Special-Relativistic Derivation of Generally Covariant Gravitation Theory*

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The Newtonian gravitation theory is generalized to an inhomogeneous wave equation for a tensor gravitational potential in Euclidean space-time by invoking the special relativity postulates of Lorentz invariance and equivalence of mass and energy. Under the assumption of Lagrangian derivability, this is found to lead uniquely to the generally covariant field theories (including the general relativity theory) augmented by four auxiliary conditions. Appendices treat the general definition of the energy tensor, and an empirically disqualified special relativistic scalar generalization of the Newtonian theory.

INTRODUCTION

THE general theory of relativity has long occupied a position of isolation with respect to the rest of contemporary fundamental physics. The special theory of relativity has been intimately and indispensibly amalgamated with quantum mechanics in evolving current theoretical representations of elementary processes, but the general theory, despite the elegance of its concepts, has not exhibited any real relation whatever to quantum physics. The attempts which have been made to connect general relativity and quantum mechanics have been directed largely at showing to what degree the two disciplines may be *compatible* rather than seeking a basic interdependence. Thus formalisms have been developed in which the fundamental equations of quantum theory are written in generally covariant form, and, on the other hand, both approximate and exact methods for subjecting the Einstein metric field to quantization have been proposed.

There seems a distinct possibility that the wide gap between general relativity theory and quantum mechanics is in part due to a difference in language. In the relativity theory the paths of particles or light rays are defined by invariant geometric properties and the coordinate representation of the paths is secondary and, except in regions where the field is asymptotically Euclidean, ambiguous. The formalism of quantum mechanics, on the other hand, is based on a classical Hamiltonian theory in which fields, potentials, and trajectories have a perfectly explicit coordinate representation, apart from Lorentz transformation. As a prerequisite to making a detailed conceptual translation from one discipline to the other, it would seem necessary to decide what coordinate conditions, so to speak, are implicit in the quantum-mechanical equations of motion at small distances, and there is no clue as to how to do this in the present form of the relativity theory. The question is not resolved by writing the quantum-mechanical wave equations and commutation relations in generally covariant form, for this procedure has no more *a priori* justification than naively rewriting the field equations of pre-Einsteinian gravitation theory in generally covariant form. The latter is easy to do but it does not lead in itself to the correct results, since the essential characteristic of the Einstein theory is not the formal covariance of the equations to coordinate transformation but the relationship of the metric tensor and the gravitational potential.

The present paper is intended as a preliminary step in exploring one possible relationship of the generally covariant theories to the immediate classical basis of quantum theory, and to the quantum formalism itself. It tries to show that there exists a physically logical basis for the generally covariant theories without an *a priori* conceptual identification of the field tensor with the metric of space. The principle of equivalence and the associated geometrical postulates of general relativity are abandoned as a foundation, and, instead, a starting point is taken which employs concepts closer in some respects to those which have been found pertinent to microscopic physics. The Newtonian gravitation theory is generalized in accordance with the special relativity postulates of Lorentz invariance and the equivalence of mass and energy. This amounts to postulating an ordinary Euclidean second-order wave equation for a tensor potential $f_{\mu\lambda}$, in which the source term, however, is the total Lagrangian derived energy-momentum tensor as defined in special relativistic field theories. It is then found that the requirement that such a wave equation be consistent with the Lagrangian field equations leads to a set of four auxiliary conditions and a unique class of Lagrangians which are the Riemannian invariants built on a *single* tensor function $g_{\mu\lambda}$, which is in turn a function of $f_{\mu\lambda}$ and the Euclidean metric tensor $\eta_{\mu\lambda}$. With an appropriate choice of the Riemannian invariant, $g_{\mu\lambda}$ may be identified with the Einstein field tensor. The four auxiliary conditions are not unique, and depend on an arbitrary assignment of covariancy or contravariancy, and of tensor density weight, to $f_{\mu\lambda}$.

In a paper to follow, it is suggested that the generally covariant theories without auxiliary conditions, including the customary Einstein gravitation theory, may possibly allow such a wealth of solutions, in addition to familiar ones, as would permit a basic reinterpretation of the generally covariant theories as a means of kinematical representation comparable in important respects to the classical Hamilton-Jacobi formalism.

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THE GRAVITATIONAL LAGRANGIAN

In the Newtonian theory of attraction, the gravitational potential V obeys the equation

$$\nabla^2 V - \lambda\rho = 0, \tag{1}$$

where ρ is the density of matter and λ is proportional to the constant of gravitation. We now seek to modify (1) in accordance with the special relativity postulates of Lorentz invariance and equivalence of mass and energy. The obvious Lorentz-invariant generalizations of (1) are the scalar equation

$$\Box^2\phi - \lambda T_{\mu\lambda}\eta^{\mu\lambda} = 0, \tag{2}$$

and the tensor equation

$$\Box^2 f_{\mu\lambda} - \lambda T_{\mu\lambda} = 0, \tag{3}$$

where ϕ and $f_{\mu\lambda}$ are a scalar and tensor field respectively, $T_{\mu\lambda}$ is the energy-momentum tensor, $\eta_{\mu\lambda}$ is the (Euclidean) metric tensor and $\Box^2$ is the contracted second covariant derivative with respect to $\eta_{\mu\lambda}$. In accordance with the postulate of the equivalence of mass and energy, we now require that $T_{\mu\lambda}$ represent the *total* energy-momentum tensor, *including the contribution associated with the ϕ field or $f_{\mu\lambda}$-field.*[1]

The theory of the scalar equation is treated in Appendix II. This equation does not lead to astronomically correct relativistic corrections to the Newtonian gravitation law. We shall consider here the tensor equation (3). If the "matter variables" are represented formally by u (we do not specify here the number or nature of these variables except to require that they have well-defined covariance properties under coordinate transformation), an invariant Lagrangian leading to the field equations (3) will have the general form

$$L = \int \mathfrak{L}(f,u,\eta)d^4x, \tag{4}$$

where f symbolizes the tensor $f_{\mu\lambda}$ and η the metric tensor $\eta_{\mu\lambda}$. $\mathfrak{L}$ will transform under coordinate transformation as a tensor density of weight $+1$. The field equations for $f_{\mu\lambda}$ resulting from the usual Lagrangian variational procedure are then

$$\delta L/\delta f_{\mu\lambda} = 0, \tag{5}$$

and the contravariant energy-momentum tensor-density corresponding to this Lagrangian is

$$\mathfrak{T}^{\mu\lambda} = \delta L/\delta\eta_{\mu\lambda}. \tag{6}$$

When all the field equations for $f_{\mu\lambda}$ and u are satisfied, $\mathfrak{T}^{\mu\lambda}$ obeys the conservation equations[2]

$$\mathfrak{T}^{\mu\lambda}{}_{|\lambda} = 0, \tag{7}$$

where $|\lambda$ represents covariant differentiation with respect to $\eta_{\mu\lambda}$.

Multiplying (3) through by $\eta^{\sigma\mu}\eta^{\lambda\tau}|\eta|^{\frac{1}{2}}$ where $|\eta| = \det|\eta_{\mu\lambda}|$, the contravariant density form of the $f_{\mu\lambda}$ equation is

$$\Box^2\mathfrak{f}^{\sigma\tau} - \lambda\mathfrak{T}^{\sigma\tau} = 0, \tag{8}$$

where $\mathfrak{f}^{\sigma\tau}$ denotes $\eta^{\sigma\mu}\eta^{\lambda\tau}|\eta|^{\frac{1}{2}}f_{\mu\lambda}$ and we have commuted $\Box^2$ and $\eta^{\alpha\beta}$ since the covariant derivative of a tensor with respect to itself vanishes. In view of (7), any field $\mathfrak{f}^{\sigma\tau}$ which satisfies (8) and obeys customary boundary conditions[3] must also satisfy

$$\mathfrak{f}^{\sigma\tau}{}_{|\tau} = 0. \tag{9}$$

It follows that (8) is equivalent to (9) plus the set of equations

$$\mathfrak{D}^{\sigma\tau}(f) - \lambda\mathfrak{T}^{\sigma\tau} = 0, \tag{10}$$

where

$$\mathfrak{D}^{\sigma\tau}(f) = |\eta|^{\frac{1}{2}}D^{\sigma\tau}(f) = |\eta|^{\frac{1}{2}}(\Box^2 f^{\sigma\tau} - f^{\sigma\alpha}{}_{|\alpha\beta}\eta^{\beta\tau} - f^{\alpha\tau}{}_{|\alpha\beta}\eta^{\sigma\beta} + f^{\alpha\beta}{}_{|\alpha\beta}\eta^{\sigma\tau}) \tag{11}$$

obeys $[\mathfrak{D}^{\sigma\tau}(f)]_{|\tau} = 0$ identically.

Now instead of seeking solutions of the implicit equations (3) directly, we seek the class of Lagrangians leading to the less restrictive field equations (10). The solutions of these equations which obey the auxiliary conditions (9) will then satisfy (3). The requirement that (10) be the Lagrange-Euler equations resulting from variation of the Lagrangian L with respect to $f_{\mu\lambda}$ may be written according to (5) and (6) as

$$\delta L/\delta f_{\sigma\tau} = \mathfrak{D}^{\sigma\tau}(f) - \lambda(\delta L/\delta\eta_{\sigma\tau}). \tag{12}$$

Let $L = L_1 + L_2$ where

$$L_2 = -(2/\lambda)\int f_{\mu\lambda}\mathfrak{R}^{\mu\lambda}(\eta)d^4x, \tag{13}$$

and $\mathfrak{R}^{\mu\lambda}(\eta)$ is the contracted Riemann-Christoffel tensor density derived from $\eta_{\mu\lambda}$ as fundamental tensor. Since $\mathfrak{R}^{\mu\lambda}(\eta)$ vanishes for the Euclidean $\eta_{\mu\lambda}$, $\delta L_2/\delta f_{\mu\lambda} = 0$. It is easily verified that for infinitesimal variations of $\eta_{\sigma\tau}$ from the Euclidean values,

$$\lambda(\delta L_2/\delta\eta_{\sigma\tau}) = \mathfrak{D}^{\sigma\tau}(f), \tag{14}$$

so that (12) is equivalent to the following equation for L_1:

$$\delta L_1/\delta f_{\sigma\tau} = -\lambda(\delta L_1/\delta\eta_{\sigma\tau}). \tag{15}$$

It is clear that the general solution of (15) is

$$L_1 = L_1(\eta_{\sigma\tau} - \lambda f_{\sigma\tau}, u). \tag{16}$$

That is, L_1 must be a function of the matter variables and the single tensor argument $\eta_{\sigma\tau} - \lambda f_{\sigma\tau}$. Thus, denoting $\eta_{\sigma\tau} - \lambda f_{\sigma\tau}$ by $g_{\sigma\tau}$, the general solution for L in

[1] Some elementary consequences of this requirement on the Euclidean gravitational field equation, and its approximate realization, were treated by the author in a thesis submitted in partial fulfillment of the requirements for the S.B. degree at the Massachusetts Institute of Technology, January 15, 1947, and similarly, but independently, by S. N. Gupta, Phys. Rev. **96**, 1683 (1954).

[2] The definition and some properties of the energy tensor are discussed in Appendix I.

[3] We imply the exclusion of the half-advanced, half-retarded solutions of the homogeneous equation $\Box^2\mathfrak{f}^{\sigma\tau}{}_{|\tau} = 0$.

empty space is

$$L=\int \mathfrak{R}(g)d^4x-(2/\lambda)\int f_{\mu\lambda}\mathfrak{R}^{\mu\lambda}(\eta)d^4x, \qquad (17)$$

where $\mathfrak{R}(g)$ is a Riemannian invariant-density constructed from $g_{\sigma\tau}$ alone; that is, a "general invariant" in the language of relativity theory. Since the terms in $2/\lambda$ vanish due to the assumption of Euclidean $\eta_{\mu\lambda}$, the field equations for $g_{\sigma\tau}$ are simply those resulting from variation of $\int \mathfrak{R}(g)d^4x$ alone. The simplest nontrivial form $\mathfrak{R}$ is $\mathfrak{R}=\mathfrak{R}^{\mu\lambda}(g)g_{\mu\lambda}$, where $\mathfrak{R}^{\mu\lambda}(g)$ is the contracted contravariant Riemann-Christoffel tensor-density constructed from the fundamental tensor $g_{\mu\lambda}$. This is a form of the Lagrangian for the Einstein gravitational equations so that $g_{\mu\lambda}$ may then be identified with Einstein's metric tensor in empty space. In nonempty regions, $\mathfrak{R}(g)$ will be replaced by some invariant function of both $g_{\mu\lambda}$ and u. Its form will of course depend of the covariance properties of the u variables and on the type of coupling assumed.

NONUNIQUENESS OF THE AUXILIARY CONDITIONS

In the preceding analysis, we have assumed that the tensor $f_{\mu\lambda}$, which is the generalization of the Newtonian potential, is a covariant tensor. There is nothing in the original equation (1) to force this choice, and, we might as well have started with a contravariant tensor, or either a covariant or contravariant tensor density of arbitrary weight. We shall now show that these alternative choices are closely related to the theory already developed, and may be realized by a change in the auxiliary conditions. Since the notation of the tensor calculus requires a definite choice of covariant or contravariant character in an expression, we shall carry through the analysis for a contravariant tensor density of arbitrary weight, and then indicate the minor changes appropriate to the covariant choice.

If we replace L by a new Lagrangian

$$\bar{L}=L_1-(2/\lambda)\int \psi^{\mu\lambda}\mathfrak{R}_{\mu\lambda}(\eta)|\eta|^{-N/2}d^4x, \qquad (18)$$

where $\psi^{\mu\lambda}$ is any tensor density function of $f_{\mu\lambda}$ and $\eta_{\mu\lambda}$ of weight N, connected to $f_{\mu\lambda}$ by an invertible transformation, the field equations for $f_{\mu\lambda}$ are unaltered because of the vanishing of $\mathfrak{R}_{\mu\lambda}(\eta)$. The new energy tensor is

$$\begin{aligned}\delta\bar{L}/\delta\eta_{\sigma\tau}&=\lambda^{-1}|\eta|^{(1-N)/2}D^{\sigma\tau}(\psi)-2\lambda^{-1}\\ &\quad\times\int(\partial\psi^{\mu\lambda}/\partial\eta_{\sigma\tau})\mathfrak{R}_{\mu\lambda}(\eta)|\eta|^{-N/2}d^4x+\delta L_1/\delta\eta_{\sigma\tau}\\ &=\delta L/\delta\eta_{\sigma\tau}-\lambda^{-1}\mathfrak{D}^{\sigma\tau}(f)\\ &\qquad+\lambda^{-1}|\eta|^{(1-N)/2}D^{\sigma\tau}(\psi), \qquad (19)\end{aligned}$$

since $\mathfrak{R}_{\mu\lambda}(\eta)=0$. Hence, the field equations (10) may be rewritten

$$|\eta|^{(1-N)/2}D^{\sigma\tau}(\psi)-\lambda(\delta\bar{L}/\delta\eta_{\sigma\tau})=0. \qquad (20)$$

Now let us rewrite the new Lagrangian in terms of $\eta_{\mu\lambda}$ and the new variables $\psi^{\mu\lambda}$. Denoting $\bar{L}$ expressed in this form by $\bar{L}^*$,

$$\frac{\delta\bar{L}^*}{\delta\eta_{\mu\lambda}}=\frac{\delta\bar{L}}{\delta\eta_{\mu\lambda}}+\frac{\delta\bar{L}}{\delta f_{\alpha\beta}}\left(\frac{\partial f_{\alpha\beta}}{\partial\eta_{\mu\lambda}}\right)_{\psi^{\sigma\tau}=\text{const}}=\frac{\delta\bar{L}}{\delta\eta_{\mu\lambda}}, \qquad (21)$$

when the field equations $\delta\bar{L}/\delta f_{\alpha\beta}=0$ are satisfied. Hence, the field equations for $\psi^{\mu\lambda}$ (20) may be written

$$|\eta|^{(1-N)/2}D^{\sigma\tau}(\psi)-\lambda(\delta\bar{L}^*/\delta\eta_{\sigma\tau})=0. \qquad (22)$$

If now we impose the auxiliary conditions

$$\psi^{\mu\lambda}{}_{|\lambda}=0, \qquad (23)$$

(22) becomes

$$\Box^2\psi^{\mu\lambda}-\lambda|\eta|^{(N-1)/2}(\delta\bar{L}^*/\delta\eta_{\sigma\tau})=0, \qquad (24)$$

which is as valid a generalization of (1) as (8). The actual Lagrange-Euler equations resulting from variation of $\bar{L}^*$ with respect to $\psi^{\alpha\beta}$ may be written

$$\frac{\partial f_{\mu\lambda}}{\partial\psi^{\alpha\beta}}\left[D^{\mu\lambda}(\psi)-\lambda|\eta|^{(N-1)/2}\frac{\delta\bar{L}^*}{\delta\eta_{\mu\lambda}}\right]=0, \qquad (25)$$

which although equivalent to (22) are not formally identical as was assumed in the analysis which led to (16). If the analysis is carried through for a covariant tensor density $\psi_{\mu\lambda}$, of weight N, the equations analogous to (23) and (24) are

$$\psi_{\mu\lambda|\sigma}\eta^{\lambda\sigma}=0, \qquad (26)$$

$$\Box^2\psi_{\mu\lambda}-\lambda|\eta|^{(N-1)/2}\eta_{\sigma\mu}\eta_{\tau\lambda}(\delta\bar{L}^*/\delta\eta_{\sigma\tau})=0. \qquad (27)$$

It thus appears that there is associated with any given general invariant a family of Lagrangians leading to equations of the type (10) and differing only by terms which do not affect the field equations but which have the effect of altering the energy tensor by expressions with identically vanishing divergencies. Although the field equations for this family of Lagrangians are all identical in the sense that they can be obtained from each other by variable transformations, the auxiliary conditions (23) or (26) required to satisfy the wave equations of the form (24) or (27) for each choice of variables $\psi(\sigma\tau)$ will not be equivalent in general.

We have thus been led to the result that generalization of the Newtonian potential equation, in accordance with the postulates of special relativity, to the tensor field equation of the form (3) leads uniquely to the class of generally covariant field theories, augmented by auxiliary conditions. There seems to be no way of distinguishing between the various choices of auxiliary conditions on the basis of the assumptions we have made thus far in generalizing the Newtonian law (1). In the usual interpretation of Einstein's gravitation theory as a law regulating the metric structure of space, auxiliary conditions of the type introduced are considered merely to fix the form of the (arbitrary) coordinate system and are not regarded as having any physical content. In a paper to follow, however, the question of the physical significance of the auxiliary conditions will be reexamined.

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APPENDIX I. DEFINITION OF THE ENERGY TENSOR

The method of constructing the Lagrangian derived energy tensor used in this paper is due to Eddington[4] and been discussed by Chang[5] and others. The general type of Lagrangian considered is of the form

$$L=\int \mathfrak{L}(\eta_{\mu\lambda},w^{(\sigma)})d^4x, \tag{I.1}$$

where $\mathfrak{L}$ is any function of the Euclidean metric tensor $\eta_{\mu\lambda}$, the dynamic variables $w^{(\sigma)}$ and their derivatives of any order, which transforms as an invariant density of weight $+1$ under an arbitrary change of coordinate system. L will then be an invariant. The dynamic variables $w^{(\sigma)}$ are not specified in number or kind except that they have well defined covariance properties under coordinate transformation. Since L is an invariant, the variation

$$\delta L=\int [(\delta L/\delta\eta_{\mu\lambda})\delta\eta_{\mu\lambda}+\textstyle\sum_\sigma(\delta L/\delta w^{(\sigma)})\delta w^{(\sigma)}]d^4x \tag{I.2}$$

will vanish when $\delta\eta_{\mu\lambda}$ and $\delta w^{(\sigma)}$ are variations due to an arbitrary infinitesimal coordinate transformation. The Lagrange equations for the dynamic variables $w^{(\sigma)}$ are

$$\delta L/\delta w^{(\sigma)}=0, \tag{I.3}$$

so that when the field equations are satisfied, (I.2) is simply

$$\int (\delta L/\delta\eta_{\mu\lambda})\delta\eta_{\mu\lambda}d^4x=0. \tag{I.4}$$

Under the infinitesimal coordinate transformation $x'^\mu=x^\mu+\delta x^\mu$, the change in the covariant tensor $\eta_{\mu\lambda}$ is

$$-\delta\eta_{\mu\lambda}=\eta_{\mu\alpha}\delta x^\alpha{}_{,\lambda}+\eta_{\alpha\lambda}\delta x^\alpha{}_{,\mu}-\eta_{\mu\lambda,\alpha}\delta x^\alpha, \tag{I.5}$$

where the $\eta_{\mu\lambda}$ are expressed in terms of the new coordinates. Assuming the δx^μ vanish outside a finite region, (I.4) may be transformed by partial integration to the form

$$\int (\delta L/\delta\eta_{\mu\lambda})_{|\lambda}\eta_{\mu\alpha}\delta x^\alpha=0, \tag{I.6}$$

where $|\lambda$ indicates covariant differentiation with respect to $\eta_{\mu\lambda}$. Since this must hold for arbitrary δx^α,

$$(\delta L/\delta\eta_{\mu\lambda})_{|\lambda}=0, \tag{I.7}$$

and in view of this conservation property the energy tensor density may be identified as

$$\mathfrak{T}^{\mu\lambda}=\delta L/\delta\eta_{\mu\lambda}. \tag{I.8}$$

This definition of the energy tensor is unique to the extent that the form of the Lagrangian leading to given field equations (I.3) is unique. However the Lagrangian may be altered by the formal addition of terms of the form

$$\int A\mathfrak{R}(\eta)d^4x,\quad \int A_{\mu\lambda}\mathfrak{R}^{\mu\lambda}(\eta)d^4x,\quad \text{etc.}, \tag{I.9}$$

(where A, $A_{\mu\lambda}$, $\cdots$ are scalar or tensor functions of the dynamic variables and $\eta_{\mu\lambda}$ and $\mathfrak{R}(\eta)$, $\mathfrak{R}^{\mu\lambda}(\eta)$, $\cdots$ are contractions of the Riemann-Christoffel tensor density built on $\eta_{\mu\lambda}$) without affecting these equations since the Riemann-Christoffel tensor vanishes. Denoting an expression of the form (I.9) by K, since $\delta K/\delta w^{(\sigma)}=0$ for all $w^{(\sigma)}$ values, (I.7) becomes the *identity*

$$(\delta K/\delta\eta_{\mu\lambda})_{|\lambda}\equiv 0. \tag{I.10}$$

Hence the effect of such formal additions to the Lagrangian is to change the energy tensor by an expression whose divergence vanishes identically. This ambiguity is unavoidable since the requirement on an energy tensor is simply that it be conserved when the field equations are satisfied.

A special illustration of the effect of the freedom of definition of the energy tensor is provided by the Lagrangian of the general relativity theory. If the Einstein field tensor $g_{\mu\lambda}$ is regarded as the dynamic variable, the field equations may be obtained from the Lagrangian

$$L=\int \mathfrak{R}(g)d^4x, \tag{I.11}$$

where $\mathfrak{R}(g)$ is the invariant density formed by contraction of the Riemann-Christoffel tensor formed on $g_{\mu\lambda}$. Since this Lagrangian does not contain the metric tensor $\eta_{\mu\lambda}$ at all, the energy tensor as defined above vanishes. However, let us replace each derivative $g_{\mu\lambda,\sigma}$, $g_{\mu\lambda,\sigma\tau}$ occurring in (I.11) by $g_{\mu\lambda|\sigma}$ or $g_{\mu\lambda|\sigma\tau}$ where $|$ indicates covariant differentiation with respect to $\eta_{\mu\lambda}$. This will not destroy the invariance of the Lagrangian to coordinate transformations, and since the form of the Lagrangian will be unaltered in Cartesian coordinates, the field equations will not be affected in any coordinate system. The alteration of the Lagrangian

[4] A. S. Eddington, *Mathematical Theory of Relativity* (The University Press, Cambridge, 1937), second edition, pp. 140–141.
[5] T. S. Chang, Proc. Cambridge Phil. Soc. 44, 76 (1948).

can be represented as the addition of terms of form similar to (I.9). The resultant energy tensor identically obeys the conservation conditions (I.7). A tensor which obeys the conservation conditions only when the field equations are satisfied can be formally derived by changing to new dynamic variables $f_{\mu\lambda}=(g_{\mu\lambda}-\eta_{\mu\lambda})$. The two energy tensors differ only by the expression $\mathfrak{R}^{\mu\lambda}(g)-\frac{1}{2}g^{\mu\lambda}\mathfrak{R}(g)$ and hence are identical when the field equations are satisfied. In that case, they are closely related, in Cartesian coordinates, to the usual "pseudo energy-tensor" as defined, for example, by Eddington.[6]

APPENDIX II. SCALAR THEORY

The scalar gravitation theory corresponding to the field equation

$$\Box^2\phi-\lambda T_{\mu\lambda}\eta^{\mu\lambda}=0 \quad (2)$$

may be developed in close analogy to the treatment of the tensor theory. If the energy tensor $T_{\mu\lambda}$ is derived from a Lagrangian $L(\phi,\eta_{\mu\lambda},u)$, where u represents the matter variables, then (2) is equivalent to the Lagrange-Euler equation if

$$\delta L/\delta\phi=|\eta|^{\frac{1}{2}}\Box^2\phi-\lambda(\delta L/\delta\eta_{\mu\lambda})\eta_{\mu\lambda}. \quad \text{(II.1)}$$

Let $L=L_1+L_2$, where

$$L_2=-(3\lambda)^{-1}\int\phi\mathfrak{R}(\eta)d^4x, \quad \text{(II.2)}$$

where $\mathfrak{R}(\eta)$ is the Riemannian curvature invariant. For Euclidean $\eta_{\mu\lambda}$, $\delta L_2/\delta\phi$ then vanishes and

$$\lambda(\delta L_2/\delta\eta_{\mu\lambda})\eta_{\mu\lambda}=|\eta|^{\frac{1}{2}}\Box^2\phi. \quad \text{(II.3)}$$

Hence the resulting equation for L_1 is

$$\delta L_1/\delta\phi+\lambda\eta_{\mu\lambda}(\delta L_1/\delta\eta_{\mu\lambda})=0. \quad \text{(II.4)}$$

Now make the transformation $\gamma_{\mu\lambda}=e^{-\lambda\phi}\eta_{\mu\lambda}$, so that

$$L_1(\phi,\eta_{\mu\lambda},u)=L_1^*(\phi,\gamma_{\mu\lambda},u). \quad \text{(II.5)}$$

Then, (II.4) may be written

$$0=\frac{\delta L_1^*}{\delta\phi}+\frac{\delta L_1^*}{\delta\gamma_{\mu\lambda}}\frac{\partial\gamma_{\mu\lambda}}{\partial\phi}+\lambda\frac{\delta L_1^*}{\delta\gamma_{\alpha\beta}}\frac{\partial\gamma_{\alpha\beta}}{\partial\eta_{\mu\lambda}}\eta_{\mu\lambda}=\frac{\delta L_1^*}{\delta\phi}, \quad \text{(II.6)}$$

which shows that L_1^* must be a function of u and the tensor $\gamma_{\mu\lambda}$ alone. Hence, Eq. (2) leads to the general Lagrangian,

$$L=\int\mathfrak{K}(e^{-\lambda\phi}\eta_{\mu\lambda})d^4x+\int\mathfrak{J}(e^{-\lambda\phi}\eta_{\mu\lambda},u)d^4x-(3\lambda)^{-1}\int\phi\mathfrak{R}(\eta)d^4x, \quad \text{(II.7)}$$

where $\mathfrak{K}$ and $\mathfrak{J}$ are any general invariant densities built on the fundamental tensor $e^{-\lambda\phi}\eta_{\mu\lambda}$. Since the last term vanishes for Euclidean $\eta_{\mu\lambda}$, the equations of motion will be unaltered by dropping it and hence, making the change of variable $\psi=e^{-\lambda\phi}$, the field equations may be obtained by variation of the Lagrangian

$$L=\int\mathfrak{K}(\psi\eta_{\mu\lambda})d^4x+\int\mathfrak{J}(\psi\eta_{\mu\lambda},u)d^4x. \quad \text{(II.8)}$$

It will be noted that if $g_{\mu\lambda}=\psi\eta_{\mu\lambda}$, this is the general Lagrangian for the tensor theory, and it follows then that if there do exist solutions of the tensor theory of this form, then ψ must obey the field equation of the scalar theory.

If $\mathfrak{K}(\psi\eta_{\mu\lambda})=\mathfrak{R}(\psi\eta_{\mu\lambda})$, it may be verified that for Euclidean $\eta_{\mu\lambda}$, (II.8) reduces to

$$L=\int\psi^{-1}\psi_{,\alpha}\psi_{,\beta}\eta^{\alpha\beta}|\eta|^{\frac{1}{2}}d^4x+\int\mathfrak{J}(\psi\eta_{\mu\lambda},u)d^4x. \quad \text{(II.9)}$$

This form bears an interesting resemblance to the Lagrangian of the general relativity theory, which Eddington[7] has pointed out can be written as an expression homogeneous of order 2 in the quantities $\mathfrak{g}^{\mu\lambda}{}_{,\alpha}$ and homogeneous of order -1 in the $\mathfrak{g}^{\mu\lambda}$. However, in distinction to the general relativity case, a simple variable transformation renders this scalar theory linear.

It is readily seen that the class of scalar theories just developed can give no bending of light by a gravitational field. In the absence of gravitational field, the Lagrangian of the electromagnetic field may be written

$$L_{\text{e.m.}}=\int(A_{\mu,\lambda}-A_{\lambda,\mu})(A_{\sigma,\tau}-A_{\tau,\sigma})\eta^{\mu\sigma}\eta^{\lambda\tau}|\eta|^{\frac{1}{2}}d^4x. \quad \text{(II.10)}$$

This expression is homogeneous of degree 0 in the $\eta_{\mu\lambda}$, and hence it is apparent that it will be unaltered if $\eta_{\mu\lambda}$ is replaced by $\psi\eta_{\mu\lambda}$.

The Lagrangian (II.9) leads in first approximation to the Newtonian law of gravitation, but it predicts a motion of the perihelion of Mercury different from the observed advance. The motion can be obtained exactly.

[6] A. S. Eddington, reference 4, pp. 134–137.

[7] A. S. Eddington, reference 4, pp. 131–134.