalisms, one has still to decouple the two-particle amplitude from amplitudes allowing for the presence of mesons and/or nucleon-antinucleon pairs.

Let us then return to Eq. (10) and examine first the circumstances under which it may be expected that the decoupling of the two-nucleon (no-meson) amplitude from the two-nucleon-plus-one-meson amplitude will have physical meaning. We reiterate at this point that we shall not take into account that contribution to the nuclear force which arises from the emission and subsequent absorption of a meson by the same nucleon; accordingly, we consider that the interaction between two nucleons comes about solely as a consequence of

[11] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

[12] See references 5 and 6; also, W. Macke, Phys. Rev. **91**, 195 (1953); F. J. Dyson, Phys. Rev. **91**, 1543 (1953).

the continual interchange of mesons between the nucleons.

We denote the meson and nucleon masses by μ and M, respectively, and work with natural units for which $\hbar=c=1$. Also, it will be convenient, for what follows, to refer to the momentum p of a particle (which may be a nucleon or meson) as small or large according as $p\lesssim\mu$ or $>\mu$, respectively. Similarly, we shall say that the energy of the system is small or large according as the quantity $|E-2M|\lesssim\mu^2/M$ or $>\mu^2/M$, respectively.

Now, there is a nominal energy gap of width μ between the no-meson and one-meson subspaces. In point of fact, however, the actual gap between the two subspaces can be much smaller than μ, and the extent to which it is smaller may be measured by the amount of high-momentum component contained in ϕ_0. Thus, if the energy of the system is large to begin with, the no-meson amplitude ϕ_0 will generally contain important high-momentum components, so that one can then hardly speak of an energy separation between the two subspaces at all. Even if the energy of the system is small, one cannot immediately conclude that there are only low-momentum components in ϕ_0, unless the coupling to the high-momentum components of ϕ_1 can be characterized as weak. Thus, only if the energy of the system is small, and if there is weak coupling between ϕ_0 and ϕ_1 in the sense just described, can one be certain that ϕ_0 will contain solely low-momentum components, or, equivalently, that there exists a real energy separation $\sim\mu$ between the two subspaces.

The importance of having such a sharp energy gap consists in the fact that it is only then that it appears physically significant to characterize the system by a two-particle Schrödinger equation. This can be seen in the following way. In the quantum-field theoretical description of the system, the particles are point particles and the interaction is a point interaction. When one proceeds to try to simplify the two-nucleon problem by eliminating the meson-field variables, one immediately introduces a new complication in that the interaction between the nucleons generally becomes very complicated, being nonlocal in character. This complication can be considered to occur in consequence of the fact that the nucleons now have a structure with radius of the order of the nucleon Compton wavelength $1/M$. It is then evident that the transition to the two-nucleon Schrödinger picture is pointless unless nucleon-structure effects are small, and indeed this will be the case only if the recoil momentum p of a nucleon on emitting or absorbing a meson is small compared to M. This last condition will be satisfied if $p\lesssim\mu$ which corresponds to the existence of an energy separation between the no-meson and one-meson subspaces of width $\sim\mu$.

We see therefore that, only when we have to deal with essentially nonrelativistic problems, can the description of the two-nucleon system by a two-particle Schrödinger equation be expected to be useful. We shall accordingly assume, for all that follows, that the energy of our system is small and the coupling to high momenta weak. In order to develop a consistent procedure for carrying out the reduction to the two-particle Schrödinger picture, we shall, in fact, impose the stronger restriction that the over-all coupling between the nucleon and meson fields is weak.[13] Under these circumstances, it becomes possible to decouple Eq. (10) in a systematic way by applying a sequence of canonical transformations. The resultant two-particle interaction operator, valid for internucleon separations $\gtrsim 1/\mu$, will then appear as a double series expansion in the coupling constant and nucleon recoil. We shall presuppose that this series expansion is asymptotic in both parameters.

We proceed to exhibit the transformations which will decouple Eq. (10). Let us use, for the moment, the more compact notation of Eq. (1),

$$(E-\mathcal{H})\Psi=\mathcal{K}\Psi, \tag{1}$$

it being understood that the lowest-order Tamm-Dancoff approximation has already been made, and that, in detailed calculations, we will use a momentum-space representation in the no-meson and one-meson configuration subspaces. We apply the canonical transformation

$$\Psi=(\exp-i\mathcal{S})\Phi, \tag{13}$$

where $\mathcal{S}$ will be determined so as to eliminate the first-order coupling between ϕ_0 and ϕ_1. If we recall the theorem that, for an arbitrary operator $\mathcal{A}$,

$$(\exp i\mathcal{S})\,\mathcal{A}\,(\exp-i\mathcal{S}) \\ =\mathcal{A}+i[\mathcal{S},\mathcal{A}]+(i^2/2!)[\mathcal{S},[\mathcal{S},\mathcal{A}]]+\cdots, \tag{14}$$

it becomes evident that we must choose $\mathcal{S}$ so that

$$-i[\mathcal{S},\mathcal{H}]=\mathcal{K}. \tag{15}$$

Upon examination of the matrix representation of $\mathcal{H}$ and $\mathcal{K}$ as given in (9), it is apparent that one can take $\mathcal{S}$ to be

$$\mathcal{S}=-i\mathcal{K}', \tag{16a}$$

where

$$\mathcal{K}'=\begin{pmatrix} 0 & K_{01}' \\ K_{10}' & 0 \end{pmatrix}, \tag{16b}$$

and where K_{01}' and K_{10}' are, in turn, operators which link the no-meson and one-meson Fock subspaces. Since $\mathcal{S}$ must be Hermitian, $\mathcal{K}'$ will be skew-Hermitian so that $(K_{01}')^\dagger=-K_{10}'$. The momentum-space representations of K_{01}' and K_{10}' can be directly determined

[13] One must be careful to note that the supposition that the meson-nucleon coupling constant is small is made for mathematical convenience; the condition that nucleon recoil be small is, on the other hand, a physical requirement which must be satisfied in order to be able to go over to the two-particle Schrödinger picture at all.

from (15) and (16). One finds

$$\left.\begin{aligned}(P|K_{01}'|Q)&=(P|K_{01}|Q)[W_0(P)-W_1(Q)]^{-1},\\(Q|K_{10}'|P)&=(Q|K_{10}|P)[W_1(Q)-W_0(P)]^{-1};\end{aligned}\right\}\quad(17)$$

here, P and Q denote the momenta, spins, and isotopic spins of free-particle no-meson and one-meson states, respectively.

It is a straightforward matter to carry through in detail the $\mathcal{S}$-transformation defined by (13), (16), and (17). Equation (1) will go over into a new Schrödinger equation for the vector Φ of the form

$$(E-\mathcal{H})\Phi=\mathcal{L}\Phi, \tag{18a}$$

where $\mathcal{L}$, the new interaction Hamiltonian, will appear as a power series in the coupling constant with the leading term of second order; thus,

$$\mathcal{L}=\mathcal{L}^{(2)}+\mathcal{L}^{(3)}+\mathcal{L}^{(4)}+\cdots. \tag{18b}$$

Both the second- and fourth-order terms, $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$, respectively, will be diagonal in that subregion of Fock space to which we have restricted ourselves on having applied the lowest-order Tamm-Dancoff approximation; these terms accordingly do not couple the no-meson and one-meson subspaces. The third-order term $\mathcal{L}^{(3)}$ is off-diagonal, but can be removed by a second canonical transformation analogous to (13). However, if we agree to restrict our considerations to terms which are at most of fourth order in the coupling constant, we need not bother with this second canonical transformation since it will contribute terms of sixth and higher order.

The no-meson and one-meson Fock subspaces are now completely decoupled from one another to fourth order. We are accordingly in a position to identify the transformed two-nucleon (no-meson) Tamm-Dancoff amplitude (which we now call ϕ) with the two-particle Schrödinger wave function for the system, and the matrix element $(\mathcal{L}^{(2)}+\mathcal{L}^{(4)})_{00}$ with the interaction Hamiltonian. One finds, for the equation satisfied by ϕ, the following:

$$(E-W_0)\phi=(V^{(2)}+V^{(4a)})\phi, \tag{19}$$

where

$$V^{(2)}=\tfrac{1}{2}(K_{01}'K_{10}-K_{01}K_{10}') \tag{20a}$$

and

$$V^{(4a)}=\tfrac{1}{8}(K_{01}'K_{10}'K_{01}'K_{10}-3K_{01}'K_{10}'K_{01}K_{10}'+3K_{01}'K_{10}K_{01}'K_{10}'-K_{01}K_{10}'K_{01}'K_{10}'). \tag{20b}$$

Equation (19) is a *bona fide* two-particle Schrödinger equation with a Hermitian Hamiltonian since, to fourth order at least, the possibility of coupling to the one-meson subspace has been removed.

The interaction operators $V^{(2)}$ and $V^{(4a)}$ have a well-defined physical meaning. Within the limitations of the lowest-order Tamm-Dancoff approximation, the most general type of nucleon-nucleon interaction consists in the interchange of an arbitrary number of mesons between the two nucleons subject only to the proviso that any given meson shall be absorbed before another is created. In $V^{(2)}$ and $V^{(4a)}$, therefore, we exhibit explicitly that part of the total nucleon-nucleon interaction which is contributed by one- and two-meson interchanges, respectively, when there is, at most, one meson in the field. Of course, $V^{(2)}$ and $V^{(4a)}$ also contain self-energy terms which we shall always discard.

It is instructive, at this point, to compare $V^{(2)}$ and $V^{(4a)}$ with the corresponding second- and fourth-order interactions which appear when one reduces the Tamm-Dancoff equations to a single "two-particle" equation by algebraic elimination of all amplitudes other than the no-meson amplitude. The total "potential," as given by (12), has the form

$$V_{TD}=K_{01}(E-W_1)^{-1}K_{10}. \tag{21}$$

As it stands, it involves E. In order to eliminate this dependence on the energy of the system, we make use of an adaptation of the iteration procedure of Lévy[5] and Klein.[6]

Let us first write out Eq. (12) in terms of a momentum-space representation. We obtain:

$$[E-W_0(P)]\phi_0(P)=\sum_{Q,R}(P|K_{01}|Q)[E-W_1(Q)]^{-1}\times(Q|K_{10}|R)\phi_0(R), \tag{22}$$

where we have used P and R to label the free-particle no-meson states, and Q the free-particle one-meson states. Assuming that the energy is small and the coupling to high-momentum states is weak, we can express the energy denominator in the right-hand side of (22) in the following way:

$$\begin{aligned}[E-W_1(Q)]^{-1}&=-\{[W_1(Q)-W_0(R)]-[E-W_0(R)]\}^{-1}\\&=-\frac{1}{W_1(Q)-W_0(R)}\sum_{n=0}^{\infty}\left(\frac{E-W_0(R)}{W_1(Q)-W_0(R)}\right)^n.\end{aligned} \tag{23}$$

Upon substituting (23) into (22) and taking the over-all coupling between the meson and nucleon fields to be small, one is in a position to carry through a sequence of successive iterations of the resultant equation by replacing $[E-W_0(R)]\phi_0(R)$, wherever it appears on the right-hand side, by its value as given by the equation itself. The result is to generate an interaction which is no longer explicitly dependent on the energy and which has the form of a power series in the coupling constant. For the second- and fourth-order interactions, one readily finds:

$$V_{TD}^{(2)}=-K_{01}K_{10}', \tag{24a}$$

$$V_{TD}^{(4a)}=K_{01}(K_{10}'K_{01}K_{10}')'; \tag{24b}$$

the meaning of a prime when affixed to an operator is precisely as given in Eq. (17).

We now observe that, not only is there a lack of correspondence between the two sets of interactions as given by (20a), (20b), (24a), and (24b), but that $V_{TD}^{(2)}$ and $V_{TD}^{(4a)}$ are not even necessarily Hermitian. Accordingly, one cannot generally identify $V_{TD}^{(2)}+V_{TD}^{(4a)}$ as the potential (up to fourth order) which follows from the lowest-order Tamm-Dancoff approximation.

Under certain limiting circumstances, however, this identification can be justified. We recall that the interaction operators which we are considering here make sense only in the limit of small nucleon recoil and so can be expressed as power series in μ/M. Given a typical interaction operator V, there are various ways of approximating it for small nucleon recoil momenta, and we shall want to distinguish these possibilities carefully from one another. In the *fixed-source* approximation, we let the nucleon mass become infinitely large, so that all nucleon recoil effects go out. In the *adiabatic* limit, on the other hand, we retain the leading term of V in its expansion in μ/M; the remaining terms accordingly constitute *nonadiabatic* corrections. It is evident that, from a practical standpoint, the difference between the fixed-source and adiabatic approximations consists in the treatment of nucleon-spin matrix elements, since retardation effects are completely neglected in both cases. Finally, by the *nonrelativistic* limit, we shall imply the retention of all terms which are compatible with the designation of $M+p^2/2M$ as the energy of a free particle.

Returning to our comparison of (20a) and (20b) with (24a) and (24b), we now notice that, for the case of the second-order interaction, the Hermitian character of $V_{TD}^{(2)}$ will be restored and the interaction itself will, in fact, coincide with $V^{(2)}$ if retardation effects are neglected.[14] However, the Hermitian property of $V_{TD}^{(2)}$ is lost as soon as one takes into account nonadiabatic corrections.[15]

Insofar as the fourth-order interaction is concerned, the situation is somewhat more complicated. Upon neglect of retardation, we have

$$V^{(4a)}=\tfrac{1}{2}(K_{01}'K_{10}'K_{01}'K_{10}-K_{01}K_{10}'K_{01}'K_{10}')$$

while $V_{TD}^{(4a)}=K_{01}'K_{10}'K_{01}'K_{10}$, so that the Hermitian character of $V_{TD}^{(4a)}$ as well as its equivalence to $V^{(4a)}$ requires that the operators $K_{01}'K_{10}'$ and $K_{01}'K_{10}$ commute with one another. In our consideration of several cases of particular interest in Sec. III, we shall see that this requirement is satisfied.[16] These results suggest that, to lowest-order approximation in nucleon recoil (adiabatic approximation), the solution of the Tamm-Dancoff equations by canonical transformation or by algebraic substitution will lead to equivalent results.

III. APPLICATIONS

(a) Neutral Scalar Theory

By way of illustration of the formalism of the preceding section, let us evaluate $V^{(2)}$ and $V^{(4a)}$ for the neutral scalar theory, assuming at first that we have to deal with fixed point sources. The interaction Hamiltonian $\mathcal{H}$ will then read

$$\mathcal{H}=f\sum_i \phi(\mathbf{r}_i),\tag{25}$$

where f measures the strength of the coupling between a point nucleon located at $\mathbf{r}_i$ and the meson field $\phi(\mathbf{r}_i)$; the running index i can assume but two values, 1 and 2. Upon expanding ϕ in terms of plane waves in the customary way, we have

$$\phi(\mathbf{r})=\sum_{\mathbf{k}}[2\omega(k)]^{-\frac{1}{2}}[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}+a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}],\tag{26}$$

where $\omega(k)=(\mu^2+k^2)^{\frac{1}{2}}$ and where we have taken the volume of the system to be unity; the coefficients $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are the usual meson absorption and emission operators, respectively. From (25) and (26), it then follows that

$$K_{01}=(K_{10})^\dagger=f\sum_{\mathbf{k}}[2\omega(k)]^{-\frac{1}{2}}[a(\mathbf{k})]_{01}\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i};\tag{27}$$

in view of (17), one has also

$$K_{01}'=-(K_{10}')^\dagger=-f\sum_{\mathbf{k}}2[2\omega(k)]^{-\frac{3}{2}}\times[a(\mathbf{k})]_{01}\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i}.\tag{28}$$

Now, as has already been noted,[14] the second-order interaction (20a), upon neglect of retardation effects, can be simplified to read

$$V^{(2)}=K_{01}'K_{10};\tag{29}$$

if we use the bold-face notation $\mathbf{V}$ to distinguish the matrix representatives of the nucleon-nucleon potential, we then have

$$\mathbf{V}^{(2)}\equiv(0|V^{(2)}|0)=(0|K_{01}'K_{10}|0).\tag{30}$$

It then follows from (27), (28), and (30), together with

$$(0|[a(\mathbf{k}')]_{01}[a^\dagger(\mathbf{k})]_{10}|0)=(0|a(\mathbf{k}')a^\dagger(\mathbf{k})|0)=\delta_{\mathbf{k},\mathbf{k}'},\tag{31}$$

that

$$\mathbf{V}^{(2)}=-(f^2/2)\sum_{\mathbf{k}}[\omega(k)]^{-2}\sum_{i,j}e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}.\tag{32}$$

[14] For we may then replace W_0 by $2M$, whence we have the equality $K_{01}'K_{10}=-K_{01}K_{10}'$.

[15] By way of illustration, see the detailed discussions of nonadiabatic corrections to nuclear forces for the scalar and pseudoscalar theories as given by Lévy (reference 5) and Klein (reference 6); in particular, note that certain nonadiabatic terms appear which are proportional to the difference of the kinetic energies of the initial and final no-meson states and which are non-Hermitian [see, e.g., Eq. (35) of reference 6]. Non-Hermitian terms will also appear in the Bethe-Salpeter theory [see, e.g., Eq. (39) of reference 11].

[16] To see how this result follows, we note that, for the usual meson theories, the operators $K_{01}'K_{10}'$ and $K_{01}'K_{10}$, in the adiabatic limit, will be *static* in that they will depend on the relative coordinates of the two particles as well as their spins and isotopic spins, and so can be expressed as a sum of invariant forms involving these variables [see, e.g., L. Eisenbud and E. P. Wigner, Proc. Natl. Acad. Sci. **27**, 281 (1941); D. Feldman, Phys. Rev. **92**, 824 (1953)]. Under these circumstances, the required commutativity is easily demonstrated.

Upon dropping self-energy terms and converting the summation over $\mathbf{k}$ into an integral $(\sum_{\mathbf{k}}\rightarrow(2\pi)^{-3}\int d\mathbf{k})$, we obtain the well-known result[17]

$$\mathbf{V}^{(2)}=-f^2(2\pi)^{-3}\int d\mathbf{k}\omega^{-2}e^{i\mathbf{k}\cdot\mathbf{r}}=-(f^2/4\pi)r^{-1}e^{-\mu r}, \quad (33\text{a})$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$.

The corresponding fourth-order interaction $\mathbf{V}^{(4a)}\equiv(0|V^{(4a)}|0)$ can be easily evaluated in a similar way from Eq. (20b). After an elementary calculation, one finds

$$\mathbf{V}^{(4a)}=f^4(2\pi)^{-6}\int d\mathbf{k}\omega^{-2}e^{i\mathbf{k}\cdot\mathbf{r}}\int d\mathbf{k}'(\omega')^{-3}e^{i\mathbf{k}'\cdot\mathbf{r}},$$

or

$$\mathbf{V}^{(4a)}=(f^2/4\pi)^2r^{-1}e^{-\mu r}(2/\pi)K_0(\mu r), \quad (33\text{b})$$

where K_0 is the zeroth-order Hankel function of imaginary argument.[18]

Now, the neutral scalar theory with fixed sources serves as a particularly useful example since, for this case, the interaction (33a) is known to be rigorously correct to all orders in the coupling constant.[19] We can therefore conclude, at least when the coupling constant is small, that the lowest-order Tamm-Dancoff approximation is, in fact, not an improvement over the use of second-order weak-coupling perturbation theory, for it fails to include certain fourth-order terms which are of the same order of magnitude as (and which, for the case at hand, precisely cancel) those fourth-order terms which are taken into account.[20]

To obtain the complete fourth-order interaction, we must allow for the possibility of having two mesons present in the field at the same time. Thus, let us extend the lowest-order Tamm-Dancoff approximation by retaining the two-meson amplitude ϕ_2 in addition to the no-meson and one-meson amplitudes, ϕ_0 and ϕ_1 respectively, which formed the basis of our previous considerations. The generalization of the formalism of Sec. II to this case is essentially straightforward and is carried through in the following way.

The states Ψ of our system can now be written as three-component column matrices,

$$\Psi=\begin{bmatrix}\phi_0\\ \phi_1\\ \phi_2\end{bmatrix}, \quad (34)$$

with the normalization

$$|\phi_0|^2+|\phi_1|^2+|\phi_2|^2=1. \quad (35)$$

The free-field and interaction Hamiltonians, $\mathcal{H}$ and $\mathcal{K}$, respectively, will accordingly appear as three-by-three matrices having the following forms:

$$\mathcal{H}=\begin{bmatrix}W_0 & 0 & 0\\ 0 & W_1 & 0\\ 0 & 0 & W_2\end{bmatrix}, \quad (36\text{a})$$

$$\mathcal{K}=\begin{bmatrix}0 & K_{01} & 0\\ K_{10} & 0 & K_{12}\\ 0 & K_{21} & 0\end{bmatrix}; \quad (36\text{b})$$

those off-diagonal elements of $\mathcal{K}$ which correspond to a direct coupling of the no-meson and two-meson Fock subspaces are identically zero since, by hypothesis, $\mathcal{K}$ is linear in the meson field variables.

We proceed as before to apply a canonical transformation

$$\Psi=(\exp-i\mathcal{S})\Phi \quad (13)$$

to the Schrödinger equation

$$(E-\mathcal{H})\Psi=\mathcal{K}\Psi, \quad (1)$$

where $\mathcal{S}$ is now chosen so as to remove the first-order coupling between ϕ_0 and ϕ_1, and ϕ_1 and ϕ_2, i.e., we require that

$$-i[\mathcal{S},\mathcal{H}]=\mathcal{K}. \quad (15)$$

By analogy with Eqs. (16a) and (16b), we can write

$$\mathcal{S}=-i\mathcal{K}', \quad (37\text{a})$$

where

$$\mathcal{K}'=\begin{bmatrix}0 & K_{01}' & 0\\ K_{10}' & 0 & K_{12}'\\ 0 & K_{21}' & 0\end{bmatrix}. \quad (37\text{b})$$

The operators K_{st}' $(s,t=0,1,2)$ have the momentum-space representations

$$(P|K_{st}'|Q)=(P|K_{st}|Q)[W_s(P)-W_t(Q)]^{-1}, \quad (38)$$

where P and Q denote the momenta, spins, and isotopic spins of free-particle s-meson and t-meson states, respectively.

On carrying through the unitary transformation defined by (13), (37), and (38), Eq. (1) will once again go over into a new Schrödinger equation for the vector Φ of the form given by (18a) and (18b). However, unlike our previous discussion, the new second-order interaction Hamiltonian will now no longer be diagonal, but will, in fact, contain off-diagonal terms coupling the no-meson and two-meson Fock subspaces directly, *viz.*,

$$\begin{aligned}(\mathcal{L}^{(2)})_{02}&=\tfrac{1}{2}(K_{01}'K_{12}-K_{01}K_{12}'),\\ (\mathcal{L}^{(2)})_{20}&=\tfrac{1}{2}(K_{21}'K_{10}-K_{21}K_{10}').\end{aligned} \quad (39)$$

Accordingly, to fourth order in the coupling constant, we have not as yet attained a complete decoupling of the no-meson amplitude from all others.

Let us write $\mathcal{L}^{(2)}$ as the sum of a diagonal and an off-diagonal matrix which we denote by $\mathfrak{D}$ and $\mathfrak{O}$, respectively; in particular, the only nonvanishing

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[17] See, for example, G. Wentzel, *Einführung in die Quantentheorie der Wellenfelder* (Franz Denticke, Vienna, 1943), p. 44.

[18] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1945), second edition, pp. 78, 172; for details of the integration procedure, see the appendix of the second part of reference 5.

[19] Reference 17, p. 46.

[20] A similar argument has been given by Henley and Ruderman (reference 9).

matrix elements of $\mathcal{O}$ are given by (39). Then, to remove the off-diagonal terms (39) from the interaction Hamiltonian, we need to apply a second unitary transformation

$$\Phi=(\exp-i\mathcal{T})\Xi \tag{40}$$

to (18a), where it is evident that $\mathcal{T}$ must be determined so that

$$-i[\mathcal{T},\mathcal{H}]=\mathcal{O}. \tag{41}$$

This requirement will be satisfied if $\mathcal{T}$ is given by

$$\mathcal{T}=-i\mathcal{O}', \tag{42a}$$

where the only nonzero matrix elements of $\mathcal{O}'$ are

$$\begin{aligned}(\mathcal{O}')_{02}&=\tfrac{1}{2}(K_{01}'K_{12}-K_{01}K_{12}')',\\(\mathcal{O}')_{20}&=\tfrac{1}{2}(K_{21}'K_{10}-K_{21}K_{10}')';\end{aligned} \tag{42b}$$

the meaning of the primed operators is given by (38).

On carrying through the $\mathcal{T}$-transformation defined by (40), (42), and (38), we obtain the Schrödinger equation for the transformed vector Ξ, *viz.*,

$$(E-\mathcal{H})\Xi=\mathfrak{M}\Xi, \tag{43a}$$

where

$$\mathfrak{M}=\mathfrak{M}^{(2)}+\mathfrak{M}^{(3)}+\mathfrak{M}^{(4)}+\cdots, \tag{43b}$$

and where the second-order term $\mathfrak{M}^{(2)}$ has the form of a diagonal three-by-three matrix. Although the third- and fourth-order terms, $\mathfrak{M}^{(3)}$ and $\mathfrak{M}^{(4)}$, contain off-diagonal matrix elements, these will not contribute to the fourth-order nucleon-nucleon interaction and so can be ignored.

To fourth order in the coupling constant, therefore, the new no-meson Tamm-Dancoff amplitude, which we again denote by ϕ, is completely decoupled from all the others; hence, we identify the equation for ϕ with the two-particle Schrödinger equation for the system in which the matrix element $(\mathfrak{M}^{(2)}+\mathfrak{M}^{(4)})_{00}$ will appear as the (Hermitian) interaction Hamiltonian. The result is to find that ϕ satisfies

$$(E-W_0)\phi=(V^{(2)}+V^{(4a)}+V^{(4b)})\phi, \tag{44}$$

where explicit expressions for $V^{(2)}$ and $V^{(4a)}$ have already been given in Eqs. (20a) and (20b), and where[21]

$$V^{(4b)}=V^{(4b')}+V^{(4b'')}, \tag{45a}$$

with

$$\begin{aligned}V^{(4b')}=\tfrac{1}{8}(&K_{01}'K_{12}'K_{21}'K_{10}-3K_{01}'K_{12}'K_{21}K_{10}'\\&+3K_{01}'K_{12}K_{21}'K_{10}'-K_{01}K_{12}'K_{21}'K_{10}')\end{aligned} \tag{45b}$$

and

$$\begin{aligned}V^{(4b'')}=\tfrac{1}{8}[&(K_{01}'K_{12}-K_{01}K_{12}')'(K_{21}'K_{10}-K_{21}K_{10}')\\&-(K_{01}'K_{12}-K_{01}K_{12}')(K_{21}'K_{10}-K_{21}K_{10}')'].\end{aligned} \tag{45c}$$

[21] The general expression for $V^{(4b)}$ can be readily shown to be equivalent, in the adiabatic limit, to $V_{TD}{}^{(4b)}=-K_{01}(K_{12}(K_{21}K_{10}')')'$ which is the corresponding contribution to the fourth-order potential which is obtained by eliminating the one-meson and two-meson amplitudes by algebraic substitution.

Let us denote the entire fourth-order interaction by $V^{(4)}$, i.e.,

$$V^{(4)}=V^{(4a)}+V^{(4b)}. \tag{46}$$

Then, it is clear from our previous discussion that $V^{(4a)}$ represents that part of the fourth-order interaction which comes about from the interchange of two mesons by the nucleons when there is never more than one meson present in the field. On extending the lowest-order Tamm-Dancoff approximation so as to allow for the simultaneous presence of two mesons in the field, we obtain an additional contribution to the fourth-order potential which we have called $V^{(4b)}$. The sum of these two contributions yields $V^{(4)}$ which is clearly equivalent to the usual perturbation-theoretic result.

Returning to the neutral scalar theory with fixed sources and making use of the additional relations

$$\begin{aligned}K_{12}&=(K_{21})^\dagger=f\textstyle\sum_{\mathbf{k}}[2\omega(k)]^{-\frac{1}{2}}\\&\qquad\times[a(\mathbf{k})]_{12}\textstyle\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i},\\K_{12}'&=-(K_{21}')^\dagger=-f\textstyle\sum_{\mathbf{k}}2[2\omega(k)]^{-\frac{3}{2}}\\&\qquad\times[a(\mathbf{k})]_{12}\textstyle\sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i},\end{aligned} \tag{47}$$

and

$$\begin{aligned}(0|a(\mathbf{k})a(\mathbf{k}')a^\dagger(\mathbf{k}'')a^\dagger(\mathbf{k}''')|0)&\\=\delta_{\mathbf{k},\mathbf{k}'''}\delta_{\mathbf{k}',\mathbf{k}''}+\delta_{\mathbf{k},\mathbf{k}''}\delta_{\mathbf{k}',\mathbf{k}'''},&\end{aligned} \tag{48}$$

one finds easily that

$$\mathbf{V}^{(4b')}=-(f^2/4\pi)^2r^{-1}e^{-\mu r}(2/\pi)K_0(\mu r)=-\mathbf{V}^{(4a)} \tag{49a}$$

and

$$\mathbf{V}^{(4b'')}=0. \tag{49b}$$

Hence, as anticipated, the two-meson contribution $\mathbf{V}^{(4b)}$ to the fourth-order potential precisely cancels the one-meson contribution $\mathbf{V}^{(4a)}$ as given by (33b), i.e.,

$$\mathbf{V}^{(4)}=0. \tag{50}$$

This then reaffirms our previous statement that, when one deals with essentially weak coupling, perturbation theory seems to yield a better result than the Tamm-Dancoff approximation.

Two additional remarks are appropriate at this point, and they both concern the term $\mathbf{V}^{(4a)}$. In the first place, it is evident that, for the case at hand, we have assumed fixed point-source nucleons from the very beginning so that the ladder corrections $\mathbf{V}^{(4a)}$ to the second-order potential are adiabatic and static. Nevertheless, various authors,[5,6,10] when they have treated the Tamm-Dancoff equations by algebraic elimination of all amplitudes except for the two-nucleon amplitude, have referred to the corresponding contributions to the over-all potential as "nonadiabatic and velocity-dependent" corrections. In point of fact, this nomenclature is misleading. Indeed, it is clear, from our earlier discussion, that we would have obtained precisely the same result for $\mathbf{V}^{(4a)}$ had we used $V_{TD}{}^{(4a)}$ [Eq. (24b)] as the starting point of our considerations instead of $V^{(4a)}$ [Eq. (20b)] since the nucleons are taken to be at rest. It must be stressed that the iteration procedure which was used to deduce the general form of $V_{TD}^{(4a)}$ goes through in exactly the same way even if one deals with fixed sources, provided one takes care not to approximate the energy of the system E which appears explicitly in V_{TD} [Eq. (21)] by $2M$.

Secondly, it has been argued, particularly by Brueckner and Watson,[10] that the iteration procedure which was used to express V_{TD} in the form of a power series is invalid, and that one ought in fact not expand the energy denominator in V_{TD} at all. Then, one would expect that the total potential, to fourth order, would be better approximated by omitting the term $\mathbf{V}^{(4a)}$ altogether, so that one would be left with the net interaction $\mathbf{V}^{(2)}+\mathbf{V}^{(4b)}$ instead of $\mathbf{V}^{(2)}$. However, since $\mathbf{V}^{(2)}$ is also known to be the exact result for the neutral scalar theory with fixed sources, this procedure of dropping the "nonadiabatic velocity-dependent" corrections is suspect, at least when the coupling is weak.[20] In any case, it is clear that, within the framework of the method of canonical transformations, there appears to be no justification for treating $\mathbf{V}^{(4a)}$ on a different footing from the contributions $\mathbf{V}^{(4b)}$ which appear upon introduction of the two-meson amplitude.

We shall complete our discussion of the neutral scalar theory by considering to what extent the fixed-source potential [Eqs. (33a) and (50)] must be modified so as to constitute the complete nonrelativistic interaction. Now, for an attractive Yukawa potential of the type (33a), the nucleon velocities v, in a bound system like the deuteron, will be of the order of μ/M. Also, the binding energy is small compared to μ^2/M so that the mean kinetic energy is of the order of the depth of the potential well, i.e., $\mu^2/M\sim(f^2/4\pi)\mu$. Hence we have $v\sim\mu/M\sim f^2/4\pi$.

To obtain an interaction which is consistent with the retention of terms of order μ^2/M for the kinetic energy of the nucleons, we consequently need to compute the second-order interaction $\mathbf{V}^{(2)}$ more carefully and look for nonadiabatic corrections which are of order μ/M with respect to the adiabatic potential given by (33a). The fourth-order potential is already given correctly by (50). It is therefore sufficient to confine our considerations to the equation

$$(E-W_0)\phi=V^{(2)}\phi, \tag{51}$$

with $V^{(2)}$ given by (20a).

The interaction Hamiltonian $\mathcal{H}$ will now be of the form

$$\mathcal{H}=f\int d\mathbf{r}\bar{\psi}\psi\phi; \tag{52}$$

here, ψ is the nucleon field variable and $\bar{\psi}=\psi^\dagger\gamma_4$. The latter quantities are conveniently expanded in terms of plane waves, leading to the usual expressions

$$\psi(\mathbf{r})=\sum_{\mathbf{p}}\sum_{\lambda=1}^{4} b(\mathbf{p}\lambda)u(\mathbf{p}\lambda)e^{i\mathbf{p}\cdot\mathbf{r}}, \tag{53a}$$

$$\bar{\psi}(\mathbf{r})=\sum_{\mathbf{p}}\sum_{\lambda=1}^{4} b^\dagger(\mathbf{p}\lambda)\bar{u}(\mathbf{p}\lambda)e^{-i\mathbf{p}\cdot\mathbf{r}}, \tag{53b}$$

where $b(\mathbf{p}\lambda)$ and $b^\dagger(\mathbf{p}\lambda)$ are the nucleon absorption and emission operators, respectively, and where $u(\mathbf{p}\lambda)$ is the free-particle Dirac spinor which is characterized by the momentum $\mathbf{p}$ and spin-energy index λ. For definiteness, let $\lambda=1,2$ and $\lambda=3,4$ denote positive- and negative-energy solutions, respectively. The $u(\mathbf{p}\lambda)$ are normalized so that

$$\sum_{\rho=1}^{4} u_\rho^\dagger(\mathbf{p}\lambda)u_\rho(\mathbf{p}\lambda')=\delta_{\lambda,\lambda'}. \tag{54}$$

Upon substituting (26) and (53) into (52), we find that K_{01} and K_{10} are given by

$$K_{01}=(K_{10})^\dagger=f\sum_{\mathbf{p},\mathbf{k}}\sum_{\lambda,\lambda'=1}^{2}[(2\omega(k)]^{-\frac{1}{2}}\bar{u}(\mathbf{p}+\mathbf{k}\lambda')\times u(\mathbf{p}\lambda)[b^\dagger(\mathbf{p}+\mathbf{k}\lambda')b(\mathbf{p}\lambda)a(\mathbf{k})]_{01}; \tag{55}$$

we have restricted the possible values of the spin-energy indices to 1 and 2 since only positive-energy spinors are relevant in the calculation of the second-order interaction. Similarly, we have

$$K_{01}'=-(K_{10}')^\dagger=-f\sum_{\mathbf{p},\mathbf{k}}\sum_{\lambda,\lambda'=1}^{2}[2\omega(k)]^{-\frac{1}{2}}\times[\omega(k)+E(p)-E(\mathbf{p}+\mathbf{k})]^{-1}\bar{u}(\mathbf{p}+\mathbf{k}\lambda')u(\mathbf{p}\lambda)\times[b^\dagger(\mathbf{p}+\mathbf{k}\lambda')b(\mathbf{p}\lambda)a(\mathbf{k})]_{01}, \tag{56}$$

where $E(p)=(M^2+p^2)^{\frac{1}{2}}$.

Now, in momentum space, Eq. (51) will have the form

$$[E-E(1)-E(2)]\phi(1,2)=\tfrac{1}{2}\sum_{3,4}(1,2|V^{(2)}|(3,4)\phi(3,4), \tag{57}$$

where, for brevity, we have simply written i for $\mathbf{p}_i\lambda_i$. The factor $\frac{1}{2}$ which appears on the right-hand side of (57) arises on account of the fact that we are dealing with a system of two identical particles.[22] The mo-

[22] In making the transcription from (51) to (57), we expand ϕ in terms of the complete set of two-nucleon free-particle states $\Psi(i,j)$ where these, in turn, can be defined by the relation $\Psi(i,j)=b^\dagger(i)b^\dagger(j)\Psi_{\text{vac}}$. However, since $\Psi(i,j)$ and $\Psi(j,i)$ are not independent of one another but, in fact, satisfy $\Psi(j,i)=-\Psi(i,j)$, we set $\phi=\sum_{i>j}\phi(i,j)\Psi(i,j)$ where it is assumed that the single-particle states are ordered. On substitution into (51), we then find

$$[E-E(1)-E(2)]\phi(1,2)=\sum_{3>4}(1,2|V^{(2)}|3,4)\phi(3,4). \tag{57'}$$

Alternatively, we may put $\phi=\frac{1}{2}\sum_{i,j}\phi(i,j)\Psi(i,j)$ with $\phi(j,i)=-\phi(i,j)$, in which case we are led to the form given in the text.

mentum-space representation of the second-order nucleon-nucleon potential $\mathbf{V}^{(2)}$ will accordingly be determined by requiring that

$$\sum_{3,4}(1,2|\mathbf{V}^{(2)}|3,4)\phi(3,4)$$
$$=\tfrac{1}{2}\sum_{3,4}(1,2|V^{(2)}|3,4)\phi(3,4)$$
$$=\tfrac{1}{4}\sum_{3,4}(1,2|K_{01}'K_{10}-K_{01}K_{10}'|3,4)\phi(3,4). \quad (58)$$

The explicit evaluation of the matrix element $(1,2|\mathbf{V}^{(2)}|3,4)$, including nonadiabatic corrections, proceeds in a rather straightforward way upon insertion of (55) and (56) into (58) along with application of (31) and the analogous relation

$$(1,2|b^\dagger(j')b(i')b^\dagger(j)b(i)|3,4)$$
$$=(\delta_{j1}\delta_{j'2}-\delta_{j2}\delta_{j'1})(\delta_{i3}\delta_{i'4}-\delta_{i4}\delta_{i'3}) \quad (59)$$

(ignoring self-energy terms, as always). The final result is to find[23]

$$(\mathbf{p}_1\lambda_1,\mathbf{p}_2\lambda_2|\mathbf{V}^{(2)}|\mathbf{p}_1+\mathbf{k}\lambda_3,\ \mathbf{p}_2-\mathbf{k}\lambda_4)$$
$$=-\frac{f^2}{2}\left(\frac{1}{2\pi}\right)^3\left\{\frac{1}{\omega^2(k)-[E(p_1)-E(\mathbf{p}_1+\mathbf{k})]^2}\right.$$
$$\left.+\frac{1}{\omega^2(k)-[E(p_2)-E(\mathbf{p}_2-\mathbf{k})]^2}\right\}\bar{u}(\mathbf{p}_1\lambda_1)$$
$$\times u(\mathbf{p}_1+\mathbf{k}\lambda_3)\bar{u}(\mathbf{p}_2\lambda_2)u(\mathbf{p}_2-\mathbf{k}\lambda_4). \quad (60)$$

It is now immediately evident from (60) that, in the adiabatic limit, one will simply regain the fixed-source second-order potential (33a); also, that the first nonvanishing nonadiabatic corrections are of order $(\mu/M)^2$ with respect to the adiabatic potential.[24] Accordingly, the nonrelativistic interaction is given completely by (33a).

(b) Charge-Symmetric *Ps-Ps* Theory

In this subsection we shall consider the nuclear-force problem for the case of greatest physical interest, *viz.*, the charge-symmetric pseudoscalar theory with pseudoscalar coupling. The interaction Hamiltonian is then given by

$$\mathcal{K}=iG\int d\mathbf{r}\bar{\psi}\gamma_5\tau_\alpha\phi_\alpha\psi, \quad (61)$$

where G is the coupling constant, γ_5 is the Dirac matrix $\gamma_1\gamma_2\gamma_3\gamma_4$, and the τ_α ($\alpha=1, 2, 3$) are the usual isotopic-spin matrices. The meson field components ϕ_α now constitute a vector in isotopic-spin space, while the nucleon field variable ψ is an eight-component Dirac spinor. We adopt the convention that the index α is to be summed from 1 to 3 whenever it appears twice in an expression.

In point of fact, we shall find it more convenient to work with the equivalent form

$$\mathcal{K}=\mathcal{K}^{(pr)}+\mathcal{K}^{(pv)}+\cdots, \quad (62a)$$

with

$$\mathcal{K}^{(pr)}=(G^2/2M)\int d\mathbf{r}\bar{\psi}\psi\phi_\alpha\phi_\alpha, \quad (62b)$$

$$\mathcal{K}^{(pv)}=(G/2M)\int d\mathbf{r}\psi^\dagger\boldsymbol{\sigma}\cdot(\nabla\phi_\alpha)\tau_\alpha\psi, \quad (62c)$$

which is obtained from (61) upon application of the Dyson-Foldy transformation.[25] The pseudoscalar interaction is thus equivalent to the sum of the pair and pseudovector interactions, $\mathcal{K}^{(pr)}$ and $\mathcal{K}^{(pv)}$, respectively, plus other terms of order $(G/2M)^2$ which will not influence any of the following considerations and which will accordingly be completely ignored.

Now, it is evident from (62) that the leading second- and fourth-order terms of the nucleon-nucleon potential will be proportional to $(G^2/4\pi)(\mu/2M)^2$ and $(G^2/4\pi)^2\times(\mu/2M)^2$, respectively. If we assume that the coupling constant is large, say, $G^2/4\pi\sim10$,[26] we would expect the fourth-order interaction to predominate over that of the second order.[27] We shall accordingly take for the nonrelativistic nucleon-nucleon potential the leading fourth-order interaction, which goes as $(G^2/4\pi)^2(\mu/2M)^2$, plus correction terms of order $(G^2/4\pi)(\mu/2M)^2$ and $(G^2/4\pi)^2(\mu/2M)^3$;[28] terms which involve powers of G higher than the fourth will not be considered.

We shall now need to extend our previous formalism somewhat since the interaction Hamiltonian (62) involves the meson field both linearly and bilinearly. For the Schrödinger equation, we have

$$(E-\mathcal{K})\Psi=(\mathcal{K}^{(pr)}+\mathcal{K}^{(pv)})\Psi, \quad (63)$$

[23] A similar result has been derived by M. Jean, Compt. rend. **232**, 2045 (1951), using Schwinger's covariant perturbation theory.

[24] In Lévy's discussion of this problem (reference 5), there do appear nonadiabatic corrections which are of order μ/M with respect to the fixed-source potential (and which we have alluded to earlier in reference 15); these terms are discarded, however, when it is shown, by means of a variational principle, that they lead to corrections to the adiabatic coupling constant which are of order $(\mu/M)^2$.

[25] See, for example, S. D. Drell and E. M. Henley, Phys. Rev. **88**, 1053 (1952).

[26] Bethe, Dyson, Mitra, Ross, Salpeter, Schweber, Sundaresen, and Visscher, Phys. Rev. **90**, 372 (1953).

[27] H. A. Bethe, Phys. Rev. **76**, 191 (1949); K. M. Watson and J. V. Lepore, Phys. Rev. **76**, 1157 (1949).

[28] If we take the perturbation-theoretic approach seriously so that the dominant term in the potential is the leading fourth-order interaction, then, to have a bound system like the deuteron, one can estimate roughly that $G^2/4\pi\sim4(M/\mu)^{\frac{1}{2}}$. This is in accord with the order of magnitude of the coupling constant assumed in the text and justifies, formally, the selection of terms to be retained in the nonrelativistic potential.

where the state vector Ψ of the system is once again represented by a three-component column matrix (34) and the free-field Hamiltonian $\mathcal{H}$ appears as a diagonal three-by-three matrix (36a). On the other hand, the pair and pseudovector interaction Hamiltonians will have the following structure:

$$\mathcal{H}^{(pr)}=\begin{bmatrix} 0 & 0 & K_{02} \\ 0 & K_{11} & 0 \\ K_{20} & 0 & 0 \end{bmatrix}, \tag{64a}$$

$$\mathcal{H}^{(pv)}=\begin{bmatrix} 0 & K_{01} & 0 \\ K_{10} & 0 & K_{12} \\ 0 & K_{21} & 0 \end{bmatrix}. \tag{64b}$$

In (64a) we have dropped the matrix elements K_{00} and K_{22}; the former contains self-energy terms only, while the latter will not contribute to the nuclear force until the sixth order.

We proceed next to apply a series of canonical transformations so as to decouple the no-meson amplitude ϕ_0 from ϕ_1 and ϕ_2. Since the pair term is the larger of the two interaction terms, we first set

$$\Psi=(\exp-i\mathcal{R})\Phi, \tag{65}$$

where $\mathcal{R}$ is chosen so as to eliminate the off-diagonal part of $\mathcal{H}^{(pr)}$ from the interaction Hamiltonian; we therefore take

$$\mathcal{R}=-i\begin{bmatrix} 0 & 0 & K_{02}' \\ 0 & 0 & 0 \\ K_{20}' & 0 & 0 \end{bmatrix}. \tag{66}$$

On carrying through the unitary transformation defined by (65) and (66), we find that Φ satisfies the equation

$$(E-\mathcal{H})\Phi=\mathcal{L}\Phi, \tag{67a}$$

where

$$\mathcal{L}=\mathcal{H}^{(pv)}+\mathcal{H}_D{}^{(pr)}+\mathcal{L}^{(3)}+\mathcal{L}^{(4)}+\cdots; \tag{67b}$$

we denote by $\mathcal{H}_D{}^{(pr)}$ the diagonal part of the pair interaction. The third- and fourth-order terms, $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$, involve the pair interaction once and twice, respectively; in particular, the fourth-order two-pair contribution to the nuclear force is already completely contained in $\mathcal{L}^{(4)}$.

In order to obtain the fourth-order one-pair interaction as well as the second-order potential, we apply a second transformation of the form

$$\Phi=(\exp-i\mathcal{S})\Xi, \tag{68}$$

where $\mathcal{S}$ is given by (37a) and (37b), thereby removing $\mathcal{H}^{(pv)}$ from the interaction Hamiltonian. The resultant equation satisfied by Ξ will then have the form

$$(E-\mathcal{H})\Xi=(\mathfrak{M}^{(2)}+\mathfrak{M}^{(3)}+\mathfrak{M}^{(4)}+\cdots)\Xi. \tag{69}$$

While $\mathfrak{M}^{(2)}$ still contains off-diagonal matrix elements [they are, in fact, precisely the same as those listed in (39)], their removal, by a third canonical transformation, will only serve to lead to a contribution to the potential of order $(G^2/4\pi)^2(\mu/2M)^4$. It is also once again clear that the off-diagonal matrix elements contained in $\mathfrak{M}^{(3)}$ and $\mathfrak{M}^{(4)}$ will not affect the fourth-order nuclear force at all. Accordingly, we can conclude that, to the order of interest in the calculation of the nonrelativistic nuclear force, the new no-meson Tamm-Dancoff amplitude ϕ is completely decoupled from all the others so that the equation satisfied by ϕ can be identified as the two-particle Schrödinger equation for the system [the interaction Hamiltonian will be given by $(\mathfrak{M}^{(2)}+\mathfrak{M}^{(4)})_{00}$]. After an elementary calculation, one finds that this equation is given by

$$(E-W_0)\phi=(V^{(2)}+V^{(4a)}+V^{(4b)})\phi, \tag{70}$$

where

$$V^{(2)}=\tfrac{1}{2}(K_{01}'K_{10}-K_{01}K_{10}'), \tag{71a}$$

$$V^{(4a)}=\tfrac{1}{2}(K_{02}'K_{20}-K_{02}K_{20}'), \tag{71b}$$

$$V^{(4b)}=-(K_{01}'K_{11}K_{10}'+K_{01}'K_{12}K_{20}'+K_{02}'K_{21}K_{10}'). \tag{71c}$$

It is evident that $V^{(4a)}$ and $V^{(4b)}$ involve the pair interaction twice and once, respectively. All three terms in (71) are, of course, Hermitian.[29]

In proceeding to apply (71) to the charge-symmetric *ps-ps* theory, we shall find it convenient, at first, to treat the interaction Hamiltonian (62) in the fixed-source approximation. We can then write

$$\begin{aligned}\mathcal{H}^{(pr)}&=(G^2/2M)\textstyle\sum_i \boldsymbol{\phi}_\alpha(\mathbf{r}_i)\boldsymbol{\phi}_\alpha(\mathbf{r}_i),\\ \mathcal{H}^{(pv)}&=(G/2M)\textstyle\sum_i \boldsymbol{\sigma}^{(i)}\cdot\boldsymbol{\nabla}\boldsymbol{\phi}_\alpha(\mathbf{r}_i)\tau_\alpha{}^{(i)}.\end{aligned} \tag{72}$$

Next, we expand $\boldsymbol{\phi}_\alpha$ in a Fourier series, *viz.*,

$$\boldsymbol{\phi}_\alpha(\mathbf{r})=\textstyle\sum_{\mathbf{k}}[2\omega(k)]^{-\frac{1}{2}}[a_\alpha(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}+a_\alpha{}^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}], \tag{73}$$

whereupon we may write

$$\begin{aligned}K_{n,\,n+1}&=(K_{n+1,\,n})^\dagger=i(G/2M)\sum_{\mathbf{k}}[2\omega(k)]^{-\frac{1}{2}}\\ &\quad\times[a_\alpha(\mathbf{k})]_{n,\,n+1}\sum_i(\boldsymbol{\sigma}^{(i)}\cdot\mathbf{k})\tau_\alpha{}^{(i)}e^{i\mathbf{k}\cdot\mathbf{r}_i},\\ K'_{n,\,n+1}&=-(K'_{n+1,\,n})^\dagger=-i(G/2M)\sum_{\mathbf{k}}2[2\omega(k)]^{-\frac{3}{2}}\\ &\quad\times[a_\alpha(\mathbf{k})]_{n,\,n+1}\sum_i(\boldsymbol{\sigma}^{(i)}\cdot\mathbf{k})\tau_\alpha{}^{(i)}e^{i\mathbf{k}\cdot\mathbf{r}_i};\end{aligned} \tag{74}$$

[29] If one eliminates the one-meson and two-meson amplitudes from the Tamm-Dancoff equations by algebraic substitution, one finds, for the fourth-order potential, $V_{TD}{}^{(4a)}=-K_{02}K_{20}'$ and $V_{TD}{}^{(4b)}=K_{01}(K_{11}K_{10}')'+K_{01}(K_{12}K_{20}')'+K_{02}(K_{21}K_{10}')'$. It is then clear that, in the adiabatic limit, these expressions will agree with those for $V^{(4a)}$ and $V^{(4b)}$ which are given in the text.

also,

$$K_{02}=(K_{20})^{\dagger}=(G^2/4M)\sum_{\mathbf{k},\mathbf{k}'}(\omega\omega')^{-\frac{1}{2}}\times[a_\alpha(\mathbf{k})a_\alpha(\mathbf{k}')]_{02}\sum_i e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_i},$$

$$K_{02}'=-(K_{20}')^{\dagger}=-(G^2/4M)\sum_{\mathbf{k},\mathbf{k}'}(\omega\omega')^{-\frac{1}{2}}(\omega+\omega')^{-1}\times[a_\alpha(\mathbf{k})a_\alpha(\mathbf{k}')]_{02}\sum_i e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}_i}; \tag{75}$$

and finally

$$K_{11}=(G^2/2M)\sum_{\mathbf{k},\mathbf{k}'}(\omega\omega')^{-\frac{1}{2}}[a_\alpha^{\dagger}(\mathbf{k}')a_\alpha(\mathbf{k})]_{11}\times\sum_i e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_i}; \tag{76}$$

in (76), we have discarded a self-energy term.

It is now a straightforward matter to insert (74)–(76) into (71) and so obtain the desired nuclear force; it is also necessary to make use of (31) and (48), suitably generalized for the charge-symmetric theory. One then finds directly, for the second-order nuclear force,

$$\begin{aligned}\mathbf{V}^{(2)}&=-(G/2M)^2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(2\pi)^{-3}\times\int d\mathbf{k}(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{k})(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{k})\omega^{-2}e^{i\mathbf{k}\cdot\mathbf{r}}\\&=(G^2/4\pi)(2M)^{-2}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\nabla})\times(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{\nabla})(r^{-1}e^{-\mu r});\end{aligned} \tag{77a}$$

for the fourth-order two-pair interaction,[18]

$$\begin{aligned}\mathbf{V}^{(4a)}&=-3(G^2/2M)^2(2\pi)^{-6}\times\int d\mathbf{k}d\mathbf{k}'[\omega\omega'(\omega+\omega')]^{-1}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\\&=-3(G^2/4\pi)^2(\mu/2M)^2\times[(2/\pi)(\mu r^2)^{-1}K_1(2\mu r)];\end{aligned} \tag{77b}$$

and for the fourth-order one-pair interaction,

$$\begin{aligned}\mathbf{V}^{(4b)}&=-6(G^2/2M)(G/2M)^2(2\pi)^{-6}\times\int d\mathbf{k}d\mathbf{k}'(\mathbf{k}\cdot\mathbf{k}')(\omega\omega')^{-2}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\\&=6(G^2/4\pi)^2(2M)^{-3}[\boldsymbol{\nabla}(r^{-1}e^{-\mu r})]^2\\&=6(G^2/4\pi)^2(\mu/2M)^3[1+(\mu r)^{-1}]^2(\mu r^2)^{-1}e^{-2\mu r}.\end{aligned} \tag{77c}$$

It will be noticed that the fourth-order potential $\mathbf{V}^{(4a)}+\mathbf{V}^{(4b)}$, which has been derived here in the fixed-source approximation, is in agreement with Lévy's result (see second part of reference 5) as modified by Klein.[6,30]

[30] That the Lévy-Klein potential can be deduced by applying perturbation theory in the limit of fixed point sources has also been noted by S. D. Drell and K. Huang, Phys. Rev. **91**, 1527 (1953) and by K. J. Le Couteur, Report of the Birmingham Conference on Nuclear Physics, 1953, Birmingham, England (unpublished).

In order to complete our discussion of the nonrelativistic nuclear force, we need to investigate the nonadiabatic corrections of order μ/M with respect to the leading fourth-order two-pair term. For this purpose, we return to the original form of the pair interaction $\mathcal{K}^{(pr)}$ as given by (62b), and, by analogy with (57), use, as our point of departure, the Schrödinger equation

$$[E-E(1)-E(2)]\phi(1,2)=\tfrac{1}{2}\sum_{3,4}(1,2|V^{(4a)}|3,4)\phi(3,4), \tag{78}$$

where $V^{(4a)}$ is given by (71b). The fourth-order two-pair potential $\mathbf{V}^{(4a)}$ can therefore be identified from the expression

$$\begin{aligned}\sum_{3,4}(1,2|\mathbf{V}^{(4a)}|3,4)\phi(3,4)&=\tfrac{1}{2}\sum_{3,4}(1,2|V^{(4a)}|3,4)\phi(3,4)\\&=\tfrac{1}{4}\sum_{3,4}(1,2|K_{02}'K_{20}-K_{02}K_{20}'|3,4)\phi(3.4).\end{aligned} \tag{79}$$

If we insert (53), appropriately generalized to take into account the isotopic-spin formalism, and (73) into (62b), we find

$$K_{02}=(K_{20})^{\dagger}=(G^2/4M)\sum_{\mathbf{p},\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'=1}^{4}[\omega(k)\omega(k')]^{-\frac{1}{2}}\times\bar{u}(\mathbf{p}+\mathbf{k}+\mathbf{k}'\lambda')u(\mathbf{p}\lambda)\times[b^{\dagger}(\mathbf{p}+\mathbf{k}+\mathbf{k}'\lambda')b(\mathbf{p}\lambda)a_\alpha(\mathbf{k})a_\alpha(\mathbf{k}')]_{02}, \tag{80}$$

and

$$\begin{aligned}K_{02}'=-(K_{20}')^{\dagger}=-(G^2/4M)\sum_{\mathbf{p},\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'=1}^{4}[\omega(k)\omega(k')]^{-\frac{1}{2}}&\times[\omega(k)+\omega(k')+E(p)-E(\mathbf{p}+\mathbf{k}+\mathbf{k}')]\\&\times\bar{u}(\mathbf{p}+\mathbf{k}+\mathbf{k}'\lambda')u(\mathbf{p}\lambda)[b^{\dagger}(\mathbf{p}+\mathbf{k}+\mathbf{k}'\lambda')\\&\times b(\mathbf{p}\lambda)a_\alpha(\mathbf{k})a_\alpha(\mathbf{k}')]_{02};\end{aligned} \tag{81}$$

λ is now an isotopic-spin spin-energy index which, for positive energies, we have taken to range from 1 to 4. Then, upon substituting (80) and (81) into (79), we obtain, after a brief calculation,

$$\begin{aligned}(\mathbf{p}_1\lambda_1,\,\mathbf{p}_2\lambda_2|\mathbf{V}^{(4a)}|\,\mathbf{p}_1+\mathbf{K}\lambda_3,\,\mathbf{p}_2-\mathbf{K}\lambda_4)&=-\frac{3}{2}\left(\frac{G^2}{2M}\right)^2\left(\frac{1}{2\pi}\right)^6\int_{\mathbf{k}+\mathbf{k}'=\mathbf{K}}d\mathbf{k}d\mathbf{k}'\left[\frac{\omega(k)+\omega(k')}{\omega(k)\omega(k')}\right]\\&\times\left\{\frac{1}{[\omega(k)+\omega(k')]^2-[E(p_1)-E(\mathbf{p}_1+\mathbf{K})]^2}+\frac{1}{[\omega(k)+\omega(k')]^2-[E(p_2)-E(\mathbf{p}_2-\mathbf{K})]^2}\right\}\\&\times\bar{u}(\mathbf{p}_1\lambda_1)u(\mathbf{p}_1+\mathbf{K}\lambda_3)\bar{u}(\mathbf{p}_2\lambda_2)u(\mathbf{p}_2-\mathbf{K}\lambda_4).\end{aligned} \tag{82}$$

It will now be noticed that, in the adiabatic limit, (82) simply goes over into the form (77b) which was derived in the fixed-source approximation. It is also evident, from the form of the energy denominators in (82), that the nonadiabatic corrections which are of order μ/M with respect to the leading two-pair terms vanish.[31] We are hence justified in identifying the fixed-source potential (77) as the nonrelativistic interaction for the *ps-ps* theory (apart from the question of radiative corrections which we have consistently ignored); as is well known, this potential is wholly inadequate on account of the near-cancellation of the attractive two-pair term $\mathbf{V}^{(4a)}$ by the repulsive one-pair term $\mathbf{V}^{(4b)}$.[32]

(c) Charge-Symmetric *Ps-Pv* Theory

As a final illustration, we consider briefly the charge-symmetric pseudoscalar theory with pseudovector coupling. This case is of particular interest for the nuclear-force problem in view of the accumulation of evidence pointing toward the suppression of the effects of the pair term (62b) in *ps-ps* theory when radiative and higher-order corrections are taken into account[33]; under these circumstances, we have left the pseudovector interaction (62c) which we now rewrite as

$$\mathcal{H}=(g/\mu)\int d\mathbf{r}\psi^\dagger\boldsymbol{\sigma}\cdot(\nabla\phi_\alpha)\tau_\alpha\psi, \tag{83}$$

where the coupling constants g and G are connected by the relation $g=G(\mu/2M)$.

The calculation of the nonrelativistic nuclear force for the *ps-pv* theory can now be carried through in a manner which is quite analogous to our previous discussion of the neutral scalar theory. First, we estimate that $v\sim\mu/M\sim g^2/4\pi$, so that we shall need to compute the second- plus fourth-order nuclear force in adiabatic (or fixed-source) approximation, as well as the nonadiabatic corrections of order μ/M with respect to the leading term in the second-order potential.

In the fixed-source limit, the interaction Hamiltonian is given by

$$\mathcal{H}=(g/\mu)\sum_i\boldsymbol{\sigma}^{(i)}\cdot\nabla\phi_\alpha(\mathbf{r}_i)\tau_\alpha{}^{(i)}. \tag{84}$$

The second- and fourth-order potentials may therefore be calculated by inserting (74) into (20) and (45). The result for the second-order force has already been listed in Eq. (77a). For the fourth-order interaction, we find

$$\begin{aligned}\mathbf{V}^{(4a)}&=(g/\mu)^4(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(2\pi)^{-6}\int d\mathbf{k}d\mathbf{k}'\\&\quad\times\{(\mathbf{k}\cdot\mathbf{k}')^2-[\boldsymbol{\sigma}^{(1)}\cdot(\mathbf{k}\times\mathbf{k}')][\boldsymbol{\sigma}^{(2)}\cdot(\mathbf{k}\times\mathbf{k}')]\}\\&\qquad\times\omega^{-2}(\omega')^{-3}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\\&=(g^2/4\pi)^2\mu^{-4}(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\\&\quad\times(\{(\nabla\cdot\nabla')^2-[\boldsymbol{\sigma}^{(1)}\cdot(\nabla\times\nabla')][\boldsymbol{\sigma}^{(2)}\cdot(\nabla\times\nabla')]\}\\&\qquad\times r^{-1}e^{-\mu r}(2/\pi)K_0(\mu r'))_{\mathbf{r}'=\mathbf{r}};\end{aligned} \tag{85a}$$

$$\begin{aligned}\mathbf{V}^{(4b')}&=-\mathbf{V}^{(4a)}-\tfrac{3}{2}(g/\mu)^4(2\pi)^{-6}\int d\mathbf{k}d\mathbf{k}'\\&\quad\times\{2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(\mathbf{k}\cdot\mathbf{k}')^2+3[\boldsymbol{\sigma}^{(1)}\cdot(\mathbf{k}\times\mathbf{k}')]\\&\qquad\times[\boldsymbol{\sigma}^{(2)}\cdot(\mathbf{k}\times\mathbf{k}')]\}\omega^{-2}(\omega')^{-3}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\\&=-\mathbf{V}^{(4a)}-\tfrac{3}{2}(g^2/4\pi)^2\mu^{-4}(\{2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(\nabla\cdot\nabla')^2\\&\quad+3[\boldsymbol{\sigma}^{(1)}\cdot(\nabla\times\nabla')][\boldsymbol{\sigma}^{(2)}\cdot(\nabla\times\nabla')]\}\\&\qquad\times r^{-1}e^{-\mu r}(2/\pi)K_0(\mu r'))_{\mathbf{r}'=\mathbf{r}};\end{aligned} \tag{85b}$$

$$\begin{aligned}\mathbf{V}^{(4b'')}&=\tfrac{1}{2}(g/\mu)^4(2\pi)^{-6}\int d\mathbf{k}d\mathbf{k}'\{2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(\mathbf{k}\cdot\mathbf{k}')^2\\&\quad+3[\boldsymbol{\sigma}^{(1)}\cdot(\mathbf{k}\times\mathbf{k}')][\boldsymbol{\sigma}^{(2)}\cdot(\mathbf{k}\times\mathbf{k}')]\}\\&\qquad\times(\omega-\omega')[\omega^3(\omega')^2(\omega+\omega')]^{-1}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\\&=-\tfrac{1}{2}(g^2/4\pi)^2\mu^{-4}(\{2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(\nabla\cdot\nabla')^2\\&\quad+3[\boldsymbol{\sigma}^{(1)}\cdot(\nabla\times\nabla')][\boldsymbol{\sigma}^{(2)}\cdot(\nabla\times\nabla')]\}\\&\quad\times\{(4/\pi)(r^{-1}+(r')^{-1})K_0[\mu(r+r')]\\&\qquad-3r^{-1}e^{-\mu r}(2/\pi)K_0(\mu r')\})_{\mathbf{r}'=\mathbf{r}}.\end{aligned} \tag{85c}$$

It is evident from (45) that $\mathbf{V}^{(4a)}$ represents that part of the fourth-order potential which is contained in the lowest-order Tamm-Dancoff approximation; on the other hand, $\mathbf{V}^{(4b)}=\mathbf{V}^{(4b')}+\mathbf{V}^{(4b'')}$ makes its appearance when we allow for the presence of two mesons in the field.

The total fourth-order potential $\mathbf{V}^{(4)}=\mathbf{V}^{(4a)}+\mathbf{V}^{(4b)}$, which evidently is the perturbation-theoretic result, has been derived by many authors.[6,9,34] It contains a strong repulsive force in triplet-even states which makes this potential quite unsuitable for a description of the

[31] In Klein's treatment of this problem (reference 6), one does encounter nonvanishing nonadiabatic corrections of order μ/M with respect to the leading fourth-order potential (and also the second-order potential); these terms are dropped, however, after Klein invokes a variational-principle argument similar to that noted in reference 24.

[32] See, for example, Fig. 2 in Drell and Huang's paper (reference 30).

[33] For a summary of the case for pair suppression, see A. Klein, Phys. Rev. **95**, 1061 (1954).

[34] Taketani, Machida, and Ohnuma, Progr. Theoret. Phys. Japan **6**, 638 (1951); K. Nishijima, Progr. Theoret. Phys. Japan **6**, 815, 911 (1951); J. L. Lopes and R. P. Feynman, *Symposium on New Research Techniques in Physics* (Academia Brasileira de Ciencias, Rio de Janeiro, 1954), p. 251.

deuteron, say. Brueckner and Watson, in their discussion of the nuclear-force problem for *ps-pv* theory, have suggested that the fourth-order interaction is better approximated by the term $\mathbf{V}^{(4b)}$, in which case the repulsion, mentioned above, is turned into an attraction. However, this procedure is difficult to justify in the light of our previous discussion of the neutral scalar theory [immediately following Eq. (50)].

Finally, we need to consider the nonadiabatic corrections of order μ/M with respect to the second-order potential (77a). For this purpose, we can essentially take over the results of our earlier calculation for the neutral scalar theory [Eq. (51) ff.] suitably modified for the *ps-pv* theory. The final result is to find[35]

[35] This result has been obtained previously by L. Van Hove, Phys. Rev. **75**, 1519 (1949).

$$(\mathbf{p}_1\lambda_1,\,\mathbf{p}_2\lambda_2|\mathbf{V}^{(2)}|\,\mathbf{p}_1+\mathbf{k}\lambda_3,\,\mathbf{p}_2-\mathbf{k}\lambda_4)$$
$$=-\frac{1}{2}\left(\frac{g}{\mu}\right)^2\left(\frac{1}{2\pi}\right)^3\left\{\frac{1}{\omega^2(k)-[E(p_1)-E(\mathbf{p}_1+\mathbf{k})]^2}+\frac{1}{\omega^2(k)-[E(p_2)-E(\mathbf{p}_2-\mathbf{k})]^2}\right\}$$
$$\times u^\dagger(\mathbf{p}_1\lambda_1)(\boldsymbol{\sigma}\cdot\mathbf{k})\tau_\alpha u(\mathbf{p}_1+\mathbf{k}\lambda_3)$$
$$\times u^\dagger(\mathbf{p}_2\lambda_2)(\boldsymbol{\sigma}\cdot\mathbf{k})\tau_\alpha u(\mathbf{p}_2-\mathbf{k}\lambda_4). \quad (86)$$

It is clear from (86) that the nonadiabatic corrections of order μ/M vanish so that the nonrelativistic potential for *ps-pv* theory is given by the fixed-source calculation [(77a) and (85)].[31]

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Effect of Renormalization on Meson-Nucleon S-Scattering*

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Renormalization prescriptions are given for the covariant integral equations of meson-nucleon scattering, taking into account the difficulties of overlapping divergences. The covariant wave equations, corresponding to the iteration of second-order irreducible processes, are solved approximately and renormalized in closed form (in the case of the pseudoscalar theory with pseudoscalar coupling). The S-phase shifts corresponding to the states of isotopic spin 1/2 and 3/2 are computed, and their variation with energy is compared with experiment. The only parameter which can be adjusted is the meson-nucleon coupling constant G. It is found that a good agreement with experiment is obtained when $G^2/4\pi=7.5$. The possibility of this agreement being purely coincidental cannot be ruled out, but other interpretations of this result are discussed.

I. INTRODUCTION

In a foregoing paper,[1] a covariant treatment of meson-nucleon scattering has been presented, which permits, in principle, the elimination of special renormalization difficulties arising in this problem. The main result was that—once the wave integral equation corresponding only to the finite processes is solved—it is possible to express and to renormalize in closed form all the remaining contributions to the scattering cross sections.[2]

The renormalization prescriptions which have to be applied to the closed expressions yielded by the theory were, however, incorrectly stated in that paper,[3] mainly because the difficulties coming from the so-called "overlapping" divergences[4] were not properly taken into account. Fortunately, it is possible to reformulate those prescriptions without losing the advantage of having a closed expression for the corresponding part of the renormalized S-matrix elements. This reformulation is given in Sec. II of the present paper.

Once this formal work has been done, however, there still remains the problem of actually calculating the scattering differential cross sections, in order to compare them with experiment. The first difficulty, here, is that the kernel of the partial wave-equation corresponding to the finite processes is still expressed as a series of powers of the large coupling constant G. This series does not seem easy to sum, and its first few terms do not appear to yield a good approximation. However,

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[1] M. Lévy, Phys. Rev. **94**, 460 (1954). This paper will be, in the following, referred to as (I). References to its equations will be given as Eq. (I,···).
[2] This applies, of course, only to the special divergences mentioned above. The "normal" radiative corrections must be handled by means of the well-known methods of Feynman and Dyson.
[3] The correct results were stated without proof in a note added in proof to paper (I). Most of the results contained in Sec. I of the present paper have already been reported in a letter from the author to Prof. N. M. Kroll, which has been reproduced, together with the answer from N. M. Kroll, in an Appendix to the *Proceedings of the Fourth Annual Rochester Conference on High Energy Physics* (University of Rochester, Rochester, 1954). Our special thanks are due to Prof. N. M. Kroll for this interesting correspondence.
[4] A. Salam, Phys. Rev. **82**, 217 (1951).