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Model for Asymmetric Fission*

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On the basis of shell effects, asymmetric fission is attributed to symmetrical division except for a non-fissionable core. Distributions of fission-fragment yields, *vs* fragment charges and masses, are obtained from simple statistical considerations of the division.

THE mechanism of asymmetric fission, as distinct from that of fissionability, does not appear as yet to have been clearly explained. As is well known, the customary analysis[1] leads to the conclusion that the two fission fragments should be of approximately equal mass. In order to resolve this disagreement with experimental facts it has been suggested on several occasions[2] that the binding energies of the fission fragments are affected by nuclear shell-structure effects and the most favorable division is an asymmetrical one.[3]† Hill and Wheeler,[4] however, consider that neither energy release nor shell-structure effects constitute an explanation of asymmetric fission. Another explanation[5] of asymmetrical fission based on the variation of barrier penetration with fission fragment mass has also been considered by Hill and Wheeler as untenable.

In the following note, we have looked into the possibility that there are still shell-structure effects but that their influence on fission might be more direct than simply affecting binding energies. We have assumed that even in the deformed nucleus the shells persist and preserve a degree of independence from each other. The act of fission is looked upon, once the critical deformation of the nuclear surface has been reached, as a symmetrical severing of the individual shells except for a core of inner shells. This core does not necessarily have to be the same group of inner shells in all fissions, but may vary somewhat according to the energy available in the fission process. The final stage of fission occurs when the core moves with one half of the nucleons of the outer shells.[6]

As, according to the present model, fission starts out symmetrically and continues symmetrically up to the point at which the core is reached, it would seem to remove the criticism raised by Hill and Wheeler that it is hard to see how a nucleus would know in the transition stage the potential shell structure of the not-yet-formed products.

An interesting aspect of the fission problem, if one is to accept the assumption of independence of the shells, is the threshold energy required for fissioning of intermediate shell-cores within the nucleus. If we employ the relationship derived by Hill and Wheeler[4] for the maximum energy of deformation, we obtain for the fission threshold of $_{20}\mathrm{Ca}^{40}$ a value of 167 Mev. It is significant that this energy is greater than the observed U^{235} symmetrical fission energy of $\sim$150 Mev and is identical with the observed maximum energy from asymmetrical fission.[6a] The value of the fission threshold energy for $_{20}\mathrm{Ca}^{40}$ should be compared with those for neighboring larger closed shells, such as $_{28}\mathrm{Ni}^{56}$ and the hypothetical $_{50}\mathrm{Sn}^{100}$, for which the calculated values are 120 and 35 Mev, respectively.‡ It seems plausible, therefore, that fission takes place beyond the $Z=N=50$ shell and up to the $Z=N=20$ core.

MASS DISTRIBUTION OF FRAGMENTS

We assume that the fissioning nucleus of $_{92}\mathrm{U}^{235}$ is not a uniform liquid droplet but rather a series of neutron and proton shells conforming[7] to the j-subshells of the spin-orbit coupling model.[8]

One might expect on the basis of a literal interpretation of the shell model, that if the nucleons were confined to subshells in accordance with Pauli's principle, then

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[1] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).

[2] M. G. Mayer, Phys. Rev. **74**, 235 (1948); G. C. Wick, Phys. Rev. **76**, 181 (1949); K. H. Kingdon, Phys. Rev. **76**, 136 (1949); G. Gamow and C. L. Critchfield, *Theory of the Atomic Nucleus* (Oxford University Press, London, 1949); L. Meitner, Nature **165**, 561 (1950); T. D. Newton, Phys. Rev. **87**, 187 (1952); P. Fong, Phys. Rev. **89**, 332 (1953).

[3] After this paper was written, the author's attention was directed to a discussion by D. Curie of asymmetric fission based on a nuclear model of a core of 50 protons and 82 neutrons, together with a corona of the remaining nucleons [D. Curie, Compt. rend. **235**, 1286 (1952); **237**, 1401 (1953)].

† *Note added in proof.*—A model of asymmetric fission rather similar to the one being presently discussed, but not based on shell-structure ideas, was given by P. F. Gast in 1947 [Phys. Rev. **72**, 1265 (1947)]. The author is indebted to the editor for drawing attention to this note.

[4] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).

[5] S. Frankel, J. Phys. (U.S.S.R.) **10**, 533 (1946); E. Bagge, Z. Naturforsch. **2a**, 565 (1947).

[6] A name for this model which has been suggested to us is "peach" model.

[6a] D. C. Brunton and G. L. Hanna, Can. J. of Research **28A**, 190 (1950); R. B. Leachman, Phys. Rev. **87**, 444 (1952).

‡ *Note added in proof.*—These fission threshold energies, evaluated for "free surface" nuclei, are clearly maximum values. The observed fission energy is similarly a maximum value as it refers to the completely separated fragments. Both values are undoubtedly modified in a deformed and incompletely fissioned nucleus.

[7] See, however, the arguments of level order in a deformed nucleus; reference 4.

[8] M. G. Mayer, Phys. Rev. **75**, 1969 (1949); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949); P. F. A. Klinkenberg, Revs. Modern Phys. **24**, 63 (1952).

the nucleons in a particular subshell would not be permitted to move randomly throughout the whole shell. Rather, it is plausible that only those nucleons which at the instant of fissioning are within the region of the neck are allowed to distribute statistically between the two fragments.

If the partition width is of the order of a nucleon diameter, it is readily estimated that the fraction of statistically-uncertain nucleons in a subshell is $\sim\Delta R/R$, where ΔR is of the order of a nucleon radius and R is of the order of the shell radius. This fraction is therefore of the order of $N^{-\frac{1}{3}}$, where N is the total number of nucleons enclosed within a particular j-subshell.

Estimates of the numbers of neutrons and protons in the various subshells of ${}_{92}\mathrm{U}^{235}$ are given in Table I.[9] The neutrons and protons are treated independently and the total number of neutrons distributed between the two fragments is assumed to be 142, two neutrons being assumed lost in the fission process.

In the calculation it is assumed that, with the exception of the statistically uncertain nucleons, the subshells tend to divide equally. The statistical probability of r uncertain nucleons of a particular subshell which choose to go with one fragment and of $(n-r)$ uncertain nucleons which choose to go with the other fragment is assumed to be $\binom{n}{r}$. By continuing the division of the subshells down to the innermost ${}_{20}\mathrm{Ca}^{40}$ core, it can readily be shown that the probabilities P_N and P_P of obtaining N neutrons in one fragment out of 29 uncertain neutrons, and P protons in one fragment out of 19 uncertain protons, respectively, are those given in Table II.

TABLE I. Numbers of neutrons and protons in the j subshells of ${}_{92}\mathrm{U}^{235}$, and estimates of the numbers of statistically-uncertain neutrons and protons in fission.

Nuclear subshell	Numbers of neutrons	Uncertain neutrons	Numbers of protons	Uncertain protons
$7i\ 11/2$	6	1		
$6g\ 9/2$	10	2		
$7i\ 13/2$	14	3		
$4p\ 1/2$	2	0		
$4p\ 3/2$	4	1		
$5f\ 5/2$	6	1		
$6h\ 9/2$	10	2	10	2
$5f\ 7/2$	8	2		
$3s\ 1/2$	2	0	2	0
$4d\ 3/2$	4	1	4	1
$6h\ 11/2$	12	3	12	3
$5g\ 7/2$	8	2	8	2
$4d\ 5/2$	6	2	6	2
$5g\ 9/2$	10	3	10	3
$3p\ 1/2$	2	0	2	0
$4f\ 5/2$	6	2	6	2
$3p\ 3/2$	4	1	4	1
$4f\ 7/2$	8	3	8	3
20–20 core	20		20	

TABLE II. Probability distributions of obtaining numbers of neutrons and protons out of statistically uncertain neutrons and protons in fission.

Neutron distribution		Proton distribution	
N	$P_N\times 2^{29}$	P	$P_P\times 2^{19}$
0 or 29	1	0 or 19	1
1 or 28	29	1 or 18	19
2 or 27	406	2 or 17	171
3 or 26	3654	3 or 16	969
4 or 25	23 751	4 or 15	3876
5 or 24	118 755	5 or 14	11 628
6 or 23	475 020	6 or 13	27 132
7 or 22	1 560 780	7 or 12	50 388
8 or 21	4 292 145	8 or 11	75 582
9 or 20	10 015 005	9 or 10	92 378
10 or 19	20 030 010		
11 or 18	34 597 290		
12 or 17	51 895 935		
13 or 16	67 863 915		
14 or 15	77 558 760		

The probability, P_A, of obtaining a fragment of particular mass value A will then be obtained by summing over all possible ways of obtaining a constant value of a particular $(P+N)$ sum. The mass values A_1 and A_2 of the two fragments will be given by

$$A_1=(20+26.5+P)+(20+46.5+N)=113+(P+N),$$
$$A_2=(26.5+19-P)+(46.5+29-N)=121-(P+N).$$

The calculated values of P_A are shown plotted as full circles in Fig. 1, where the heavy curve represents the experimental distribution[10] for the slow-neutron fission of U^{235}. The points have been normalized to the experimental curve at the maximum yields.

If we compare the calculated points with the experimental curve, two features are in evidence. Firstly, the calculated peaks occur at approximately the correct mass values and secondly, the gross fall-off in yield fits the experimental curve closely. The calculated widths of the yield curves, however, do not agree with experiment. Nevertheless, an adequate fit between calculation and experiment can be achieved by adding to the calculated full circle points a similar set of yield curves displaced slightly from the calculated ones. The second set of curves is indicated in Fig. 1 by the triangle points which have been obtained simply by subtracting the full circle points from the experimental curve. It is significant that the peaks of the subsidiary yield curves are approximately 56 mass numbers apart and also have rather closely the same form as the initially calculated curves. It would seem plausible that the subsidiary curves arise from symmetrical division of the ${}_{92}\mathrm{U}^{235}$ nuclei only to the ${}_{28}\mathrm{Ni}^{56}$ core.

The small, though significant, fission yield observed in the symmetrical trough of the yield curve cannot evidently be attributed to an overlapping of the fission yields of the ${}_{20}\mathrm{Ca}^{40}$-core type. Although accurate subtraction of the calculated points from the experimental

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[9] The levels are probably not in the order given by a spherical nuclear shell model. However, the numbers of uncertain nucleons from one level to another are small and much the same no matter what levels are involved. Moreover, the distributions are not very susceptible to the order in which the numbers are taken.

[10] Plutonium Project Report, Revs. Modern Phys. **18**, 513 (1946).

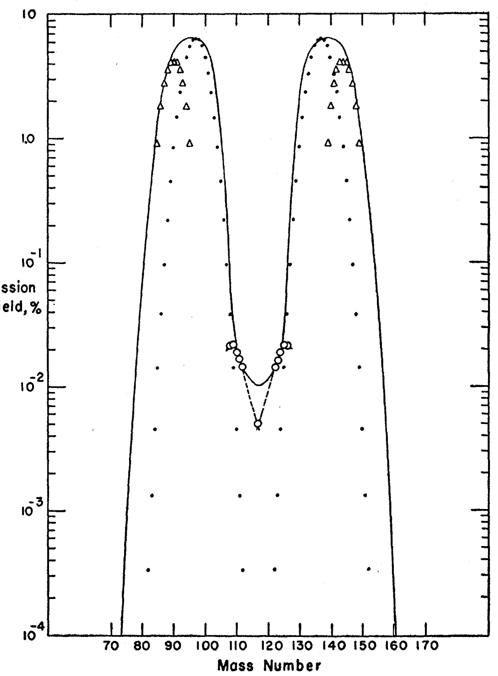

FIG. 1. U^{235} slow-neutron fission yields *vs* mass number. The full curve represents the smoothed experimental data. The full points represent the calculated probabilities, P_A, for the ${}_{20}Ca^{40}$-core type of fission. Triangle points are the differences between the curve and full points and probably represent the ${}_{28}Ni^{56}$-core type of fissions. Similarly the open-circle points and dotted curve probably represent the ${}_{8}O^{16}$-core type of fissions.

curve cannot be made along the steep sides of the trough, approximate subtractions near the base are shown by open circle points in Fig. 1. It seems possible that these subsidiary curves, which are of the order of 16 mass numbers apart, might be attributed to symmetrical fission processes occurring to the ${}_{8}O^{16}$ core. According to the analysis indicated in Fig. 1, this would occur at thermal neutron excitation energies, only to the extent of a few tenths of a percent of the normal fissions. At higher energies of excitation and deformation, it would seem feasible that the fraction of ${}_{8}O^{16}$-core type of fissions might increase, especially at the expense of the ${}_{28}Ni^{56}$-core type of fissions. Some evidence in support of these changes in the form of the fission yield curve appears to exist in experiments with higher energies of excitation.[11,12]

CHARGE DISTRIBUTION OF FRAGMENTS

The distribution of protons which might be apportioned between the two fragments is already available in Table II. Unfortunately, to date, it does not appear to have been experimentally feasible to obtain accurate data on the direct yields of particular Z-fragments. However, some notion of the distribution may be obtained from the known yields of radioactive chains and the probable decay lengths of these chains.

According to Glendenin, Coryell, and Edwards[13] and Pappas,[14] the most probable chain length of the U^{235} fragments is approximately three charge units. In Fig. 2, the experimental yields[10] of the chains from slow-neutron fission of ${}_{92}U^{235}$ are plotted *vs* the atomic numbers which are three units removed from the stable end-products of the chains. (In order to improve the continuity of available data, both light and heavy fragments have been superimposed on the same plot.)

If the Z-distributions for different chains are the same, and it will be shown later that this is very likely the case, then the points of Fig. 2 should represent approximately the Z-yield curve of the directly-produced fragments. The full curve of Fig. 2 represents the calculated distribution obtained from the values of P_P of Table II. This curve has been normalized and moved laterally a small amount to fit the experimental points at the peak. The calculated peaks actually occur at $Z=36$ and $Z=56$, if one assumes only 20–20 core type of

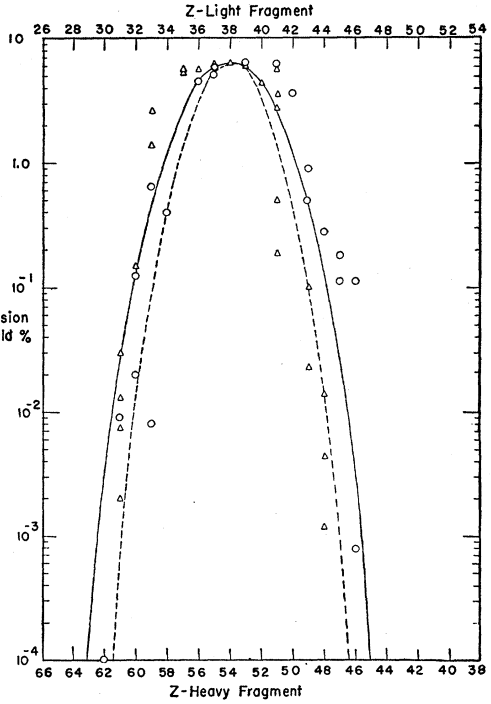

FIG. 2. U^{235} slow-neutron fission yields *vs* atomic number. The full curve represents the calculated probability distribution, P_P, for direct production of fragments of atomic number Z. The open circle points refer to experimental yields of light fragments (upper Z-scale). The triangle points refer to experimental yields of heavy fragments (lower Z-scale). The dotted curve represents the probability distribution, P_Z, for fragments of a particular atomic mass chain.

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[11] R. W. Spence, Brookhaven National Laboratory Report, No. C-9, 1949 (unpublished).

[12] R. A. Schmitt and N. Sugarman, Phys. Rev. **89**, 1155 (1953).

[13] Glendenin, Coryell, and Edwards in *Radiochemical Studies: The Fission Products* (McGraw-Hill Book Company, Inc., New York, 1951), Paper No. 52, National Nuclear Energy Series, Plutonium Project Record, Vol. 9, Div. IV.

[14] A. C. Pappas, Massachusetts Institute of Technology Technical Report No. 63, 1953 (unpublished).

fission. The experimental peaks occur at approximately $Z=38$ and $Z=54$. Apart from this small shift, the agreement of the calculated curve with experiment is excellent.[15]

In obtaining the probability distribution of fission fragments of mass value A, the probability distributions of fission fragments of atomic charge Z for a particular value of A were also necessarily obtained. In Fig. 3, these distributions P_Z are shown plotted *vs* atomic charge Z for various values of the atomic mass A. Only those values of A, between 126 and 136 for the heavy fragment and between 98 and 108 for the light fragment, have here been considered as largely representative of the 20–20 core type of fission for which the calculations were carried out. All curves of Fig. 3 are seen to be of the same form and width, which is somewhat narrower than the direct-Z distribution already considered. For comparison, the Z-distribution curve for constant A is shown by the dotted curve in Fig. 2. It should be pointed out that the calculated Z-distribution for constant A is still wider than the distribution experimentally evaluated by Pappas.[14]

The calculated yields of particular Z-fragments as fractions of the total chain yields of fragments of the same A, can now be compared with the fractional yields determined from experiment.[14] For nuclides which would appear to result largely from the 20–20 core type of fission, the above comparison is made in Table III. The fission-fragment yield, P_Z, is obtained for a particular Z from a curve of a particular A in Fig. 3. The total chain yield, P_A, is obtained from the same curve by summing the yields for all Z. From Table III, it is seen that there is reasonable agreement between calculated and experimental fractional yields.

TABLE III. Comparison of calculated and experimental values of the fractional chain yield of a particular Z-fragment.

Nuclide	A	Z	Fractional yield (expt.)	P_A (calc.)	P_Z (calc.)	Fractional yield (calc.)
Nb^{96}	96	41	9×10^{-5} 4×10^{-3}	1.1×10^{-1}	1.6×10^{-4}	1.45×10^{-3}
Te^{131}	131	52	4×10^{-1}	2.6×10^{-2}	4.0×10^{-3}	1.54×10^{-1}
Te^{133}	133	52	3.6×10^{-1}	5.95×10^{-2}	5.0×10^{-3}	0.84×10^{-1}
I^{134}	134	53	1.6×10^{-1}	7.93×10^{-2}	1.1×10^{-2}	1.4×10^{-1}
Xe^{135}	135	54	5×10^{-2}	9.73×10^{-2}	1.8×10^{-2}	1.85×10^{-1}
Cs^{136}	136	55	1.5×10^{-2} 9×10^{-4}	1.1×10^{-1}	2.4×10^{-2}	2.2×10^{-1}
La^{140}	140	57	$<3\times10^{-2}$	7.93×10^{-2}	1.8×10^{-2}	2.2×10^{-1}

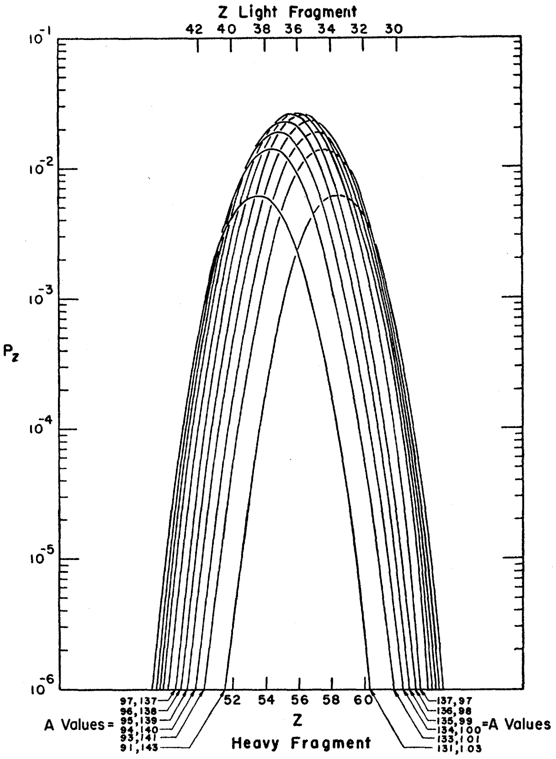

FIG. 3. Calculated probability distributions, P_Z, of fragments of atomic numbers Z, as a function of Z, for various atomic mass values A of particular radioactive chains.

Another point of contact between calculation and experiment can be made by comparing the most probable chain lengths for constant values of A. For the A values applicable to the present calculations, the most probable values of the charges, Z_P, of the fragments can be obtained immediately from Fig. 3. The values of Z_A, the charges of the end-products of the chains, have been taken from the report by Pappas.[14] The calculated lengths of the chains, Z_A-Z_P, are then given in Table IV. It is seen that for the light fragment of the listed A-values, the chain length is approximately one unit, and for the heavy fragment of the listed A-values, the chain length is approximately six to seven units. Experimentally, both of these lengths are believed to be

TABLE IV. Probable chain lengths in U^{235} fission.

Heavy fragment	A	131	133	135	137	139	141	143
	Z_P	53.5	54.5	55.5	56.0	57.0	57.5	58.5
	Z_A	54	55	56	56.5	57.5	58.5	60
	(Z_A-Z_P)	0.5	0.5	0.5	0.5	0.5	1.0	1.5
Light fragment	A	91	93	95	97	99	101	103
	Z_P	33.5	34.5	35.0	36.0	36.5	37.5	38.5
	Z_A	40	41	42	43	44	44	45
	(Z_A-Z_P)	6.5	6.5	7.0	7.0	7.5	6.5	6.5

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[15] It should be pointed out however that no account has been taken of the rather smaller possibility of 28–28 core type of fission affecting the Z-distribution. The agreement of the proton distributions with experiment would seem to indicate that the proton shells are generally split right down to the ${}_{20}Ca^{40}$ core. In view of the alternative ${}_{28}Ni^{56}$ mode of fission, already indicated from the mass distribution analysis, it would thus seem admissible that the proton and neutron shells down to ${}_{20}Ca^{40}$ are not necessarily fissioned to the same extent, and that the width of the A-distribution curve could be taken up by an indefiniteness in the splitting of only the neutron shells.

approximately three to four units. Thus, again, it would seem possible that a complementary shift of approximately two charges in the positions of the peaks of the light and heavy fragments would bring the calculations into line with experiment. It is possible, since the light fragment tends to have an excess of neutrons in the division, that some allowance for neutron-proton interactions might be able to account for this shift.

It is a pleasure to acknowledge the valuable discussions on these problems with R. B. Duffield. Thanks are also due A. T. Nordsieck and J. Weneser for criticism of the manuscript.

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Angular Momentum Coupling in Deuteron Reactions*

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Information about nuclear coupling schemes which can be derived from relative cross sections in deuteron stripping and pickup reactions in light nuclei is considered. In a few cases, experimental results are applied to give a determination of the intermediate-coupling parameter.

1. INTRODUCTION

In recent years, the intermediate-coupling shell model has received the attention of a number of authors.[1] Energy levels of p shell nuclei in particular have been studied in intermediate coupling because they are not consistent with either of the extreme coupling schemes. The most striking cases are given in the very complete and useful survey paper of Inglis[1] where approximate level schemes are given as a function of the intermediate-coupling parameter which measures the relative effective strengths of the spin-orbit and central parts of the interaction.

Our present purpose is to consider certain deuteron stripping and pickup reactions using intermediate coupling wave functions. By considering the ratio of the cross sections of a d-p reaction leading to different states of the same final nucleus one can also deduce the value of the intermediate-coupling parameter and thus obtain an independent check on the validity of the model.

We should remark that the present work was stimulated by an investigation of Christy[2] on angular momentum coupling in a certain class of resonant nuclear reactions. Christy's work confined itself to the two extreme coupling schemes and did not require the construction of explicit nuclear wave functions. These become essential, however, if we wish to depart from the extreme cases. When one does this one encounters, among others, the difficulty that the compound state is almost always quite highly excited and thus may not be a good candidate for description by a shell model. Deuteron reactions do not have this difficulty, for we may restrict ourselves to low-lying states. There is also the important fact that the essential features of these reactions are well understood and adequately described by Butler's theory.[3]

An analysis similar to ours has already been published by Lane.[1] Lane has in fact emphasized the desirability of using intermediate-coupling wave functions for many purposes beyond the elementary one of calculating level schemes. In his first-cited paper, magnetic dipole moments and transition strengths and reduced widths for nucleon emission are considered and applied with success to the low-lying states of C^{13} and N^{13}. In the present work (which was done independently of Lane) we consider stripping and pick-up reactions involving nuclei with $A=6$, 7, 13, 14.

2. DEUTERON REACTION CROSS SECTIONS

We first write the d-p cross section using the "Born approximation" theory[4] which is equivalent to, but simpler in formulation, than the original theory of Butler. Corrections such as the Coulomb effect and the interaction between proton and nucleus are neglected because of the large uncertainties in the shell model treatment of nuclear spectroscopy. The details of the Born approximation are well-known and we do not give them; we consider in detail only the overlap integral between the initial and final systems since it is from this (or rather a ratio of two of them) that we shall gain information concerning the intermediate coupling parameter. This integral can be related to a

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* Supported in part by the U. S. Atomic Energy Commission.

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[1] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953); Phys. Rev. **87**, 915 (1952); N. Zeldes, Phys. Rev. **90**, 416 (1953); G. E. Tauber and T. Y. Wu, Phys. Rev. **93**, 295 (1954); **94**, 1307 (1954); A. M. Lane, Proc. Phys. Soc. (London) **A66**, 977 (1953); Phys. Rev. **92**, 839 (1953); R. Schulten, Z. Naturforsch. **8**, 759 (1953); R. Schulten and R. A. Ferrell, Phys. Rev. **94**, 739 (1954).

[2] R. F. Christy, Phys. Rev. **89**, 839 (1953).

[3] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951).

[4] P. B. Daitch and J. B. French, Phys. Rev. **87**, 900 (1952); Bhatia, Huang, Huby, and Newns, Phil. Mag. **43**, 485 (1952).