Lower Limit for the Energy Derivative of the Scattering Phase Shift

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It is shown that the derivative of the scattering phase shift with respect to energy, $d\eta/dE$, must exceed a certain limit if the interaction of scattered particle and scatterer vanishes beyond a certain distance. This limitation of $d\eta/dE$ is, fundamentally, a consequence of the principle of causality; it is derived, however, from a property of the derivative matrix R.

I

THE cross section and its angular dependence, as functions of energy, do not seem to determine in general the phase shifts uniquely.[1] It may be useful, therefore, to derive certain general rules about the energy dependence of phase shifts which may facilitate the choice between the apparently equivalent sets of phase shifts. The relations to be derived here are based, fundamentally, on what has come to be called "the principle of causality." It states that the scattered wave cannot leave the scatterer before the incident wave has reached it. However, the calculation to be carried out will make use of a single property of the derivative matrix R which was given already by Eisenbud and the present writer.[2]

Before carrying out the very simple calculation, the general nature of the result will be illustrated by means of Eisenbud's interpretation of the energy derivative of the phase shift as time delay.[3] Let us consider, for sake of simplicity, a scattering center of radius a, i.e., assume that the incident particle behaves like a free particle outside a sphere of this radius. Let us consider then an incident beam which is the superposition of two monoenergetic beams of energy $\hbar(\nu+\nu')$ and $\hbar(\nu-\nu')$, respectively. The corresponding wave numbers will be denoted by $k+k'$ and $k-k'$. Hence,

$$\psi_{\text{inc}}=r^{-1}(e^{-i(k+k')r-i(\nu+\nu')t}+e^{-i(k-k')r-(\nu-\nu')t}). \quad (1)$$

Both k' and ν' are infinitesimally small so that (1) is a substitute for a wave packet,[4] the center of which is at the point where the two spherical waves of (1) are in phase, i.e., where

$$2k'r+2\nu't=0. \quad (1a)$$

Since $\nu'/k'=d\nu/dk$ is the velocity of the particle, the incident particle indeed moves with a velocity v toward the scattering center. If $\eta+\eta'$ and $\eta-\eta'$ are the phase shifts which correspond to the energy values $\hbar(\nu+\nu')$ and $\hbar(\nu-\nu')$, the outgoing wave will be

$$\psi_{\text{out}}=r^{-1}(e^{i(k+k')r-i(\nu+\nu')t+2i(\eta+\eta')} + e^{i(k-k')r-i(\nu-\nu')t+2i(\eta-\eta')}). \quad (1b)$$

The two waves of (1b) are in phase where

$$2k'r-2\nu't+4\eta'=0,$$

i.e., where

$$r=-2\eta'/k'+(\nu'/k')t=-2d\eta/dk+(d\nu/dk)t. \quad (1c)$$

One sees that the outgoing wave is retarded by a stretch $2d\eta/dk$; it arrives at a point $r-2d\eta/dk$ at the time it would have arrived at r without the action of the scattering center. The causality principle as formulated in the first paragraph gives no upper value for the retardation $2d\eta/dk$: if the particle is temporarily captured by the scattering center, there is no reason for it not being retained an arbitrarily long time. However, the "retardation" cannot assume arbitrarily large negative values, in classical theory it could not be less than $-2a$. It will be seen that the wave nature of the particles does permit some infringement of the relation

$$d\eta/dk>-a. \quad (2)$$

It will be shown that, nevertheless, (2) is essentially preserved also in quantum theory. It does hold, in particular, for large k.

The relation (2) gives a simple physical interpretation to the qualitative behavior of the energy dependence of η. Close to resonances, where the incident particle is in fact captured and retained for some time by the scattering center, $d\eta/dk$ will assume large positive values. On the other hand, $d\eta/dk$ will be close to $-a$ at energy values at which the incident particle hardly enters the scatterer. One would expect (on the basis of Liouville's theorem or the completeness relations) that the two effects, on the whole, balance each other, i.e., that the integral of $d\eta/dE$ over the whole energy range is close to zero, at least if the scattering can be described by a nonsingular potential. This is indeed the case: $\eta=0$ at $E=\infty$ and $\eta=b\pi$ for $E=0$, where b is the number of bound states.[5] Hence, if the cross section

[1] See, e.g., the articles of Fermi, Metropolis, and Alei, Phys. Rev. **95**, 1581 (1954); de Hoffmann, Metropolis, Alei, and Bethe, Phys. Rev. **95**, 1586 (1954); and of R. L. Martin, Phys. Rev. **95**, 1606 (1954) for the ambiguities in the case of pion-nucleon scattering.

[2] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947), see (i) on page 35.

[3] L. Eisenbud, dissertation, Princeton, June 1948 (unpublished).

[4] Instead of the superposition of only two monoenergetic waves, one can use a regular wave packet in this consideration; r in (1a) and (1c) then becomes the coordinate of the center of mass of that wave packet.

[5] I am much indebted to Dr. V. Bargmann for this observation.

shows a resonance behavior, one will expect η to decrease slowly between resonances and increase fast at resonances, increases and decreases almost exactly balancing if considered over the whole energy spectrum.

II

In order to derive a rigorous minimum value for $d\eta/dk$, it will be necessary to assume that there is a radius a outside of which the wave function is that of a free particle. This is a somewhat artificial assumption which will be hardly ever satisfied rigorously. One can, therefore, expect that the condition (4b) to be derived under this assumption will be violated occasionally. However, if a is chosen reasonably large the violations will be uncommon and will extend over narrow energy intervals.

We define as internal region the inside of a sphere of radius a and call the rest of space the external region. The wave function ψ in the external region is then given by

$$r\psi = I(r) - e^{2i\eta} I^*(r), \tag{3}$$

where $I(r)$ is the radial part of an incoming spherical wave (of arbitrary angular momentum), its conjugate complex $I^*(r)$ is the outgoing wave; η is called the phase shift. For $\eta=0$, ψ must be regular at $r=0$, which fixes the phase of I except for I's sign.[6] We shall normalize I so that

$$I(r)I'(r)^* - I(r)^*I'(r) = 2i. \tag{3a}$$

$I(r)$ then represents a wave with flux $m/\hbar$. The prime denotes the derivative with respect to r. Both I and η also depend on the energy; we shall use, instead, the wave vector k as variable and denote differentiation with respect to k by a dot.

The value of the reciprocal logarithmic derivative R of ψ, with respect to r, at $r=a$, becomes

$$R = \frac{I - e^{2i\eta} I^*}{I' - e^{2i\eta} I'^*}. \tag{3b}$$

The I, I', etc., without argument, denote the value of the corresponding expression at $r=a$. Calculation of $e^{2i\eta}$ from (3a) yields

$$e^{2i\eta} = (I - I'R)(I^* - I'^*R)^{-1}. \tag{3c}$$

It may be remarked, parenthetically, that this expression remains valid if, in addition to scattering, transmutations are also possible. The left side has to be replaced then by the collision matrix, R by the derivative matrix, and I, I', etc., by diagonal matrices, the diagonal elements of which are the $I(a)$, $I'(a)$, etc., for the channel to which the row of the collision matrix refers.

[6] It is necessary, in collision theory, to fix also the sign of $I(r)$ in terms of the decomposition of the plane wave into spherical waves. This is not necessary, however, for the derivation of (4b).

III

In the present case, all quantities in (3c) are numbers and logarithmic differentiation of (3c) with respect to k gives

$$\dot{\eta} = \frac{1}{2i}\left[\frac{\dot{I} - \dot{I}'R - I'\dot{R}}{I - I'R} - \frac{\dot{I}^* - \dot{I}'^*R - I'^*\dot{R}}{I^* - I'^*R}\right]. \tag{4}$$

The term proportional to $\dot{R}$ in (4) is

$$\frac{-I'(I^* - I'^*R) + I'^*(I - I'R)}{2i|I - I'R|^2}\dot{R} = \frac{\dot{R}}{|I - I'R|^2}. \tag{4a}$$

The last part follows from (3a). The partial fraction expansion[2] of R shows directly that $\dot{R} = (\hbar^2 k/m) dR/dE > 0$ for $k > 0$ so that omission of the term containing $\dot{R}$ will make the left side of (4) larger than the right side. Elimination of R from the resulting inequality by means of (3b) yields

$$d\eta/dk = \dot{\eta} > \tfrac{1}{2}\,\mathrm{Re}[\dot{I}'I^* - \dot{I}I'^* + (\dot{I}I' - \dot{I}'I)e^{-2i\eta}]. \tag{4b}$$

Re[$\cdots$] denotes the real part of the expression contained in the bracket. The computation leading from (4) to (4b) involves the use of (3a) and the equation obtained from it by differentiation with respect to k.[6]

It remains to point out that the partial fraction expansion of R, though obtained in reference 2 by direct calculation, has been shown[7] to follow also from the causality condition described at the beginning of this note. This establishes the relation between (4b) and the qualitative consideration based on Eisenbud's early work. The connection becomes even clearer if one inserts, into (4b), the expressions for I for $l=0$ or $l=1$:

$$I_0(r) = k^{-\frac{1}{2}} e^{-ikr}; \quad I_1(r) = k^{-\frac{1}{2}}[(kr)^{-1} + i] e^{-ikr}. \tag{5}$$

These give, in the former case,

$$\dot{\eta}_0 > -a + (2k)^{-1} \sin 2(\eta_0 + ka); \tag{5a}$$

while one has, in the latter case,

$$\dot{\eta}_1 > -a + (k^2 a)^{-1}[1 - \cos 2(\eta_1 + ka)] - (2k)^{-1} \sin 2(\eta_1 + ka). \tag{5b}$$

It will be noted that, as long as $ka \ll 1$, these equations actually entail $\dot{\eta} > 0$ for a reasonably large interval of η. However, at very low k, (5b) merely expresses the fact that η_0 and η_1 are proportional to k and k^3, respectively, without limiting the proportionality constant.

Naturally, the proper limitation of a constitutes the principal difficulty in using (4b) to select the actual

[7] In addition to Eisenbud's doctoral dissertation, see W. Schutzer and J. Tiomno, Phys. Rev. **83**, 249 (1951) and, in particular N. G. Van Kampen, Phys. Rev. **89**, 1072 (1953); **91**, 1267 (1953). Also J. S. Toll, doctoral dissertation, Princeton University, 1952 (unpublished), and the more recent article of Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954). This article also contains references to the early literature.

phase shifts among the sets which reproduce the cross section. An estimate of a can be obtained only from an at least qualitative theory of the nature of the interaction. It is commonly believed, for instance, that the pion-nucleon scattering proceeds via absorption and re-emission of the pion by the nucleon. If this is the case, a will be of the order of the pion Compton wavelength, i.e., $\hbar/\mu c$. However, it would be quite difficult to tell to what extent $a=2\hbar/\mu c$ is a permissible choice or whether it is necessary to assume $a=3\hbar/\mu c$ or an even larger a, giving less and less stringent forms to (4b). An alternative form of applying the relations of this paper, which might be somewhat more free of this ambiguity, would be to plot R, as calculated from (3b), against the energy and to judge whether any possible deviation of the R obtained this way from a regular R function can be blamed on having assumed a too low value for a.

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Scattering of Polarized Nucleon Beams*

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The polarization formulas for the general reaction $a+b\rightarrow c+d$ are given in a compact form and specialized for the scattering of polarized nucleon beams at unpolarized targets. These targets may have arbitrary spin, possibly being different before and after collision. An interesting quantity in this problem is the polarization of the scattered nucleon beam. On the basis of general invariance properties of the transition matrix, this polarization is expressed by the polarization of the incoming nucleon beam and the relative momenta before and after collision. The invariant coefficients in this relation are functions of energy and scattering angle only; they are given in terms of the parameters of the transition matrix.

By using these formulas, it is shown that with triple scattering experiments one can obtain two new relations between the parameters of the transition matrix at fixed energy and angle. Quadruple scattering leads to two further relations. These relations represent information in addition to the differential cross section and the polarization resulting from unpolarized beams. The results are specialized for targets of spin zero and spin one-half, where in the latter case also the scattering of identical particles is discussed briefly.

I. INTRODUCTION

A BEAM of identical free particles with equal momentum represents in general a quantum-mechanical mixture, i.e., a classical statistical ensemble of different pure states.[1] We can characterize such an ensemble by the contributing pure states Ψ and their relative abundance $W(\Psi)$, where $\sum_\Psi W(\Psi)=1$. According to the usual rules of probability the expectation value of any operator ω in this beam is given by

$$\langle\omega\rangle_{\text{beam}}=\sum_\Psi W(\Psi)(\Psi,\omega\Psi).$$

Decomposing Ψ with respect to a complete set of orthogonal eigenstates Φ_k, $\Psi=\sum_k a_k(\Psi)\Phi_k$, we can write this expectation value in the form

$$\langle\omega\rangle_{\text{beam}}=\operatorname{Tr}(\rho\omega)=\sum_{ik}\left(\sum_\Psi W(\Psi)a_i(\Psi)a_k^*(\Psi)\right)(\Phi_k,\omega\Phi_i).$$

Here ρ is the density matrix[1] of the beam. If the particles have spin s and differ only by their spin state, then ρ is a Hermitian $2s+1$ by $2s+1$ matrix in spin-space. From the normalization of the state vectors to one follows $\operatorname{Tr}\rho=1$.

It is useful to expand the density matrix in terms of a complete set of $(2s+1)^2$ basic Hermitian matrices ω^μ in spin space,[2] which obey the relation

$$\operatorname{Tr}(\omega^\mu\omega^\nu)=(2s+1)\delta_{\mu\nu}.$$

These ω^μ are related to the irreducible spin tensor moments $T_k^{(q)}$, where q is the rank of the tensor and $k(|k|\leq q\leq 2s)$ indicates its components.[3,4] The tensor moments are not Hermitian operators as the ω^μ, but they transform directly according to the representation $\mathfrak{D}^{(q)}$ of the three dimensional rotation group. Using the Hermitian matrices ω^μ we can express the density matrix by the expectation values of all basic matrices ω^μ in the beam:

$$\rho=\frac{1}{2s+1}\sum_{\mu=1}^{(2s+1)^2}\langle\omega^\mu\rangle_{\text{beam}}\omega^\mu.$$

We say a beam is completely unpolarized if the expectation values of all tensor moments and therefore all operators ω^μ vanish, except the expectation value of the zero rank tensor $T_0^{(0)}=\omega^1=1$.

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[1] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932 and Dover Publications, New York, 1943), p. 174; and H. Weyl, *Theory of Groups and Quantum Mechanics* (Dover Publications, New York, 1931), p. 78.

[2] See, for example, U. Fano, Phys. Rev. **90**, 577 (1953); and F. Coester and J. M. Jauch, Helv. Phys. Acta **26**, 3 (1953); these papers contain further references.

[3] E. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (F. Vieweg, Braunschweig, 1953 and Edwards Brothers, Ann Arbor, 1944), p. 263.

[4] G. Racah, Phys. Rev. **62**, 442 (1942).