

other mercury atoms produce the metastable mercury 6^3P_0 atom.

An interpretation of the experimental results made with the application of Winan's partial selection rule $\Delta J=0$ gives good explanation of the results. This work seems to give support to this selection rule.

ACKNOWLEDGMENTS

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Helium Wave Equation

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The wave function for the ground state of helium has been obtained by a method which involves iterating in one direction only, the values of the function in any one plane being made mutually consistent by solving 30 simultaneous equations in 30 unknowns. The local energy value $(H\psi)/\psi$ is approximately constant over most of space, the maximum deviations occurring at large distances between electron and nucleus. The present accuracy is not sufficient to give a good determination of the eigenvalue, but it appears to be at least within 0.5 percent of the experimental value.

DETERMINATION of the form of the wave function for the ground state of the helium atom has been a baffling problem.¹ However, rather powerful methods have now been developed, as a result of which a fairly good wave function has been obtained. It seems, therefore, appropriate to report briefly on the above methods.

The wave equation is

$$\nabla^2\psi + \frac{1}{z} \frac{\partial\psi}{\partial z} + \frac{1}{4r}(E - V)\psi = 0, \quad (1)$$

where, if r_1 and r_2 are the electron-nucleus distances and θ is the angle between the corresponding radius vectors, then $4x = 2r_1r_2 \cos\theta$, $4y = r_1^2 - r_2^2$, $4z = 2r_1r_2 \sin\theta$ and $4r = r_1^2 + r_2^2$. The potential energy V is such that

$$-\frac{1}{4}V = [2(r+y)]^{-\frac{1}{2}} + [2(r-y)]^{-\frac{1}{2}} - \frac{1}{4}(r-x)^{-\frac{1}{2}}. \quad (2)$$

Since the wave function for zero electron interaction is exponential in form, i.e., $\psi = e^{-\frac{1}{2}(r_1+r_2)}$, we make the substitution $\psi = e^F$ and consider the function F . This satisfies the nonlinear equation

$$\nabla^2F + \frac{1}{z} \frac{\partial F}{\partial z} + |\nabla F|^2 + \frac{1}{4r}(E - V) = 0. \quad (3)$$

The boundary conditions are $F(y) = F(-y)$, $F(z) = F(-z)$, $F \rightarrow -\infty$ as $r \rightarrow \infty$.

Now the function

$$F^{(0)} = -\frac{1}{2}\sqrt{2}[(r+y)^{\frac{1}{2}} + (r-y)^{\frac{1}{2}}] + \frac{1}{4}(r-x)^{\frac{1}{2}}$$

satisfies the equation

$$\left(\nabla^2 + \frac{1}{z} \frac{\partial}{\partial z} \right) F^{(0)} - \frac{V}{4r} = 0. \quad (4)$$

Subtracting (4) from (3), and letting $G = F - F^{(0)}$, the equation for G is

$$\nabla^2G + \frac{1}{z} \frac{\partial G}{\partial z} + |\nabla G|^2 + 2(\nabla F^{(0)} \cdot \nabla G) + |\nabla F^{(0)}|^2 + \frac{E}{4r} = 0. \quad (5)$$

Previous work¹ had shown that the Laplacian must be approximated by an adequate difference operator. To this end, we have assumed that G at any point may be represented by a fourth-degree polynomial in x , y , and z . If G be given at five successive mesh points in, say, the x -direction, then the coefficients of the polynomial may be determined. Also, the derivatives of G may be found by differentiating the polynomial.

For the early work, it was necessary to choose a mesh with variable intervals, and this mesh has been retained. At first the boundary was taken to be defined by $|x| = 31$, $|y| = 31$, $|z| = 31$, and G was assumed to be zero there. Later work with the boundary at 63 gives much the same result and this will be reported here. The present mesh points are at $x = \pm 63, \pm 31, \pm 15, \pm 7, \pm 3, \pm 1$, and 0; $y = 0, 3, 7, 15, 31$, and 63; and $z = 0, 1, 3, 7, 15, 31$, and 63.

When Eq. (5) is approximated by a difference equation, it relates the value of G at a given point to values at surrounding points. An attempt was made to use

¹ J. H. Bartlett, Phys. Rev. **88**, 525 (1952).

TABLE I. Values of the wave function ψ .

z	y	0	3	7	15	31
$x = -31$	0	0.00156	0.00155	0.00151	0.00137	0.00098
	1	0.00156	0.00155	0.00151	0.00137	0.00098
	3	0.00154	0.00153	0.00149	0.00135	0.00096
	7	0.00143	0.00143	0.00139	0.00127	0.00091
	15	0.00109	0.00108	0.00106	0.00098	0.00073
	31	0.00045	0.00045	0.00045	0.00042	0.00034
$x = -15$	0	0.01116	0.01097	0.01021	0.00787	0.00397
	1	0.01110	0.01091	0.01016	0.00783	0.00396
	3	0.01066	0.01048	0.00977	0.00758	0.00386
	7	0.00882	0.00869	0.00817	0.00648	0.00343
	15	0.00462	0.00457	0.00438	0.00371	0.00222
	31	0.00103	0.00102	0.00100	0.00092	0.00069
$x = -7$	0	0.04636	0.04395	0.03627	0.02183	0.00828
	1	0.04562	0.04328	0.03580	0.02162	0.00823
	3	0.04041	0.03855	0.03241	0.02008	0.00782
	7	0.02536	0.02458	0.02172	0.01475	0.00630
	15	0.00839	0.00828	0.00781	0.00625	0.00335
	31	0.00130	0.00130	0.00127	0.00116	0.00084
$x = -3$	0	0.13231	0.11236	0.07581	0.03687	0.01189
	1	0.12508	0.10736	0.07340	0.03607	0.01171
	3	0.08919	0.08039	0.05929	0.03116	0.01060
	7	0.03860	0.03693	0.03131	0.01966	0.00772
	15	0.00982	0.00967	0.00908	0.00716	0.00372
	31	0.00137	0.00136	0.00133	0.00122	0.00087
$x = -1$	0	0.30399	0.19543	0.11035	0.04765	0.01420
	1	0.24117	0.17236	0.10173	0.04512	0.01367
	3	0.11954	0.10362	0.07187	0.03570	0.01165
	7	0.04224	0.04031	0.03388	0.02093	0.00808
	15	0.01004	0.00989	0.00928	0.00731	0.00378
	31	0.00137	0.00136	0.00134	0.00122	0.00087
$x = 0$	0	0.91710	0.25696	0.13230	0.05402	0.01550
	1	0.28638	0.19085	0.10913	0.04742	0.01416
	3	0.12243	0.10602	0.07333	0.03629	0.01179
	7	0.04216	0.04027	0.03392	0.02101	0.00811
	15	0.00999	0.00984	0.00924	0.00730	0.00378
	31	0.00137	0.00136	0.00133	0.00122	0.00087
$x = 1$	0	0.23527	0.18496	0.10773	0.04718	0.01414
	1	0.21115	0.16341	0.09930	0.04466	0.01362
	3	0.11157	0.09868	0.07001	0.03530	0.01160
	7	0.04066	0.03895	0.03305	0.02067	0.00804
	15	0.00982	0.00968	0.00911	0.00722	0.00376
	31	0.00135	0.00135	0.00132	0.00121	0.00087
$x = 3$	0	0.09089	0.09512	0.07000	0.03564	0.01173
	1	0.09390	0.09130	0.06778	0.03487	0.01156
	3	0.07376	0.06967	0.05470	0.03007	0.01045
	7	0.03453	0.03338	0.02907	0.01892	0.00760
	15	0.00921	0.00909	0.00859	0.00689	0.00364
	31	0.00131	0.00131	0.00128	0.00118	0.00085
$x = 7$	0	0.02793	0.03147	0.02972	0.01992	0.00798
	1	0.02923	0.03117	0.02935	0.01972	0.00793
	3	0.02838	0.02866	0.02672	0.01830	0.00753
	7	0.01993	0.01966	0.01829	0.01343	0.00604
	15	0.00725	0.00718	0.00687	0.00571	0.00319
	31	0.00119	0.00119	0.00117	0.00108	0.00079
$x = 15$	0	0.00578	0.00647	0.00691	0.00630	0.00273
	1	0.00601	0.00647	0.00688	0.00627	0.00357
	3	0.00622	0.00642	0.00669	0.00607	0.00348
	7	0.00575	0.00578	0.00580	0.00522	0.00308
	15	0.00344	0.00343	0.00337	0.00305	0.00198
	31	0.00085	0.00085	0.00084	0.00079	0.00060
$x = 31$	0	0.00066	0.00072	0.00079	0.00085	0.00072
	1	0.00068	0.00073	0.00079	0.00085	0.00072
	3	0.00071	0.00074	0.00079	0.00084	0.00071
	7	0.00073	0.00074	0.00077	0.00080	0.00067
	15	0.00064	0.00064	0.00064	0.00064	0.00053
	31	0.00030	0.00030	0.00030	0.00029	0.00024

the Liebmann iteration procedure, advancing one point at a time, and neglecting $|\nabla G|^2$. This was unsuccessful, for the new value of the function at any point was expressible as the quotient of two quantities, where the numerator depended on the neighboring values of G and the denominator depended on the mesh characteristics at the point in question. Since the denominator became small for certain points, and since the correct neighboring values could not be known beforehand, such an iterative procedure results in rapidly increasing values of G at these points. The next method there was to solve a set of six simultaneous equations in six unknowns, the values of G at the mesh points $z=0, \frac{1}{3}, 1, 3, 7, 15$ (x and y constant). This diverged after a time, but was an improvement. Consequently, there was next tried the idea of treating a whole plane ($x=\text{constant}$) at a time, and this, which corresponds to block relaxation, was successful. For any one plane, one uses the values of G on four neighboring planes, so that a large part of space is being correlated at each such step.

As a test to see how well any trial function ϕ satisfies the differential equation, we have calculated the quantity $(H\phi)/\phi = E_{100}$ as a function of position. For a true eigenfunction ψ , we would have $(H\psi)/\psi = E = \text{constant}$.

We have used $E = -1.4516$ in the iterations, and have started with $G=0$ everywhere. After the sixteenth iteration of G , there were still large differences between E and E_{100} for large z . At these points $|\nabla G|^2$ is not negligible compared with the other terms. In order to include this term in the calculations, another function was defined as

$$J = F - F^0 - G^0,$$

where G^0 was taken to denote the G values from the sixteenth iteration. The equation satisfied by J is

$$\nabla^2 J + \frac{1}{z} \frac{\partial J}{\partial z} + |\nabla J|^2 + 2(\nabla F^0 + \nabla G^0) \cdot \nabla J + \frac{E - E_{100}(G^0)}{4r} = 0. \quad (6)$$

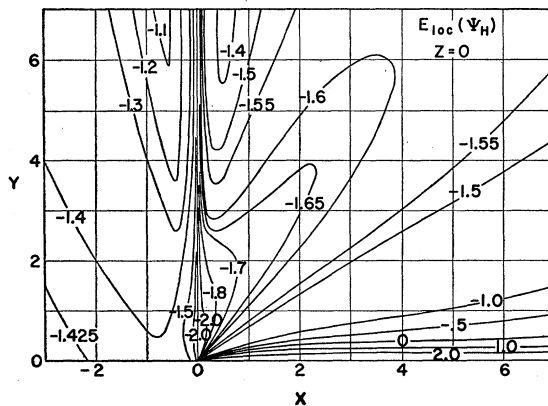


FIG. 1. Contour map of E_{100} , Hylleraas six-term function ψ_H , for $z=0$.

TABLE II. Values of $|E_{100}|$.

z	y	0	3	7	15	31
$x = -31$	1	1.4498	1.4503	1.4502	1.4509	1.4528
	3	1.4501	1.4503	1.4501	1.4512	1.4533
	7	1.4500	1.4504	1.4503	1.4515	1.4533
	15	1.4505	1.4505	1.4506	1.4512	1.4530
	31	1.4506	1.4505	1.4502	1.4508	1.4515
$x = -15$	1	1.4490	1.4491	1.4496	1.4514	1.4559
	3	1.4491	1.4492	1.4495	1.4515	1.4552
	7	1.4492	1.4494	1.4498	1.4515	1.4553
	15	1.4500	1.4501	1.4501	1.4513	1.4542
	31	1.4492	1.4498	1.4497	1.4499	1.4520
$x = -7$	1	1.4474	1.4474	1.4482	1.4520	1.4617
	3	1.4476	1.4476	1.4484	1.4520	1.4608
	7	1.4483	1.4484	1.4490	1.4521	1.4597
	15	1.4492	1.4492	1.4498	1.4517	1.4566
	31	1.4472	1.4473	1.4475	1.4488	1.4518
$x = -3$	1	1.4456	1.4450	1.4449	1.4535	1.4784
	3	1.4461	1.4458	1.4461	1.4531	1.4753
	7	1.4466	1.4466	1.4473	1.4530	1.4707
	15	1.4466	1.4467	1.4477	1.4528	1.4667
	31	1.4360	1.4363	1.4373	1.4410	1.4524
$x = -1$	1	1.4447	1.4398	1.4359	1.4566	1.5395
	3	1.4429	1.4407	1.4384	1.4559	1.5299
	7	1.4375	1.4373	1.4383	1.4563	1.5245
	15	1.4299	1.4307	1.4347	1.4546	1.5140
	31	1.3730	1.3743	1.3787	1.3968	1.4480
$x = 0$	1	1.4472	1.4379	1.4287	1.4577	1.6065
	3	1.4383	1.4336	1.4276	1.4577	1.6060
	7	1.4233	1.4225	1.4237	1.4590	1.6030
	15	1.4042	1.4059	1.4144	1.4565	1.5850
	31	1.2822	1.2847	1.2946	1.3351	1.4473
$x = 1$	1	1.4504	1.4483	1.4457	1.4520	1.4927
	3	1.4481	1.4469	1.4451	1.4518	1.4928
	7	1.4439	1.4436	1.4438	1.4521	1.4929
	15	1.4382	1.4388	1.4408	1.4521	1.4889
	31	1.4081	1.4088	1.4112	1.4226	1.4543
$x = 3$	1	1.4512	1.4510	1.4505	1.4514	1.4595
	3	1.4510	1.4508	1.4504	1.4514	1.4597
	7	1.4502	1.4501	1.4500	1.4514	1.4597
	15	1.4491	1.4491	1.4494	1.4516	1.4594
	31	1.4446	1.4447	1.4453	1.4475	1.4538
$x = 7$	1	1.4517	1.4516	1.4516	1.4516	1.4531
	3	1.4516	1.4516	1.4515	1.4517	1.4533
	7	1.4515	1.4515	1.4516	1.4514	1.4530
	15	1.4514	1.4513	1.4517	1.4517	1.4531
	31	1.4513	1.4516	1.4518	1.4518	1.4533
$x = 15$	1	1.4518	1.4518	1.4517	1.4517	1.4521
	3	1.4519	1.4519	1.4520	1.4517	1.4522
	7	1.4519	1.4518	1.4517	1.4519	1.4520
	15	1.4517	1.4519	1.4518	1.4518	1.4519
	31	1.4520	1.4518	1.4517	1.4522	1.4527
$x = 31$	1	1.4516	1.4518	1.4515	1.4519	1.4518
	3	1.4516	1.4520	1.4518	1.4516	1.4518
	7	1.4520	1.4517	1.4520	1.4521	1.4518
	15	1.4518	1.4520	1.4519	1.4521	1.4519
	31	1.4521	1.4520	1.4520	1.4524	1.4516

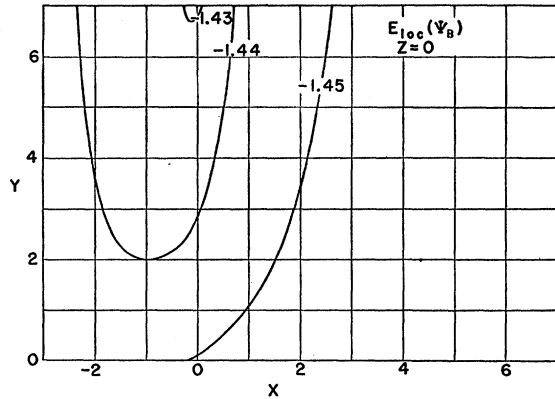


FIG. 2. Contour map of E_{100} , Bartlett function ψ_B , $z=0$.

earlier. The improvement was small, however, and it was thought that another value of E might result in faster convergence of E_{100} toward E . Therefore the two iterations were repeated with $E = -1.444$ and $E = -1.459$. $E = -1.444$ improved E_{100} at $x=0, y=0, z=31$ considerably, but at the expense of worse values at $x=0, y=31, z=1$. The reverse was true for $E = 1.459$. These trends continued for a third iteration at each of these eigenvalues. It was concluded that the true eigenvalue must lie between these. Smaller variations in E , within the range -1.450 to -1.453 , made no significant changes in E_{100} . Therefore this process does not limit E more closely than ± 0.5 percent.

After the ninth iteration of J , the E_{100} values began to oscillate without improving. The values of ψ at this stage are presented in Table I, together with corresponding values of E_{100} (Table II).

In order that one may appreciate how much better the present function (ψ_B) is than that due to Hylleraas (ψ_H), contour maps of E_{100} at $z=0$ are given for each function. Figure 1 shows the contours for the Hylleraas six-term function, out as far as $y=7$. Along the $+x$ -axis, $E_{100} \rightarrow +\infty$, and along the y -axis, $E_{100} \rightarrow -\infty$. The values of E_{100} are near to their average value of -1.4516 in only a *small* region of space. Figure 2 shows the con-

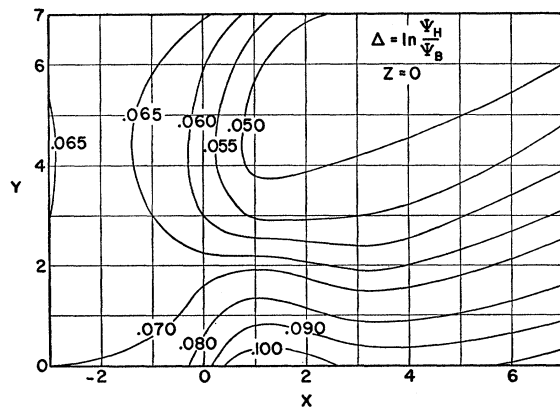


FIG. 3. Contour map of $\ln(\psi_H/\psi_B)$, $z=0$.

The same iterative method was used for J as for G . The E_{100} from the new function showed an immediate improvement over E_{100} from G^0 .

When the E_{100} for the fourth and fifth iterations of J were calculated, improvement was found in the values at all points of the later iteration compared with the

tours for our function, also only out to $y=7$. In this region, E_{loc} ranges between -1.429 and -1.452 , which demonstrates clearly that the present function is excellent. Over the whole range of space, i.e., out to $|x|=31$, $|y|=31$, $|z|=31$, the value of E_{loc} ranges from -1.2822 to -1.6065 , the extreme values being assumed at large distances in the $x=0$ plane.

Another way of comparing the Hylleraas function with ours is to plot the logarithm of the ratio ψ_H/ψ_B , and this is shown in Fig. 3. No normalization has been carried out, but if we assume that the correct value of the logarithm is about 0.07, then we see that the Hylleraas function deviates from ours by roughly ± 3 percent, but this is of course sufficient to cause marked changes in E_{loc} .

Further work may improve ψ and E_{loc} still more, especially at large y and z , and it may also lead to a more precise determination of the eigenvalue.

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High-Frequency Ionization Coefficients in Neon-Argon Mixtures*

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Breakdown electric fields at 2800-Mc/sec frequency are presented for a number of neon-argon mixtures for a large range of pressure variation. High-frequency ionization coefficients for mixtures varying from pure neon to pure argon have been computed from the breakdown fields. The collision phenomena in the gas mixtures are discussed and a theoretical derivation of breakdown fields is compared with experiment.

IONIZATION coefficients of neon-argon mixtures are of interest chiefly because of the energy exchange which takes place when excited neon atoms collide with argon atoms—the Penning effect.^{1,2} The lowest excited state of neon, 2^3S , at 16 volts is metastable, and is about three-tenths of a volt above the ionization potential of argon. A neon atom in this state exists long enough so that it makes many thousands of collisions at moderate gas concentrations. A small amount of argon in the neon will ensure that an excited neon atom will collide with an argon atom. In such a collision there is a very high probability that the neon will return to the ground state and give up its energy in ionizing the argon. We then have in effect a gas in which no energy is lost by excitation and in which ionization results whenever a neon atom is excited at its lowest level. The neon atoms are excited as a result of collisions with electrons which gain their energy from the electric field. Extensive studies of dc discharges in such mixtures have been carried out by Kruithof³ and Penning,² who published tables of the dc ionization coefficient η as a function of E/p (E being the electric field and p the gas pressure), for several concentrations of argon in the neon. The present paper presents the

high-frequency ionization coefficient ζ as a function of E/p for a number of different mixtures.

The dc ionization coefficient η is equal to α/E , the Townsend coefficient divided by the electric field. The Townsend coefficient is the number of ions produced by an electron per cm of travel. η is a function of E/p and is perhaps more significant than α . Townsend's α may be written $\nu/\mu E$; ν is the ionization rate per electron and μ the electron mobility so that $\eta = \nu/\mu E^2$. Because diffusion replaces mobility in high-frequency discharges as the controlling electron loss mechanism, the high-frequency ionization coefficient ζ has been defined as ν/DE^2 , where D is the diffusion coefficient.³ The ratio of η to ζ is equal to the ratio of the diffusion coefficient to the mobility, D/μ , which is equal to the average electron energy.

A high-frequency gas discharge is one in which the electric field alternates so rapidly that electrons are not swept out of the field during each cycle. The electrons gain energy from the electric field between elastic collisions and produce ionization by inelastic collisions in the gas. At low ion densities, the probabilities of recombination and attachment are very small and practically all electrons are lost by diffusion to the container walls. Breakdown takes place when the ionization rate equals the electron loss rate. Since the

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¹ M. J. Druyvesteyn and F. M. Penning, *Revs. Modern Phys.* **12**, 87 (1940).

² A. A. Kruithof and F. M. Penning, *Physica* **4**, 430 (1937).

³ M. A. Herlin and S. C. Brown, *Phys. Rev.* **74**, 291 (1948).