with pions. If so, this could indicate a mode of production for heavy mesons different from that for pions. It may be noted, however, that heavy mesons of greater energy have been observed in cloud chamber work, but little is known about the energy spectrum at present. Powell[4] conjectures that the hyperons and heavy mesons may often originate from secondary interactions of pions (produced in nucleon-nucleon collisions) with other nucleons in the same nucleus. Evidence that hyperons and K-mesons are created in the interaction of pions with hydrogen nuclei has been obtained in work with the Brookhaven Cosmotron.[8] Recent work on a very high-energy nuclear shower by Koshiba and Kaplon[9] gives a production ratio of neutral mesons to charged shower particles of 0.50 ± 0.11 indicating little, if any, production of heavy mesons in such an event as opposed to what might be expected from the statistical theories of multiple-meson production. This apparent lack of an abundant production of heavy mesons at high energies could also be in accordance with a secondary mode of production for the heavy mesons.

(4) We are indebted to the office of Naval Research for enabling our emulsions to be exposed on "Skyhook" balloon flights, to Miss Margaret Stott for some scanning of the emulsion used, and to Miss Jacqueline Dazé for the drawings.

[8] Fowler, Shutt, Thorndike, and Whittemore, Phys. Rev. **91**, 1287 (1953).

[9] M. Koshiba and M. F. Kaplon, Phys. Rev. **97**, 193 (1955).

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Spin Polarization of the Deuteron*†

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Methods of specifying the state of polarization of a particle of spin 1 are discussed. Selection rules for polarization effects in simple nuclear reactions are derived; in general four parameters are needed to describe the deuteron polarization due to such reactions. Methods of determining these parameters include the use of magnetic deflection. A rough analysis is made of the polarization of deuterons scattered by carbon.

INTRODUCTION

RECENTLY many successful experiments have been done with polarized beams of protons and neutrons. The present paper deals with the theoretical possibilities of extending such experiments to spin 1 particles, in particular, the deuteron. Considerable care must be taken in defining the state of polarization of a spin 1 particle, and this is discussed in Sec. 1. Sections 2 and 3 present general theorems applicable to experiments involving polarized spin 1 particles, whereas Sec. 4 presents a rough analysis of polarization effects for the special case of scattering of deuterons by a nucleus with zero spin and zero isotopic spin, such as carbon. The first three sections apply to any particle of spin 1 and may be of interest in considering the possibility that some of the heavy mesons have spin 1.

1. POLARIZATION STATES OF THE DEUTERON

The spin state of a particle (or nucleus) taking part in a nuclear reaction in general must be described as a statistical mixture of the pure spin states in which the particle may be found. If the description consists of weighting equally all members of any basis set of mutually orthogonal spin functions, the mixture is spatially isotropic and describes *unpolarized particles*. Any different distribution will describe anisotropic states and refers to *polarized particles*.[1] States which may be described by a single wave function will be called *completely polarized*. In the case of particles of spin $\frac{1}{2}$ the most general spin state may be described as a mixture of a completely polarized state with statistical weight P and an unpolarized state with weight $(1-P)$, where P is the percentage polarization. No such simple picture exists for particles of spin greater than $\frac{1}{2}$.

The von Neumann density matrix ρ is a convenient starting point in discussing polarization.[2] It may be expressed as a linear combination of independent Hermitian matrices, whose number equals the square of

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[1] The term polarization as we use it includes both "polarization" and "alignment" in the sense of Bleaney: B. Bleaney, Proc. Phys. Soc. (London) **A64**, 315 (1951); Simon, Rose, and Jauch, Phys. Rev. **84**, 1155 (1951). *Alignment* may be considered as special cases of the *tensor* type of polarization discussed later, whereas *polarization* corresponds to the *vector* type.

[2] L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952); R. H. Dalitz, Proc. Phys. Soc. (London) **A65**, 175 (1952). Our methods and notation generally follow those of the former paper.

the dimensionality of the spin space, and whose expectation values determine ρ and thus the state of the system. One of these may be the identity matrix, the expectation value of which specifies the normalization; that is, the trace of ρ. The remaining expectation values determine the polarization of the system.

We choose operators which form components of irreducible tensors[3] of rank zero (the identity matrix) one, and two. The operator which transforms under rotations like the spherical harmonic Y_{JM} is denoted by T_{JM}. This set is clearly equivalent to a set of independent Hermitian base matrices, namely that set formed by the Hermitian and anti-Hermitian parts of these operators. The orthogonality and normalization conditions are chosen:

$$\mathrm{Tr}T_{JM}T_{J'M'}{}^{\dagger}=3\delta_{JM,\,J'M'}. \tag{1.1}$$

The matrix elements of T_{JM} are determined by the tensorial properties and normalization (1.1):

$$(m'|T_{JM}|m)=\sqrt{3}C_{11}(J-M|-m'm), \tag{1.2}$$

where $C_{j_1j_2}(jm|m_1m_2)$ is a Clebsch-Gordan coefficient for combining angular momentum.[4] Explicit forms of the tensor operators in terms of components of the spin operator $\mathbf{S}$ are

$$\begin{aligned}
T_{11}&=-\tfrac{1}{2}\sqrt{3}(S_x+iS_y),\\
T_{10}&=(\tfrac{3}{2})^{\frac{1}{2}}S_z,\\
T_{22}&=\tfrac{1}{2}\sqrt{3}(S_x+iS_y)^2,\\
T_{21}&=-\tfrac{1}{2}\sqrt{3}[(S_x+iS_y)S_z+S_z(S_x+iS_y)],\\
T_{20}&=(\tfrac{1}{2})^{\frac{1}{2}}(3S_z{}^2-2),\\
T_{J,-M}&=(-1)^MT_{JM}{}^{\dagger}.
\end{aligned} \tag{1.3}$$

The density matrix may be written

$$\rho=\tfrac{1}{3}\textstyle\sum_{JM}\langle T_{JM}\rangle T_{JM}{}^{\dagger}, \tag{1.4}$$

where angular brackets denote statistical expectation values. One of the inequalities associated with ρ is that the trace of ρ is less than or equal to unity, or

$$\textstyle\sum_{JM}'|\langle T_{JM}\rangle|^2\leq 2, \tag{1.5}$$

where the prime on the summation indicates that $J=0$ (the identity) is omitted. The equality is a necessary and sufficient condition for complete polarization and the left hand side equals zero only for the unpolarized state. Therefore, one may define a percentage polarization

$$P=(\tfrac{1}{2}\textstyle\sum_{JM}'|\langle T_{JM}\rangle|^2)^{\frac{1}{2}}. \tag{1.6}$$

The components of $\mathbf{S}$ (if they do not all vanish) define a pseudovector whose direction may be used as the Z-axis so the $\langle T_{1,\pm1}\rangle$ vanish. In Sec. 2 we shall show that in all states produced in the simple reactions of interest to us, the $\langle T_{2,\pm1}\rangle$ vanish too. We now restrict our consideration to this class of states. Geometrically this class of states is characterized by the fact that the direction of the spin is also a principal axis of the symmetrical second-rank tensor formed by $\langle T_{2M}\rangle$. [That this follows from the vanishing of $\langle T_{2,\pm1}\rangle$ may be seen directly from Eqs. (1.3).] Included in this characterization are singular cases in which it is possible but not necessary to choose the spin direction parallel to a principal axis; namely, cases in which the expectation value of the spin vanishes or in which the principal axes of the tensor are not determined uniquely. This class obviously includes the unpolarized state and may be shown to include all completely polarized states.

A useful method of describing the state of polarization is to consider the representation in which ρ is diagonal. Any state then may be described as a statistical mixture of the three pure mutually orthogonal states ψ_n which form the basis of this representation, and the corresponding eigenvalues λ_n of ρ are the weights. Our restricted class of polarization states is a statistical mixture of a restricted set of pure states, which may be shown to be

$$\begin{aligned}
\psi_1&=\chi_0,\\
\psi_2&=A\chi_1+B\chi_{-1},\\
\psi_3&=B^*\chi_1-A^*\chi_{-1}.
\end{aligned} \tag{1.7}$$

Here χ_m is an eigenstate of S_z, where the Z-axis is chosen in the spin direction. If the X- and Y-axes are taken as the other principal axes of the tensor, the constants A and B must be real. [As a matter of fact, since all pure states belong to our restricted class, it follows that any pure state may be represented as one of the forms in (1.7) in a suitable coordinate system. The *set* of pure states (1.7) however is restricted by the condition that the same coordinate system is used for all three states.] Only in special cases will the states ψ_2 and ψ_3 be oriented[5]: (1) when A (or B) vanishes, the states are oriented along the Z-axis; (2) when $A=B$, the three states are χ_0 states oriented along three mutually-perpendicular axes, the principal axes of the tensor. In general states ψ_2 and ψ_3 have average spin equal in magnitude but opposite in direction. They are analogous to two elliptically polarized states of light with opposite senses of rotation and crossed major axes.

A state of the class we are considering is seen to be specified by six parameters; this is in contrast to the four parameters required to specify a pure state and the eight parameters required to specify the most general state. Three of these six parameters define the principal axes, while the other three may be taken as (A/B) and two of the λ_n (the normalization $\mathrm{Tr}\rho$ is not considered as a parameter). In place of the last three, we shall use the three real numbers $\langle T_{20}\rangle$, $\langle T_{10}\rangle$, and

[3] See for example G. Racah, Phys. Rev. **62**, 438 (1942).

[4] See for example J. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 789.

[5] An oriented state is one which is an eigenstate of the operator S_z for some choice of z-axis. E. Majorana, Nuovo cimento **9**, 43 (1932).

$\sqrt{2}\langle T_{22}\rangle$[6] which are given by

$$\begin{aligned}\langle T_{10}\rangle&=(\tfrac{3}{2})^{\frac{1}{2}}(\lambda_2-\lambda_3)(A^2-B^2),\\ \langle T_{20}\rangle&=(\tfrac{1}{2})^{\frac{1}{2}}(1-3\lambda_1),\\ \sqrt{2}\langle T_{22}\rangle&=(6)^{\frac{1}{2}}(\lambda_3-\lambda_2)AB.\end{aligned}\tag{1.8}$$

Equations (1.8) yield

$$\langle T_{10}\rangle^2+[\sqrt{2}\langle T_{22}\rangle]^2\leq\tfrac{1}{3}[\langle T_{20}\rangle+\sqrt{2}]^2.\tag{1.9}$$

Each possible state may be represented by a point in a space whose coordinates are $\langle T_{10}\rangle$, $\sqrt{2}\langle T_{22}\rangle$, and $\langle T_{20}\rangle$ respectively; these points fill the interior of a cone, whose apex is at $\langle T_{20}\rangle=-\sqrt{2}$ on the $\langle T_{20}\rangle$-axis, and whose base has a radius of $\sqrt{\tfrac{3}{2}}$ and is normal to this axis at $\langle T_{20}\rangle=+\tfrac{1}{2}\sqrt{2}$. The completely polarized state ψ_1 is at the apex, while the completely polarized states ψ_2 and ψ_3 are on opposite ends of a diameter of the base. The unpolarized state, of course, is at the origin.

2. SELECTION RULES FOR SIMPLE REACTIONS

We consider reactions in which both initial and final states contain only two particles (nuclei). The states are specified by the momenta in the c.m. system (center-of-mass) and the polarization. The polarization state may be specified by a density matrix in the product of the spin spaces of the two particles. Transition amplitudes for the various pure spin states may be written as a matrix M whose rows and columns are characterized by the spin quantum numbers of final and initial states, respectively.[7] We express ρ_i and ρ_f as linear combinations of base matrices R^ν and S^μ in the spin spaces of the initial and final systems, respectively. We find, as in reference 2:

$$I\langle S^\mu\rangle_f=\frac{1}{n_i}\sum_\nu\langle R^\nu\rangle_i\operatorname{Tr}(MR^{\nu\dagger}M^\dagger S^\mu).\tag{2.1}$$

Here I is the differential cross section for this reaction, and the basic operators are normalized so that

$$\begin{aligned}\operatorname{Tr}(R^\mu R^{\nu\dagger})&=n_i\delta_{\mu\nu},\\ \operatorname{Tr}(S^\mu S^{\nu\dagger})&=n_f\delta_{\mu\nu},\end{aligned}\tag{2.2}$$

where n_i and n_f are the dimensionalities of initial and final spin spaces. If neither initial particle is polarized and one reaction product is a deuteron and the polarization of the other is not observed, the only $\langle R^\nu\rangle_i$ unequal to zero is the expectation value of the identity, and the only S^μ of interest will be the direct products of the tensor operators T_{JM} in the deuteron spin space with the identity operator in the spin space of the other reaction product. Then (2.1) gives for the polarization of the outgoing deuterons

$$I_0\langle T_{JM}\rangle_f=\frac{1}{n_i}\operatorname{Tr}(MM^\dagger T_{JM}),\tag{2.3}$$

where I_0 denotes the cross section for unpolarized incident particles.

Since M must be invariant under space rotations and reflections $MM^\dagger$ must be similarly invariant. Terms of interest must be contractions of the T_{JM} with tensor quantities formed from $\mathbf{k}_i$ and $\mathbf{k}_f$, the initial and final momenta; to be invariant under space reflection, they must be of even degree in $\mathbf{k}_i$ and $\mathbf{k}_f$. Consequently the most general form of $MM^\dagger$ is

$$\begin{aligned}MM^\dagger=J_0+J_1\textstyle\sum_M Y_{2M}(\mathbf{k}_i)T_{2M}^\dagger+J_2\sum_M Y_{2M}(\mathbf{k}_f)T_{2M}^\dagger\\ +J_3\textstyle\sum_M Y_{2M}(\mathbf{k}_i\mathbf{k}_f)T_{2M}^\dagger\\ +J_4\textstyle\sum_M Y_{1M}(\mathbf{k}_i\times\mathbf{k}_f)T_{1M}^\dagger+\cdots.\end{aligned}\tag{2.4}$$

Here the J's depend only on $\mathbf{k}_i\cdot\mathbf{k}_f$, k_i, and k_f, and must be real in order that $MM^\dagger$ be Hermitian. The Y_{JM} are solid spherical harmonics formed from the components of their indicated arguments,[8] and $+\cdots$ indicates quantities depending on operators in the spin space of the second reaction product.

Equation (2.3) shows that $\langle T_{JM}\rangle_f$ equals the coefficient of $T_{JM}^\dagger$ in (2.4) divided by I_0; the terms denoted by $+\cdots$ contribute nothing. To study the polarization state of these deuterons, we choose the Z-axis normal to the reaction plane. Then the Z-components of $\mathbf{k}_i$ and $\mathbf{k}_f$ vanish and $\mathbf{k}_i\times\mathbf{k}_f$ is in the Z-direction, hence all harmonics of the type $Y_{J,\pm1}$ in (2.4) vanish, and thus all $\langle T_{J,\pm1}\rangle_f$. This proves the assertion in Sec. 1 that the polarization states produced in simple reactions will have the spin along a principal axis; moreover, we see this preferred direction is normal to the reaction plane.

We have already discussed the general features of these polarization states. However, it is useful to examine the $\langle T_{JM}\rangle_f$ using an axis in the reaction plane as the polar axis, for if we try to detect polarization by a second reaction, it is convenient to choose the direction of incidence of this second reaction as the Z-axis. We single out the normal as the y-axis. In this coordinate system, all nonvanishing second rank harmonics involving $\mathbf{k}_i$ and $\mathbf{k}_f$ in (2.4) are real, while $Y_{1,\pm1}(\mathbf{k}_i\times\mathbf{k}_f)$ is pure imaginary. These determine the relations between the $\langle T_{JM}\rangle_f$:

$$\langle T_{2,-M}\rangle_f=(-1)^M\langle T_{2,M}\rangle_f,\tag{2.5}$$

$$\langle T_{1,-M}\rangle_f=(-1)^{M+1}\langle T_{1M}\rangle_f.\tag{2.6}$$

Equation (2.6) includes the vanishing of $\langle T_{10}\rangle_f$, which we already know, since $\langle\mathbf{S}\rangle_f$ lies on the normal.

A second problem of interest is the angular distribution of the reaction products when the initial state consists of a polarized deuteron and an unpolarized

[6] In the principal axis system $\langle T_{22}\rangle$ is real and equal to $\langle T_{2,-2}\rangle$. If we consider as our base matrices the hermitian and the anti-Hermitian parts of T_{22}, then the expectation value of the former is $\sqrt{2}\langle T_{22}\rangle$ and the expectation value of the latter vanishes.

[7] This matrix and the following equations are a straightforward extension of the corresponding ones in reference 2.

[8] $Y_{2M}(\mathbf{k}_i,\mathbf{k}_f)$ is a second-degree harmonic, bilinear, and symmetrical in $\mathbf{k}_i$ and $\mathbf{k}_f$.

TABLE I. Maximum degree in $\cos\theta$ of the polynomial functions I_0, A, B, C, and D.

Polynomial	Maximum degree in $\cos\theta$
$I_0(\theta)$	$2L_{\max}$
$A(\theta)$	$2L_{\max}+2$
$B(\theta)$	$2L_{\max}+1$
$C(\theta)$	$2L_{\max}-1$
$D(\theta)$	$2L_{\max}$

target of spin j. The general form of $M^\dagger M$ may now be found by the arguments used in deriving (2.4); substituting this form into (2.1) with S^μ equal to unity, we find a general expression for the differential cross section I:

$$I=I_0+I_2\sum_M Y_{2M}^*(\mathbf{k}_i)\langle T_{2M}\rangle_i+I_1\sum_M Y_{2M}^*(\mathbf{k}_f)\langle T_{2M}\rangle_i +I_3\sum_M Y_{2M}^*(\mathbf{k}_i\mathbf{k}_f)\langle T_{2M}\rangle_i +I_4\sum_M Y_{1M}^*(\mathbf{k}_i\times\mathbf{k}_f)\langle T_{1M}\rangle_i, \quad (2.7)$$

where the I's are real quantities depending on k_i, k_f, and $\mathbf{k}_i\cdot\mathbf{k}_f$.

If the polarized deuterons used as projectiles in a reaction are products of a previous reaction, the $\langle T_{JM}\rangle_i$ in Eq. (2.7) applied to the second reaction must satisfy (2.5) and (2.6), where the Z-axis is the direction of incidence of the second reaction and the y-axis is normal to the plane of the first. Combining terms containing $Y_{JM}^*(\mathbf{k}_f)$ and $Y_{J,-M}^*(\mathbf{k}_f)$, we obtain the dependence of I on $\mathbf{k}_f$, which gives us the angular distribution

$$I(\theta,\phi)=I_0(\theta)+\langle T_{20}\rangle_i A(\theta) +[\langle T_{21}\rangle_i B(\theta)+iC(\theta)\langle T_{11}\rangle_i]\sin\theta\cos\phi +\langle T_{22}\rangle_i D(\theta)\sin^2\theta\cos2\phi. \quad (2.8)$$

Here θ is the polar angle, ϕ the azimuthal angle measured from the plane of the first reaction, and the functions I_0, A, B, C, D, are polynomials in $\cos\theta$. If $L_{\max}$ is the maximum effective incident orbital angular momentum (in the second reaction, the degrees of these polynomials are limited as given in Table I[9]:

In elastic collisions, the polarization produced by scattering unpolarized deuterons is related to the azimuthal asymmetries obtained by scattering polarized deuterons, assuming the Hamiltonian is invariant under time reversal.[10,11] If M' is the time-reverse of M, it may be defined

$$M_{a,b}'=M_{-b,-a}, \quad (2.9)$$

where $-b$, $-a$ are the time-reversed states of b and a respectively. The invariance argument becomes

$$M'=M,$$

so that

$$(MM^\dagger)'=M^\dagger M. \quad (2.10)$$

[9] The argument is an extension of that used for unpolarized beams; see e.g., reference 4, page 535. For an extension to polarized beams see A. Simon and T. Welton, Phys. Rev. **90**, 1036 (1953); L. Wolfenstein, Phys. Rev. **92**, 123 (1953).

[10] E. Wigner, Nachr. Ges. Wiss. Göttingen **31**, 546 (1932).

[11] This is an extension of the argument of reference 2.

The operator $\mathbf{S}$ changes sign under time-reversal and $\mathbf{k}_i$ and $\mathbf{k}_f$ are transformed into $-\mathbf{k}_f$ and $-\mathbf{k}_i$ respectively. Comparing the explicit forms for $MM^\dagger$ and $M^\dagger M$, Eq. (2.10) yields the result that J_i in Eq. (2.4) equals I_i in Eq. (2.7). It follows that, if we choose the Z-axis in the direction of the outgoing deuterons and the y-axis as usual normal to the reaction plane, the values of $\langle T_{JM}\rangle$ produced by scattering unpolarized deuterons are related to the coefficients I_0, A, B, C, and D of (2.8) by

$$\begin{aligned} I_0\langle T_{20}\rangle&=A,\\ I_0\langle T_{21}\rangle&=\tfrac{1}{2}B\sin\theta,\\ I_0\langle T_{22}\rangle&=\tfrac{1}{2}D\sin^2\theta,\\ I_0\langle T_{11}\rangle&=-(i/2)C\sin\theta. \end{aligned} \quad (2.11)$$

Then in two successive scatterings of a deuteron by an infinitely heavy nucleus through the same scattering angle θ, the angular distribution is

$$I=I_0[1+\langle T_{20}\rangle^2+2(\langle T_{21}\rangle^2+|\langle T_{11}\rangle|^2)\cos\phi +2\langle T_{22}\rangle^2\cos2\phi]. \quad (2.12)$$

Thus four parameters are required to describe the result of a double scattering experiment in contrast to the one parameter needed for a spin $\frac{1}{2}$ particle; in Eq. (2.12) these parameters are taken as the expectation values of the tensor operators after the first scattering. Experimentally the value of $\langle T_{20}\rangle^2$ is obtained by comparing the total intensity averaged over ϕ with that produced by scattering unpolarized deuterons; then the value of $\langle T_{22}\rangle^2$ is obtained from the $\cos2\phi$ term in the azimuthal distribution. But the $\cos\phi$ term contains contributions from both first and second rank tensors so that a simple double scattering does not distinguish $\langle T_{11}\rangle$ from $\langle T_{21}\rangle$. (See Sec. 3.)

It is particularly interesting to examine some of the cases of double scattering where the first scattering produces completely polarized particles. If the pure state is χ_0 oriented along the y-axis or x-axis, the $\cos2\phi$ asymmetry is a maximum and

$$I=I_0(\tfrac{3}{2}+\tfrac{3}{2}\cos2\phi). \quad (2.13)$$

If the pure state is $\chi_{\pm1}$ oriented along the y-axis the $\cos\phi$ asymmetry is a maximum, but this is also true in the more general case when the pure state is any state like Ψ_2 (or Ψ_3) of Eq. (1.7) if the principal axes of the tensor in the X-plane make angles of 45° with the Z-axis. Then I becomes

$$I=I_0(1\tfrac{1}{8}+\tfrac{3}{2}\cos\phi+\tfrac{3}{8}\cos2\phi). \quad (2.14)$$

It may be noted by comparing (2.12) with (1.6), that in all cases where the first scattering produces completely polarized particles the cross section I for the second scattering at $\phi=0$ is three times the unpolarized cross section.

3. EFFECTS OF MAGNETIC DEFLECTION

In double-collision experiments, the first may take place inside the cyclotron; the magnetic field in general will alter the polarization of the outgoing deuterons. Such effects would have to be considered in interpreting the experiments; moreover, they may be useful in studying nuclear reactions. We treat the case where the interaction of the deuteron and magnetic field may be treated as the interaction of a magnetic dipole with a static field normal to the plane of the first reaction. The spin state rotates about the field with a frequency equal to $\mu\omega$, where μ is the deuteron magnetic moment in nuclear magnetons and ω the proton Larmor frequency. The deuteron's cyclotron frequency is one-half that of the proton and thus equals the Larmor frequency. Therefore the angle of rotation of the spin relative to the direction of motion equals $(\mu-1)$ times the angle of deflection. Since $(\mu-1)$ is about $-1/7$, fairly large deflections are generally required to get appreciable effects on the polarization. Since $\langle \mathbf{S}\rangle$ is in the direction of the field, it remains unchanged. The principal axes of the second rank tensor in the reaction plane are rotated relative to the direction of motion through an angle $(\mu-1)$ times the angle of deflection.

If the deuterons can be deflected through sufficiently large angles, some of the ambiguities in the double scattering experiment discussed in Sec. 2 can be resolved. The effect of the rotation of the principal axes relative to the direction of motion will be to replace the $\langle T_{2M}\rangle$ in (2.8) by $\langle T_{2M}\rangle'$

$$\langle T_{2M}\rangle' = \sum_{M'} C_{MM'}(R)\langle T_{2M}\rangle, \tag{3.1}$$

where the C's are determined by the rotation and are readily calculated. Experiments discussed in Sec. II both with and without magnetic deflection will then determine in principle $|\langle T_{20}\rangle|$, $|\langle T_{20}\rangle'|$, $|\langle T_{22}\rangle|$, $|\langle T_{22}\rangle'|$, and the signs of the primed quantities relative to the unprimed. Then (3.1) will determine all $\langle T_{2M}\rangle$ and $\langle T_{2M}\rangle$ uniquely including all relative signs. Since (3.1) is invariant under change of sign of all $\langle T_{2M}\rangle$ there remains one ambiguity in sign. The value of the $\cos\phi$ asymmetry then provides a determination of $|\langle T_{11}\rangle|$. We may translate the result for the $\langle T_{JM}\rangle$ to the coordinate system used in Sec. 1 and represent it as a point in the space with coordinates $\langle T_{10}\rangle$, $\sqrt{2}\langle T_{22}\rangle$, and $\langle T_{20}\rangle$. Then the ambiguity in the sign of $\langle T_{2M}\rangle$ corresponds to a reflection in the origin of the $\sqrt{2}\langle T_{22}\rangle-\langle T_{22}\rangle$ plane so that if one possible result corresponds to a statistical mixture with a surplus (more than $\frac{1}{3}$) of state ψ_1, the other corresponds to a deficiency. In many cases the effect of the reflection may be to put the point outside of the allowed cone [Eq. (1.9)] so that in these cases there exists no ambiguity in the signs of the $\langle T_{2M}\rangle$. The sign of $\langle T_{11}\rangle$ cannot be determined by any of these experiments.

Magnetic deflection may yield valuable partial information about nuclear reactions under conditions which are not experimentally favorable for the complete set of experiments described above. For example, if the polarized deuterons are produced in the forward direction, the selection rules show that $\langle T_{20}\rangle$ may be the only nonvanishing expectation value. If $\langle T_{20}\rangle$ is large enough, one can produce known relative amounts of the $\langle T_{2M}\rangle$ by magnetic deflection and then obtain the cross section I and the ratios of A, B, and D of (2.9) for a second reaction. This partially calibrated detector could be useful in determining relative values of the $\langle T_{2M}\rangle$ of deuteron beams.

4. ELASTIC SCATTERING OF DEUTERONS BY CARBON

The double scattering of deuterons by carbon has been studied experimentally by Chamberlain *et al.*[12] This experiment is particularly simple because the target has both spin and isotopic spin equal to zero, and the scattering amplitude may be calculated in terms of that for protons by carbon if the impulse approximation is valid.[13] The matrix M may be written

$$M = \int d_3p[f(2\mathbf{p}+\mathbf{k})(\mathbf{k}+\mathbf{p}|t_1+t_2|\mathbf{k}_0+\mathbf{p}) \times f(2\mathbf{p}+\mathbf{k}_0)]. \tag{4.1}$$

Here $\mathbf{k}_0$ and $\mathbf{k}$ are initial and final deuteron momenta respectively, the t's are scattering amplitudes from carbon for the two nucleons inside the deuteron, and f is the deuteron ground-state wave function in momentum space, which may be written[14]

$$f(\mathbf{p}) = f_0(p) + f_2(p)S_{12}(\mathbf{p}), \tag{4.2}$$

where

$$S_{12}(\mathbf{p}) = 3\boldsymbol{\sigma}_1\cdot\mathbf{p}\boldsymbol{\sigma}_2\cdot\mathbf{p} - p^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2.$$

The t_1 and t_2 have the same dependence upon their arguments according to the charge symmetry hypothesis and may be written

$$(\mathbf{k}+\mathbf{p}|t_i|\mathbf{k}_0+\mathbf{p}) = g + h\boldsymbol{\sigma}_i\cdot[(\mathbf{k}_0+\mathbf{p})\times(\mathbf{k}+\mathbf{p})], \tag{4.3}$$

where g and h are scalars.

Equation (4.1) simplifies greatly if we assume the deuteron ground state is a pure S-state. We look first at results that are independent of the form of g and h. Since M is linear in the spin operator, it has the form

$$M = G + H\mathbf{S}\cdot\mathbf{n}, \tag{4.4}$$

where $\mathbf{n}$ is a unit vector perpendicular to $\mathbf{k}_0$ and $\mathbf{k}$. Then

$$MM^\dagger = (|G|^2 + \tfrac{2}{3}|H|^2) + 2(\tfrac{2}{3})^{\frac{1}{2}}\,\mathrm{Re}(G^*H)\sum_M T_{1M}Y_{1M}^*(\mathbf{n}) + \tfrac{1}{3}\sqrt{2}|H|^2\sum_M T_{2M}Y_{2M}^*(\mathbf{n}), \tag{4.5}$$

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[12] O. Chamberlain *et al.*, Phys. Rev. **95**, 1104 (1954).

[13] G. F. Chew, Phys. Rev. **80**, 196 (1950); G. F. Chew and G. C. Wick, Phys. Rev. **83**, 636 (1952). In the present case in which the proton scattering amplitude is sharply peaked in the forward direction the neglect of multiple collisions may not be justified.

[14] W. Rarita and J. Schwinger, Phys. Rev. **59**, 436 (1941).

where Re means the real part is to be taken. The resulting polarization state is a statistical mixture of the pure states oriented along $\mathbf{n}$ and is unaltered by the magnetic deflection discussed in Sec. 3. If the y-axis is taken along $\mathbf{n}$ the nonvanishing $\langle T_{JM}\rangle$ are given by

$$\begin{aligned} I_0\langle T_{11}\rangle &= -(2i/\sqrt{3})\,\mathrm{Re}(G^*H), \\ I_0\langle T_{22}\rangle &= (\tfrac{3}{2})^{\frac{1}{2}} I_0\langle T_{20}\rangle = -(1/2\sqrt{3})\,|H|^2, \\ I_0 &= |G|^2+\tfrac{2}{3}|H|^2, \end{aligned} \tag{4.6}$$

and the angular distribution of the double scattering is given by substituting (4.6) into (2.12). A maximum left-right asymmetry of about 17 to 1 in double scattering can occur for $|G|^2$ about equal to $\frac{2}{3}|H|^2$. For $|H|\gg|G|$ we get the maximum $\cos2\phi$ asymmetry, the angular distribution approaching $(1+\frac{1}{3}\cos2\phi)$. The ratio of the $\cos\phi$ term to the azimuthally symmetric term in the experiments of Chamberlain *et al.*[12] are consistent with (4.6) if the ratio $\frac{2}{3}|H|^2/|G|^2$ is about 0.09 and G and H have the same phase; also if the phases of G and H differ by 45° and $\frac{2}{3}|H|^2/|G|^2$ is about 0.23. In both cases by choosing $(\frac{2}{3})^{\frac{1}{2}}|H|$ less than $|G|$ we have made sure that the $\cos2\phi$ term is small enough to be consistent with the experiments. Much larger values of the relative phase are not possible.

If the g and h in (4.3) depend only on the momentum transfer, we may take these quantities out of the integral in (4.1). This would be true if the Born approximation were valid for the scattering of protons by carbon, but is must be noted that the important contributions to the integral come from proton energies about half the deuteron energy and scattering angles about twice the deuteron scattering angle. Continuing to neglect the D-state part of f, we obtain

$$\begin{aligned} M=&[2g(q)+h(q)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot(\mathbf{k}_0\times\mathbf{k})] \\ &\times\int f_0(|2\mathbf{p}+\mathbf{k}|)f_0(|2\mathbf{p}+\mathbf{k}_0|)d_3p+h(q)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) \\ &\cdot\left[(-\mathbf{q})\times\int \mathbf{p}f_0(|2\mathbf{p}+\mathbf{k}|)f_0(|2\mathbf{p}+\mathbf{k}_0|)d_3p\right], \end{aligned} \tag{4.7}$$

where $\mathbf{q}$ is the momentum transfer. The second integral in (4.7) has nonvanishing components only in the direction of $(\mathbf{k}_0+\mathbf{k})$. If we make the approximation that $|\mathbf{k}|$ is large compared to momenta inside the deuteron the product of the f's in the integrand will be peaked in the region where $\mathbf{p}\cdot(\mathbf{k}_0+\mathbf{k})=-\frac{1}{4}(\mathbf{k}_0+\mathbf{k})^2$. Taking this value of $\mathbf{p}\cdot(\mathbf{k}_0+\mathbf{k})$ out of the integral, the second integral just cancels half of the spin term in the first and we have

$$M=[2g(q)+h(q)\mathbf{S}\cdot(\mathbf{k}_0\times\mathbf{k})]\times\int f_0(|2\mathbf{p}+\mathbf{k}|)f_0(|2\mathbf{p}+\mathbf{k}_0|)d_3p. \tag{4.8}$$

The spin term involves $h(q)k^2\sin\theta$, which in general is expected to increase relative to $g(q)$ with increasing θ for small angles θ. In the scattering of protons of the same momentum $\mathbf{k}$, the maximum polarization occurs when the magnitudes of $g(q)$ and $h(q)k^2\sin\theta$ are about equal; in deuteron scattering the maximum $|\langle T_{11}\rangle|$ occurs when the magnitude of $2g(q)$ and $(\frac{2}{3})^{\frac{1}{2}}h(q)k^2\sin\theta$ are about equal, or at somewhat larger θ. In order that the $\cos2\phi$ asymmetry approach its maximum, the magnitude of $(\frac{2}{3})^{\frac{1}{2}}h(q)k^2\sin\theta$ must be large compared to $2g(q)$. This most likely would require still larger angles θ if it occurs at all. We would like to relate the deuteron polarization to that for proton, but it is not possible to predict the deuteron polarization from the proton without knowing the relative phase of g and h. If we take (4.8) literally we may state, however, that for the deuteron double-scattering results[12] to be correct within experimental error the proton polarization at the same momentum and angle should be more than 0.05. Since the proton polarization appears to be somewhat smaller we may say that Eq. (4.8) predicts either a smaller $\cos\phi$ term or a larger $\cos2\phi$ term than observed. However, experimental data are too limited to show how poor Eq. (4.8) may really be.

Since large momenta are important in (4.1), the D-state part of the deuteron functions may not make a small contribution. Then M in general would contain all terms allowed by symmetry arguments even in the Born approximation. Since I_0 generally would be the sum of five positive terms while the $\langle T_{JM}\rangle$ would be sums of terms which may have different signs, polarization effects, especially the $\langle T_{2M}\rangle$, might be reduced.

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