Relativistic Cosmology. I

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The paper presents some general relations obtaining in relativistic cosmology. It appears from these that a simple change over to anisotropy without the introduction of spin does not solve any of the outstanding difficulties of isotropic cosmological models.

I. INTRODUCTION

PERHAPS the only point in which all the current theories of cosmology are found to be in agreement is the time-dependent nature of the spatial geometry. It therefore seems of considerable interest to investigate the temporal behavior of a gravitating system as observed by a member of the system itself in its neighborhood. It is true that there exists a fairly large amount of literature where a study has been made on similar problems; however they all depend on some additional assumptions of which homogeneity and (or) some symmetry postulates seem to be very common ones. While there may indeed be some great aesthetic appeal in favor of such assumptions, yet they seem nevertheless open to serious doubts even on a smoothed-out scale and very definitely do not provide an exact picture of the universe when one considers the finer details. Further, the introduction of such assumptions lead to rather ambiguous situations when one runs into some difficulties, e.g., the well-known difficulties regarding the time-scale[1] and the original singularity (a creation in the finite past?) of the isotropic cosmologic models of general relativity have been variously attributed to the assumption of homogeneity and isotropy on the one hand[2–4] and to a failure of the general theory of relativity on the other.[5]

In this paper, an attempt is therefore made to study the temporal behavior of a gravitating cloud on the basis of the Einstein gravitational equations under very general conditions. This would presumably give one an idea about the potentialities and limitations of the general theory of relativity in providing a satisfactory solution to the cosmological problem.

II. THE DIFFERENTIAL EQUATION GOVERNING THE SPATIAL EXPANSION (OR CONTRACTION)

We shall assume that there is no interaction except through the Einstein gravitational equations

$$-8\pi T^{\mu}{}_{\nu}=R^{\mu}{}_{\nu}-\tfrac{1}{2}R\delta^{\mu}{}_{\nu}+\Lambda\delta^{\mu}{}_{\nu}, \qquad (1)$$

where the symbols have their usual significance. Under this circumstance the world lines of matter will be time-like geodesics. Further, we shall consider that there is no chaotic motion. We may now take the world lines of matter as our t-lines and if the coordinate along these lines measures the proper interval, the line element can be written in the form[6]

$$ds^2=dt^2+2g_{4i}dtdx^i+g_{ik}dx^idx^k, \qquad (2)$$

where the three-space metric $d\sigma^2=g_{ik}dx^idx^k$ is negative definite. Without loss of generality we can take $g_{4k}=0$ at a particular point (say $t=x^1=x^2=x^3=0$), i.e., the $(x^1x^2x^3)$ space is orthogonal to the world line at this point. The condition that the t-lines are geodesics gives

$$g_{4k,4}=0, \qquad (3)$$

so that g_{4k}'s vanish everywhere on the t-axis. We shall take this t-axis as the world line of our observer.

It may be noted that we are not taking $g_{4k}=0$ everywhere, i.e., we are not assuming that the geodesic congruence of the world lines is normal. This, according to Gödel,[7] corresponds to the existence of a spin relative to the compass of inertia. Our considerations therefore include the spinning cosmological models as well.

The energy momentum tensor is given by

$$T^{\mu}{}_{\nu}=\rho\frac{dx^{\mu}}{ds}\frac{dx_{\nu}}{ds}=g_{\nu\alpha}\rho\frac{dx^{\mu}}{ds}\frac{dx^{\alpha}}{ds}, \qquad (4)$$

so that for our cosmic fluid whose world lines are the t-lines (i.e., $dx^4/ds=1$, $dx^i/ds=0$), $T^{\mu}{}_{\nu}$ has only one nonvanishing component at points on the t-axis (where $g_{4k}=0$), namely

$$T^4{}_4=\rho.$$

It is now easy to deduce the following relation from the field equations (1):

$$R^4{}_4=\Lambda-4\pi\rho. \qquad (5)$$

[1] Recent researches have led to a doubling of the nebular distances and thus the "age" of the universe has been correspondingly increased. However, it seems doubtful whether even this revised time scale would be consistent with the estimates of the age of the earth by A. Holmes, Nature **163**, 453 (1949) and some of the astrophysical estimates [F. Hoyle, Nature **163**, 196 (1949)].

[2] A. S. Eddington, Science Progr. **34**, 225 (1939).

[3] R. C. Tolman, Revs. Modern Phys. **21**, 374 (1949); G. Omer, Jr., Astrophys. J. **109**, 164 (1949).

[4] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon Press, Oxford, 1934), pp. 438–39.

[5] For a concise review of these theories, see H. Bondi, *Cosmology* (Cambridge University Press, Cambridge, 1952).

[6] In this paper, Greek indices run from 1 to 4 while Latin indices run from 1 to 3.

[7] K. Gödel, Revs. Modern Phys. **21**, 447 (1949).

A direct calculation of $R^4{}_4$ gives at any point on the t-axis,

$$R^4{}_4=\frac{\partial^2}{\partial t^2}[\log\surd(-g)]+\tfrac{1}{4}g^{ik}\dot{g}_{kl}g^{lm}\dot{g}_{mi} +\tfrac{1}{4}g^{ik}g^{lm}(g_{l4,k}-g_{k4,l})(g_{i4,m}-g_{m4,i}), \quad (6)$$

where dots indicate differentiation with respect to time and $g_{ik,l}$ stands for $(\partial/\partial x^l)g_{ik}$.

We can show that $\frac{1}{4}g^{ik}\dot{g}_{kl}g^{lm}\dot{g}_{mi}-\frac{1}{3}[(\partial/\partial t)\log\surd(-g)]^2$ vanishes in case of isotropic expansion (or contraction) and is positive otherwise. For, at the particular point we can diagonalize the (3×3) matrix g_{ik} by a transformation of the form

$$\bar{x}^i=f^i(x^1,x^2,x^3),\quad \bar{t}=t, \quad (7)$$

where f^i is a function of its arguments, analytic at points on the t-axis. Such a transformation does not however disturb the value of any of the terms in (6). Hence, without loss of generality we can assume g_{ik} to be diagonalized, so that

$$\begin{aligned}\tfrac{1}{4}g^{ik}\dot{g}_{kl}g^{lm}\dot{g}_{mi}-\tfrac{1}{3}\left(\frac{\partial}{\partial t}\log\surd(-g)\right)^2 &=\tfrac{1}{4}[g^{ik}\dot{g}_{kl}g^{lm}\dot{g}_{mi}-\tfrac{1}{3}g^{ik}\dot{g}_{ik}g^{lm}\dot{g}_{lm}]\\ &=\tfrac{1}{2}[g^{11}g^{22}\dot{g}_{12}{}^2+g^{11}g^{33}\dot{g}_{13}{}^2+g^{22}g^{33}\dot{g}_{23}{}^2]\\ &\quad+\frac{1}{12}\left[\left(\frac{\dot{g}_{11}}{g_{11}}-\frac{\dot{g}_{22}}{g_{22}}\right)^2+\left(\frac{\dot{g}_{22}}{g_{22}}-\frac{\dot{g}_{33}}{g_{33}}\right)^2+\left(\frac{\dot{g}_{33}}{g_{33}}-\frac{\dot{g}_{11}}{g_{11}}\right)^2\right]\\ &=\phi^2(\text{say}),\end{aligned} \quad (8)$$

where ϕ vanishes if and only if $\dot{g}_{ik}=\alpha g_{ik}$ at the point under consideration, α being independent of the pairs of indices i and k, i.e., ϕ vanishes if the expansion (or contraction) at the point be isotropic. We note, further, that the relation $\dot{g}_{ik}=\alpha g_{ik}$ is invariant under transformation (7).

If v^μ be the velocity vector of matter, then with our choice of coordinate system it is simply the unit vector along the t-line at the point and hence $v^i=0$ and $v^4=1$ so that

$$\omega_{ik}\equiv\tfrac{1}{2}(v_{i;k}-v_{k;i})\equiv\tfrac{1}{2}(v_{i,k}-v_{k,i})=\tfrac{1}{2}(g_{i4,k}-g_{k4,i}). \quad (9)$$

The vanishing of the tensor ω_{ik} is the necessary and sufficient condition for the geodesic congruence of the world lines to be normal.[8] We shall identify the antisymmetric tensor ω_{ik} with spin, as seems natural from the classical relation $\boldsymbol{\omega}=\frac{1}{2}$ curl$\mathbf{v}$.[9] Using (9), we can write,

$$\tfrac{1}{4}g^{ik}g^{lm}(g_{i4,m}-g_{m4,i})(g_{l4,k}-g_{k4,l})=g^{ik}g^{lm}\omega_{im}\omega_{lk},$$

and this expression is easily seen to be negative definite. Writing this as $-2\omega^2$ as is consistent with the classical analogy, we get from (6) and (8),

$$R^4{}_4=\frac{\partial^2}{\partial t^2}[\log\surd(-g)]+\frac{1}{3}\left[\frac{\partial}{\partial t}\log\surd(-g)\right]^2+\phi^2-2\omega^2, \quad (10)$$

so that substitution in (5) gives

$$(1/G)(\partial^2G/\partial t^2)=(\Lambda-4\pi\rho-\phi^2+2\omega^2)/3, \quad (11)$$

where we have put $G^6=-g$.

In the ordinary isotropic case, the corresponding equation is

$$(1/G)(\partial^2G/\partial t^2)=(\Lambda-4\pi\rho)/3. \quad (12)$$

Further, one can deduce an equation formally exactly similar to (12) on the basis of Newtonian mechanics,[10] where Λ corresponds to a repulsive force proportional to distance and the term involving density arises from Newtonian gravitational attraction. Considering Eq. (11), therefore, one may say that on the classical analogy the spin gives rise to a repulsive force (the centrifugal repulsion) while any anisotropy in the local expansion effectively increases the gravitational attraction.

From the divergence relation $T^\mu{}_{4;\mu}=0$, we get

$$\rho G^3=C(\text{const}). \quad (13)$$

However, the dependence of ϕ^2 and ω^2 on G remains in general arbitrary, and thus it does not seem possible to integrate (11) without introducing further assumptions.

III. THE CASE OF A NONSPINNING GRAVITATING SYSTEM

In case the spin vanishes, $\omega^2=0$ and Eq. (11) becomes

$$(1/G)(\partial^2G/\partial t^2)=(\Lambda-4\pi\rho-\phi^2)/3. \quad (14)$$

It follows at once that a necessary condition for a completely static behavior in the neighborhood is

$$\rho=\Lambda/4\pi,$$

i.e., the local density must satisfy the relation obtaining in the Einstein static universe. It is to be noted that this condition is arrived at without any assumption regarding symmetry or conditions obtaining in the distant parts of the universe. It is now easy to see that if the whole universe is static the relation must be satisfied everywhere. It therefore follows that no nonspinning nonhomogeneous universe can be static and the only static nonspinning universe is the Einstein universe.[11]

[8] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1949), p. 115.

[9] This definition of spin is slightly different from that of Gödel. While we consider an antisymmetric tensor, Gödel defines the spin as a vector constructed from this tensor, the velocity vector and the Levi-Civita tensor ϵ^{iklm}. However, so long as the field is purely gravitational, so that the velocity field forms a geodesic congruence, physical conclusions from either definition are very similar.

[10] See reference 5, Chap. IX.

[11] The empty de Sitter universe is also sometimes regarded as static. However, in the coordinate system in which this universe is static, the t-lines are not geodesics and thus the static nature is only apparent due to the absence of matter.

We note further that Eq. (14) shows that, in the absence of spin, with a given value of ρ and $\dot{G}/G$ (determinable in principle from local observations) at a certain instant, the time behavior in the locality is identical for all locally isotropically expanding systems and is given by Eq. (12).[12] Further, in general, we have as the first integral of (14):

$$\dot{G}^2-\dot{G}_A{}^2=\frac{\Lambda}{3}(G^2-G_A{}^2)+\frac{8\pi C}{3}\left(\frac{1}{G}-\frac{1}{G_A}\right)+\frac{2}{3}\int_t^{t_A}\phi^2 G dG, \quad (15)$$

where we have used Eq. (13) and the subscript A indicates the values corresponding to an assigned state. Hence, finally, the time interval between two states A and B is given by

$$\tau_{AB}=\int_{Q_B}^{1}\frac{dQ}{\left[\frac{\Lambda}{3}(Q^2-1)+\frac{8\pi\rho_A}{3Q}(1-Q)+\left(\frac{\dot{G}}{G}\right)_A^2+\Theta\right]^{\frac{1}{2}}}, \quad (16)$$

where we have written Q for G/G_A and $\Theta\equiv\frac{2}{3}\int_Q{}^1\phi^2 QdQ$; hence, if $G_A>G_B$, then over the whole range of integration $Q\leqslant 1$ and therefore $\Theta\geqslant 0$, the equality sign occurring only if ϕ^2 vanishes (i.e., isotropic expansion or contraction). We may hence enunciate the theorem that in the absence of spin, the time interval between a state A of given ρ and $\dot{G}/G$ and another state B of specified volume ratio (i.e., G_B/G_A given) is a maximum in case of isotropic expansion (or contraction) if $G_A>G_B$ and there is no zero of $\dot{G}$ in the interval considered. In particular, in the cosmological problem in the absence of spin, among models which start from the singular state $G=0$, the time scale to the present state is a maximum for the isotropic models.

It may appear therefore that so far as the difficulty regarding the short time scale of relativistic isotropic models is concerned, nothing would be gained by simply changing over to nonisotropic models without introducing spin. However, the actual situation is slightly different. The cosmological constant Λ is not independently known. For the isotropic models, the necessity of a good fit with data of second order (e.g., the departure of the velocity-distance relation from linearity and the plausible bounds to the value of pressure) sets an upper bound to Λ which proves insufficient to give a long time scale. When, however, one gives up the assumption of isotropy, the second-order data do not set any precise bound on Λ as some other arbitrary parameters come into the picture. One can thus allow much higher values of Λ and obtain correspondingly longer time scales. Thus the longer time scales are due to an increased freedom in the choice of Λ rather than to anisotropy itself.[13] However, it should be noted that the introduction of such an arbitrary parameter robs the theory of much of its appeal and indeed if one sets $\Lambda=0$, then the theorem we have just proved shows that a simple change-over to anisotropy would only decrease the time scale.

Further, Eq. (14) shows that if $\Lambda=0$, then G cannot have any minimum, so that one has to start from a singularity at a finite time in the past as in isotropic models. Thus a simple change-over to anisotropy does not solve any of the difficulties.

IV. SOME IMPORTANT RELATIONS

In this section, we shall prove some interesting relations. By direct calculations we have, at any point on the t-axis, for the contracted Riemann-Christoffel tensor components,

$$R^i{}_k=R^{*i}{}_k+\tfrac{1}{2}g^{li}\partial^2/\partial t^2 g_{lk}+g^{li}\Gamma_{lp}{}^4\Gamma_{4k}{}^p-\tfrac{1}{2}g^{li}g^{pm}\dot{g}_{pm}\Gamma_{lk}{}^4+g^{li}\Gamma_{kp}{}^4\Gamma_{4l}{}^p-g^{li}g_{4p,k}\Gamma_{4l}{}^p, \quad (17)$$

where $R^{*i}{}_k$ are the corresponding tensor components for the three-space $d\sigma^2=g_{ik}dx^i dx^k$, and the Γ's are the Christoffel 3-index symbols. Contracting Eq. (17), we get, after some simplifications (the contraction here is from 1 to 3),

$$R^i{}_i=R^{*i}{}_i+\frac{\partial^2}{\partial t^2}[\log\sqrt{(-g)}]+\left[\frac{\partial}{\partial t}\log\sqrt{(-g)}\right]^2+g^{ik}g^{lm}g_{l4,i}\dot{g}_{km}-\tfrac{1}{2}g^{lm}g_{l4,m}g^{ik}\dot{g}_{ik}-g^{li}g_{4p,i}\Gamma_{4l}{}^p. \quad (18)$$

A very great simplification in the above relation can be attained by making a further specification of the coordinate system. So far, the three-space has been taken to be only orthogonal to the t-axis. If now we make a transformation,

$$\bar{x}^i=x^i, \quad \bar{t}=t+\phi(x^i),$$

where the function ϕ satisfies the following conditions,

$$(\phi_{,i})_0=0, \quad (\phi_{,ik})_0=\tfrac{1}{2}(g_{4k,i}+g_{4i,k}),$$

the suffix 0 indicating the value at $x^i=0$. Then in the new coordinate system $(\bar{g}_{4i,k})_0$ will be antisymmetric, i.e.,

$$(\bar{g}_{4i,k})_0=-(\bar{g}_{4k,i})_0.$$

Such a coordinate transformation, however, does not affect any of the results so far obtained and (18) now

[12] Equation 12 has been shown to be valid by R. C. Tolman [Proc. Natl. Acad. Sci. **20**, 169 (1934)] at the center of symmetry in a spherically symmetric system. J. L. Synge [Proc. Natl. Acad. Sci. **20**, 635 (1934)] has obtained the same equation on the assumption of "symmetry" about the world line at the point.

[13] See, in this connection, the papers cited in reference 3.

becomes (on dropping bars)

$$R^i{}_i = R^{*i}{}_i + \frac{\partial^2}{\partial t^2}[\log\surd(-g)] + \left[\frac{\partial}{\partial t}\log\surd(-g)\right]^2 - 2\omega^2; \quad (19)$$

so that using (1), (4), and (10), we get

$$8\pi\rho = \tfrac{1}{2}R - R^4{}_4 - \Lambda = \frac{R^{*i}{}_i - \phi^2}{2} + \frac{1}{3}\left[\frac{\partial}{\partial t}\log\surd(-g)\right]^2 - \Lambda. \quad (20)$$

Differentiating (20) with respect to t and using (13), and comparing with (11), we get the interesting relation

$$\frac{\partial}{\partial t}(R^{*i}{}_i - \phi^2) = (6\phi^2 - 2R^{*i}{}_i - 8\omega^2)\frac{\dot{G}}{G}. \quad (21)$$

If the spin vanishes and the expansion also be locally isotropic, then from (17), (4), and (1), we find that $R^{*i}{}_k$ is of the form $K\delta^i{}_k$. It is easy to see that a three-space whose contracted Riemann-Christoffel tensor is of this form is locally isotropic. Hence we have the theorem that if spin be absent and if the expansion be locally isotropic, then the space is locally isotropic. Thus local spatial isotropy follows from the restricted assumption of locally isotropic expansion (or contraction) in case of vanishing spin.