Specific Directions of Longitudinal Wave Propagation in Anisotropic Media*

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Plane elastic waves can be propagated in every direction of an unbounded elastic medium. It is known that associated with each direction there are three independent waves, the displacements of which form a mutually orthogonal set. In general none of the three displacement vectors coincides with the vector of the normal to the wave front, that is, in general the waves are neither longitudinal nor transverse. The purpose of this paper is to find specific directions in a medium of given anisotropy, along which the displacement of one of the three possible waves is exactly parallel to the direction of wave propagation. A method is developed which leads to the complete set of such "longitudinal" directions, if the matrix of the elastic coefficients is known. The method is applied to several groups of crystal symmetry, namely to the trigonal, hexagonal, tetragonal, and cubic systems, and the general conditions are established under which pure longitudinal waves exist. For α quartz, for example, the numerical calculation shows that there are five such distinct directions, not counting the ones which are equivalent by symmetry properties.

As a by-product of the results, special conditions between the elastic constants are obtained under which longitudinal waves could be propagated in any direction in a hypothetical anisotropic medium fulfilling these conditions. Owing to their relation to an early investigation by G. Green in 1839, the latter are called specialized Green's conditions.

1. INTRODUCTION

THE general laws of elastic waves propagated in crystalline media are well known.[1] In an infinitely extended crystalline medium, plane elastic waves can be propagated along any direction. If $\boldsymbol{\xi}$ is the vector of particle displacement and $\mathbf{k}$ the wave-vector normal to planes of constant phase, the elastic wave motion is described by

$$\boldsymbol{\xi}=\boldsymbol{\xi}_0 e^{i(\omega t\mp\mathbf{k}\cdot\mathbf{r})}, \quad (1)$$

$\mathbf{r}$ being the position vector of any point P in the medium. The displacement vector $\boldsymbol{\xi}$ has certain specified directions, which, in general, are not parallel to the wave-vector $\mathbf{k}$.

The theory of wave propagation through infinitely extended anisotropic media reveals that for any chosen direction of the wave-normal, that is, of the vector $\mathbf{k}$, there are three possible displacement vectors $\boldsymbol{\xi}$ which are functions of the direction of $\mathbf{k}$. These three displacement vectors are independent of each other and form a mutually orthogonal set; they belong to three independent plane waves propagated, in general, with three different velocities in the direction of $\mathbf{k}$. If it happens that one of the three displacement vectors coincides with $\mathbf{k}$, the other two necessarily lie in a plane perpendicular to $\mathbf{k}$; in this case the wave-triplet consists of one purely longitudinal (i.e., compressional) and two purely transverse waves.[2]

For certain crystals, for example quartz, special directions of $\mathbf{k}$ are known, along each of which the direction of $\boldsymbol{\xi}$ coincides with that of $\mathbf{k}$, or, in other words, along each of which compressional waves can be propagated. The only strain connected with a compressional wave in an infinitely extended region is the one normal to the wave front. By superposition of two such longitudinal waves traveling in opposite directions, a standing wave pattern is formed with planes of nodes and antinodes of stress and strain. The nodal planes of strain are then nodal planes for the total set of stresses. These planes therefore can serve as free boundaries of an infinitely extended plate, which then vibrates in a pure compressional thickness mode.

It is the purpose of this paper to outline a general method for finding the complete set of directions along which compressional waves can be propagated in an anisotropic medium and to apply the method to a number of crystal classes.

In the general case, where the displacement vector is not parallel to the wave normal, it can also be shown easily that in an infinitely extended medium there exist equidistant planes parallel to the wave front, in which all strains, and therefore all stresses, disappear. Any two such planes can serve as the free boundaries of a plate, which then can be excited in thickness vibration.

In *finite* plates, however, such standing waves cannot exist except in very special cases, without being associated with parasitic vibrations. Owing to elastic cross-coupling, a special set of stresses exists throughout the vibrating plate. In order to enforce the boundary conditions of stress-free edges of the plate, other modes of vibration will be set up which are inseparably associated with the original one-dimensional standing wave pattern, thus destroying its purity.[3]

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[1] For a condensed treatment see A. E. H. Love, *The Mathematical Theory of Elasticity* (Cambridge University Press, London, 1934), p. 298; and W. G. Cady, *Piezoelectricity* (McGraw-Hill Book Company, Inc., New York and London, 1946), p. 104. The subject is dealt with in detail by R. Bechmann, Arch. d. elektr. Übertragung **6**, 361 (1952).

[2] If $\boldsymbol{\xi}$ has any arbitrary direction, it can always be decomposed into a longitudinal and a transverse component, the components *not*, however, being independent of each other.

[3] A detailed physical outline of the mechanism of mode-coupling may be found in H. G. Baerwald, U. S. Patent No. 2,485,129, October 18, 1949.

2. PLANE ELASTIC WAVES IN ANISOTROPIC MEDIA

We now summarize in brief the pertinent features of the theory of propagation of plane elastic waves in anisotropic media. The theory was first worked out completely in 1839 by Green, although modern writers refer predominantly to a paper by Christoffel in 1877.[4] Green was interested in the mechanical theory of light propagated through crystalline media, and looked for the conditions under which transverse waves may exist. He found, between the 21 elastic coefficients needed for describing the most general stress-strain relations, 14 relations which must be fulfilled in order to ensure displacement vectors normal to any chosen direction of wave propagation. Besides the two transverse waves, according to Green's theory, there is always a third purely longitudinal wave possible. Green's problem, therefore, is closely related to the one treated here. Whereas Green derived certain relations between the elastic coefficients which must be satisfied in order to obtain transverse waves in any arbitrary direction, we assume those coefficients to be known and seek the set of special axes along which purely compressional waves may be propagated. These axes are necessarily associated with the two independent transverse waves in which Green was interested. We shall obtain conditions similar to those found by Green as a by-product of our results.

In a continuous anisotropic medium under adiabatic conditions the general stress-strain relation is[5]

$$T_i=\sum_k c_{ik}{}^D S_k-\sum_l h_{il}D_l, \quad (i, k=1, 2\cdots 6; l=1, 2, 3), \tag{2}$$

where $c_{ik}{}^D=c_{ki}{}^D$. The D_l are the components of the electric displacement vector. We shall limit ourselves to purely mechanically excited waves and therefore shall assume that no external electric fields are applied to the medium. Then the vector $\mathbf{D}$ is necessarily constant in both space and time. Since we are interested only in harmonic wave motion, $\mathbf{D}$ can be assumed to be zero and Eq. (2) can be written

$$T_i=\sum_k c_{ik}{}^D S_k. \quad (i, k=1, 2\cdots 6). \tag{3}$$

We choose a unit vector $\mathbf{s}$ with direction cosines l_1, l_2, l_3 as the direction of wave propagation. Any of the three independent and mutually orthogonal displacement vectors $\boldsymbol{\xi}$ associated with the three possible plane waves propagating along $\mathbf{s}$ may be characterized by its direction cosines m_1, m_2, m_3. Then Green's theory shows that for a given $\mathbf{s}$ the direction of any $\boldsymbol{\xi}$ can be found from the following set of equations:

$$\begin{aligned} m_1\Gamma_{11}+m_2\Gamma_{12}+m_3\Gamma_{13}&=m_1q,\\ m_1\Gamma_{12}+m_2\Gamma_{22}+m_3\Gamma_{23}&=m_2q,\\ m_1\Gamma_{13}+m_2\Gamma_{23}+m_3\Gamma_{33}&=m_3q. \end{aligned} \tag{4}$$

The coefficients $\Gamma_{ik}=\Gamma_{ki}$ are given by

$$\begin{aligned} \Gamma_{11}&=l_1^2c_{11}+l_2^2c_{66}+l_3^2c_{55}+2l_2l_3c_{56}+2l_1l_3c_{15}+2l_1l_2c_{16},\\ \Gamma_{22}&=l_1^2c_{66}+l_2^2c_{22}+l_3^2c_{44}+2l_2l_3c_{24}+2l_1l_3c_{46}+2l_1l_2c_{26},\\ \Gamma_{33}&=l_1^2c_{55}+l_2^2c_{44}+l_3^2c_{33}+2l_2l_3c_{34}+2l_1l_3c_{35}+2l_1l_2c_{45},\\ \Gamma_{13}&=l_1^2c_{15}+l_2^2c_{46}+l_3^2c_{35}+l_2l_3(c_{45}+c_{36})\\ &\quad+l_1l_3(c_{13}+c_{55})+l_1l_2(c_{56}+c_{14}),\\ \Gamma_{23}&=l_1^2c_{56}+l_2^2c_{24}+l_3^2c_{34}+l_2l_3(c_{23}+c_{44})\\ &\quad+l_1l_3(c_{45}+c_{36})+l_1l_2(c_{46}+c_{25}),\\ \Gamma_{12}&=l_1^2c_{16}+l_2^2c_{26}+l_3^2c_{45}+l_2l_3(c_{46}+c_{25})\\ &\quad+l_1l_3(c_{56}+c_{14})+l_1l_2(c_{12}+c_{66}). \end{aligned} \tag{5}$$

The three (real) values of q in Eq. (4) follow from the roots of a cubic equation, given in determinant form by

$$\begin{vmatrix} \Gamma_{11}-q & \Gamma_{12} & \Gamma_{13}\\ \Gamma_{12} & \Gamma_{22}-q & \Gamma_{23}\\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33}-q \end{vmatrix}=0. \tag{6}$$

With the values l_1, l_2, l_3 of $\mathbf{s}$ given, the Γ_{ik} can be computed from Eq. (5), and the three values of q are found from Eq. (6). Each value of q belongs to one set of direction cosines m_1, m_2, m_3, which are found by inserting q in Eq. (4). The three wave equations are

$$\partial^2\xi_\mu/\partial t^2=q_\mu\partial^2\xi_\mu/\partial s^2 \quad (\mu=1, 2, 3), \tag{7}$$

the solution of which has been given in Eq. (1). The wave velocity is

$$v_\mu=\omega/k_\mu=(q_\mu/\rho)^{\frac{1}{2}}, \tag{8}$$

ρ being the density of the medium. q_μ is therefore called the stiffness coefficient of the wave with the displacement $\boldsymbol{\xi}_\mu$.

3. CONDITIONS FOR PURELY COMPRESSIONAL WAVE MOTION

For longitudinal waves the direction of the displacement vector $\boldsymbol{\xi}$ with direction cosines m_1, m_2, m_3 is to be parallel to the unit vector in the direction of wave propagation $\mathbf{s}$ with the direction cosines l_1, l_2, l_3. Let us consider the values of m_iq in Eq. (4) as the components p_i of a vector $\mathbf{p}$ which lies in the direction of the displacement vector $\boldsymbol{\xi}$. Then $\mathbf{p}$ coincides with $\mathbf{s}$, if $\mathbf{p}\times\mathbf{s}=0$, that is, if

$$l_2p_3-l_3p_2=0, \quad l_1p_3-l_3p_1=0, \quad l_1p_2-l_2p_1=0, \tag{9}$$

or,

$$l_1:l_2:l_3=p_1:p_2:p_3. \tag{10}$$

Actually the systems (9) or (10) represent only two independent equations. The components of $\mathbf{p}$ follow from Eq. (4) upon replacing m_i by l_i, since $\mathbf{p}$ and $\mathbf{s}$

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[4] G. Green, Trans. Cambridge Phil. Soc. **7**, 121 (1839); *Mathematical Papers* (Macmillan and Company, London, 1871), p. 307; E. W. Christoffel, Ann. di matematica pura ed applicata (2) **8**, 193 (1877). An excellent account of Green's work can be found in Lord Kelvin's *Baltimore Lectures* (Cambridge University Press, London, 1904), Lectures XI and XII.

[5] W. G. Cady, J. Acoust. Soc. Am. **22**, 579 (1950). For the notation see Proc. Inst. Radio Engrs. **37**, 1378 (1949).

are assumed to be parallel:

$$\begin{aligned} l_1 q = p_1 &= \Gamma_{11} l_1 + \Gamma_{12} l_2 + \Gamma_{13} l_3, \\ l_2 q = p_2 &= \Gamma_{12} l_1 + \Gamma_{22} l_2 + \Gamma_{23} l_3, \\ l_3 q = p_3 &= \Gamma_{13} l_1 + \Gamma_{23} l_2 + \Gamma_{33} l_3. \end{aligned} \tag{11}$$

When p_i and Γ_{ik} are substituted from Eqs. (11) and (5) into Eq. (9), the latter yields a set of three equations involving the elastic constants c_{ik}^D and the direction cosines l_1, l_2, l_3, which we wish to find. To each set of real values of the l_i satisfying simultaneously Eqs. (9), there belongs a direction of propagation of a longitudinal wave. By the method outlined we shall investigate now a number of crystal groups with respect to the existence of such directions.

4. GROUP OF TRIGONAL SYMMETRY, CLASSES D_3, C_{3v}, D_{3d}

This group includes α quartz. The elastic constants form the following matrix:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & \frac{1}{2}(c_{11}-c_{12}) \end{pmatrix}. \tag{12}$$

Using this scheme we compute the Γ_{ik} from Eq. (5), then insert them in Eq. (11) and by using Eq. (9) arrive at the following set of equations:

$$l_2 l_3 \{ (l_1^2+l_2^2)(-c_{11}+c_{13}+2c_{44}) + l_3^2(-c_{13}+c_{33}-2c_{44}) \} + c_{14}\{3(l_1^2 l_2^2 + l_2^2 l_3^2 - l_1^2 l_3^2) - l_2^4\} = 0, \tag{13}$$

$$l_1 l_3 \{ (l_1^2+l_2^2)(-c_{11}+c_{13}+2c_{44}) + l_3^2(-c_{13}+c_{33}-2c_{44}) \} + c_{14}\{l_1 l_2(3l_1^2 - l_2^2 - 6l_3^2)\} = 0, \tag{14}$$

$$3c_{14} l_1 l_3 (l_1^2 - 3l_2^2) = 0. \tag{15}$$

The last equation yields immediately the three solutions

$$l_3 = 0, \tag{16}$$

$$l_1 = 0, \tag{17}$$

$$l_1/l_2 = \pm\sqrt{3}. \tag{18}$$

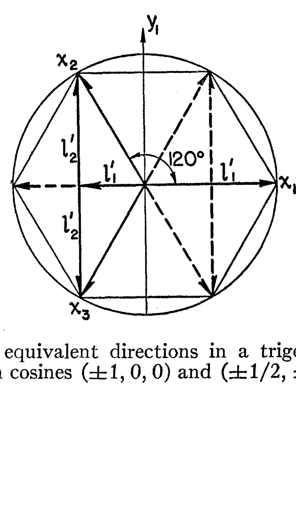

FIG. 1. Set of equivalent directions in a trigonal system with direction cosines $(\pm 1, 0, 0)$ and $(\pm 1/2, \pm\sqrt{3}/2, 0)$.

Putting each one of these solutions into Eq. (13) or (14) we have to investigate whether sets of real values l_1, l_2, l_3 exist that satisfy Eq. (13) or Eq. (14).[6]

Inserting the first value $l_3=0$ into Eq. (13), we obtain

$$c_{14} l_2^2 (3l_1^2 - l_2^2) = 0. \tag{19}$$

This equation, and therefore Eqs. (13), (14), and (15) are satisfied simultaneously by

$$l_2 = 0 \quad \text{and} \quad l_2/l_1 = \pm\sqrt{3}. \quad [l_3 = 0] \tag{20}$$

Next, inserting $l_1=0$ from Eq. (17) into Eq. (13) yields either $l_2=0$ or the following cubic equation for the ratio l_3/l_2:

$$(-c_{13}+c_{33}-2c_{44})(l_3/l_2)^3 + 3c_{14}(l_3/l_2)^2 + (-c_{11}+c_{13}+2c_{44})(l_3/l_2) - c_{14} = 0. \quad [l_1=0] \tag{21}$$

Finally, upon inserting $l_1'/l_2' = \pm\sqrt{3}$ from Eq. (18) into Eq. (13), we find that either $l_2=0$ again or that the following equation must hold:

$$(-c_{13}+c_{33}-2c_{44})(l_3'/l_2')^3 - 6c_{14}(l_3'/l_2')^2 + 4(-c_{11}+c_{13}+2c_{44})(l_3'/l_2') + 8c_{14} = 0. \quad [l_1'/l_2' = \pm\sqrt{3}] \tag{22}$$

[The primes on l_2', l_3' are added to distinguish them from l_2, l_3 of Eq. (21).]

A brief calculation shows that Eq. (21) passes into Eq. (22) by the transformation

$$l_3/l_2 = -l_3'/2l_2'. \tag{23}$$

At this point it is appropriate to take notice of the trigonal symmetry of the group under consideration. A rotation of the coordinate system in a medium of trigonal symmetry about the z-axis by $\pm 120°$ leads, as is well known, to an equivalent coordinate system. That is, any point $P'(x',y',z')$ with reference to the system so rotated is in every respect equivalent to the point $P(x,y,z)$ in the original system, if $x=x'$, $y=y'$, $z=z'$. The points P' and P pass into each other by a rotation of their position vectors by $\pm 120°$ around the z-axis. Consequently, associated with any direction in the trigonal group there are two more equivalent directions, all three directions being related, with respect to wave propagation, to each other by a 120° rotation about the z-axis. Each direction is also equivalent to its opposite direction.

Now, Eq. (20) contains a set of directions which lie in the xy-plane, as shown in Fig. 1. When $l_2=l_3=0$ we have $l_1=\pm 1$; when $l_2/l_1=\pm\sqrt{3}$, we have $l_1=\pm 1/2$, since $l_1^2+l_2^2+l_3^2=1$. The directions thus defined pass into each other by a rotation of $\pm 120°$ about the z-axis and are therefore equivalent. Hence it is sufficient to consider only one of them. Selecting the direction determined by $l_2=l_3=0$ we obtain the well-known result that in a medium of trigonal symmetry the x-axis allows the propagation of longitudinal waves.

[6] It may be noted that Eqs. (13) to (15) contain all elastic constants of the matrix (12) except c_{12}.

An infinitely extended plate with planes normal to the x-axis (X-cut) can be excited in purely longitudinal thickness vibration.

A second direction which we obtained from Eqs. (17) and (13) is the one corresponding to the direction cosines $l_1=l_2=0$, that is, $l_3=1$, or the z-axis, which consequently is also an axis allowing longitudinal waves.

Considering next $l_1=0$ according to Eq. (17), we found the corresponding values of l_3/l_2 to be determined by the cubic Eq. (21). When l_3/l_2 in this equation is replaced by $-l_3'/2l_2'$ according to Eq. (23), Eq. (21) passes into Eq. (22), which resulted from the condition $l_1'=\pm\sqrt{3}l_2'$ of Eq. (18).

It is easily seen that the conditions expressed by Eqs. (17) and (18) are equivalent through trigonal symmetry. Figure 2 shows the projection onto the xy-plane of unit vectors with components 0, l_2, l_3 corresponding to Eq. (17), and $l_1'=\pm l_2\sqrt{3}/2$, $l_2'=-l_2/2$, $l_3'=l_3$, which obviously have a 120° relation about the z-axis. Equations (18) and (22) therefore lead only to directions which by trigonal symmetry are equivalent to the directions determined by Eqs. (17) and (21), so that it will be sufficient to consider only the axes in the yz-plane ($l_1=0$) resulting from Eq. (21). From the directions thus found all other equivalent directions are obtained by applying the rules of trigonal symmetry. A cubic equation has either one or three real roots; hence, there are either one or three directions in the yz-plane resulting from Eq. (21).

Including the x- and z-axes already mentioned, we have altogether a set of three or five "longitudinal" directions in the trigonal group here considered. Counting also the equivalent directions resulting from trigonal symmetry, we have in all 7 or 13 "longitudinal" directions,[7] each of them allowing longitudinal wave propagation in either sense.

5. GROUP OF TRIGONAL SYMMETRY, CLASSES C_3, C_{3i}

There is another trigonal crystal group containing Class C_3 and C_{3i}, the elastic constants of which form the matrix

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & -c_{25} & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & c_{25} & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & c_{25} \\ -c_{25} & c_{25} & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & c_{25} & c_{14} & \tfrac{1}{2}(c_{11}-c_{12}) \end{bmatrix}. \quad (24)$$

This group evidently contains one more independent elastic constant than the group characterized by the matrix (12), namely c_{25}. Without going into a detailed discussion of the complete system of compressional axes of this group, we will treat briefly some properties of this group with respect to such axes.

[7] The z-axis is counted only once.

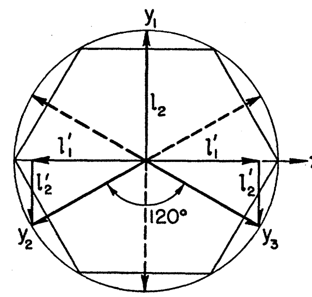

FIG. 2. Projection on the xy-plane of a second set of equivalent directions in a trigonal system.

Using the matrix (24) we find by Eqs. (5), (11), and the last equation of (9) the following condition for the direction cosines of compressional axes:

$$c_{14}l_1l_3(l_1^2-3l_2^2)+c_{25}l_2l_3(3l_1^2-l_2^2)=0. \quad (25)$$

One solution immediately found is $l_3=0$, which corresponds to directions of compressional wave propagation in the xy-plane. With $l_3=0$ one obtains from either of the other two equations in (9) the equation

$$(l_1/l_2)^3-3(c_{14}/c_{25})(l_1/l_2)^2-3(l_1/l_2)+(c_{14}/c_{25})=0. \quad (26)$$

This equation yields a set of three possible compressional axes in the xy-plane, the directions of which depend only on the ratio c_{14}/c_{25}. The three directions are mutually related by the 120° symmetry of the trigonal group.

If $c_{25}=0$, l_2 becomes zero in accordance with the result found in the previous section; in the group considered there the x-axis was found to be a possible compressional axis. This fact is due to the absence of c_{25} in the matrix (12). If $c_{14}=0$, l_1 becomes zero and the y-axis can propagate compressional waves.

If $c_{14}=-c_{25}$, it follows from Eq. (26) that $l_1=l_2=1$; in this case compressional waves may be propagated along the 45° direction in the x, y-plane.

6. COMPRESSIONAL WAVES IN α QUARTZ

As an example of the trigonal groups considered we study the complete set of longitudinal directions in α quartz. The elastic constants are assumed to have the following values[8] expressed in units of 10^{10} dyne cm^{-2}:

$$\begin{aligned} c_{11}&=87.5, \quad c_{12}=7.62, \quad c_{13}=15.1, \quad c_{14}=17.2, \\ c_{33}&=107.7, \quad c_{44}=57.3, \quad c_{66}=39.9. \end{aligned} \quad (27)$$

[8] W. G. Cady, reference 1, p. 137. The values of c_{ik}, according to Eq. (3), should be those at constant dielectric displacement **D**. The values given here hold at constant electric field **E**. The difference between c_{ik}^E and c_{ik}^D for quartz, however, is small enough to be neglected here. For simplicity the superscript E or D is omitted. The coefficients in Eq. (27) are valid for both right quartz and left quartz, if the systems of coordinate axes are chosen according to p. 408, Fig. 76, of the reference given above, that is a right-handed system of axes for a right quartz and a left-handed system for a left quartz.

With these values, Eq. (21) becomes

$$-22(l_3/l_2)^3+51.6(l_3/l_2)^2+42.2(l_3/l_2)-17.2=0. \quad (28)$$

The solutions are found to be

$$l_3/l_2=0.307, \quad 2.912, \quad -0.874. \quad [l_1=0] \quad (29)$$

From $l_2^2+l_3^2=1$ one finds the following direction-cosines:

$$\begin{array}{llll} & \text{(a)} & \text{(b)} & \text{(c)} \\ l_3= & 0.294 & 0.946 & 0.658, \\ l_2= & 0.956 & 0.325 & -0.753, \end{array} \quad (30)$$

corresponding to angles of about $-73°$, $-19°$, and $49°$ with respect to the z-axis, as indicated in Fig. 3.[9] Knowing the direction cosines we can compute the coefficients Γ_{ik} from Eq. (5) and find the stiffness coefficients q for the longitudinal waves along these directions by Eq. (4), where now $m_1=l_1$, $m_2=l_2$, $m_3=l_3$, since the wave-normal coincides with the direction of displacement. We can also use Eq. (6), which yields not only the q-values belonging to the longitudinal waves, but also the q-values of the two purely transverse waves associated with but independent of each one of the longitudinal waves.

The following values in units of 10^{10} dyne cm^{-2} have been found for the stiffness coefficients q_l of the longitudinal wave and the stiffness coefficients q_{t1} and q_{t2} for the associated transverse waves for the directions indicated by (a), (b), and (c) in Eq. (30) and Fig. 3:

$$\begin{array}{llll} \text{(a)} & q_l=76.65, & q_{t1}=51.09, & q_{t2}=60.24, \\ \text{(b)} & q_l=109.40, & q_{t1}=42.90, & q_{t2}=66.04, \\ \text{(c)} & q_l=131.34, & q_{t1}=29.25, & q_{t2}=30.41. \end{array} \quad (31)$$

For the z-axis, which we found also to be a direction allowing longitudinal waves, q_l reduces to c_{33}, as is easily proved. The values q_t of the associated transverse waves with displacements lying in the xy-plane follow from Eq. (6). With the coefficients c_{ik} from Eq. (27) we obtain

$$\text{(d)} \quad q_l=c_{33}=107.7, \quad q_{t1}=q_{t2}=c_{44}=57.3. \quad (32)$$

For transverse waves therefore rotational symmetry exists about the z-axis.

The x-axis, commonly used as a direction for pure longitudinal waves, has for the longitudinal waves $q_l=c_{11}$. The numerical values for the longitudinal and the associated transverse waves are[10]

$$\text{(e)} \quad q_l=c_{11}=87.5, \quad q_{t1}=29.32, \quad q_{t2}=67.88. \quad (33)$$

To conclude the considerations of media of trigonal symmetry we may consider the special hypothetical relations between the elastic constants c_{ik} which would have to hold in order to ensure purely longitudinal wave propagation in *any* direction. These relations, which we will call "specialized Green conditions" follow immediately from Eqs. (13) to (15), which are fulfilled for *any* set of direction-cosines l_1, l_2, l_3, if

$$c_{11}=c_{33}, \quad 2c_{44}=c_{11}-c_{13}, \quad c_{14}=0. \quad (34)$$

If there were a medium of trigonal symmetry obeying these relations between its elastic constants, it could propagate both pure longitudinal and pure transverse waves in every direction. The condition $c_{14}=0$ would convert the trigonal symmetry into hexagonal symmetry. Upon computing the Γ_{ik} from Eq. (5) by the use of the relations (34) and inserting the Γ_{ik} thus found into Eq. (11), one finds that the stiffness coefficient q for the longitudinal waves becomes $q=c_{11}$ independent of the direction. For the associated transverse waves, however, velocity and polarization are not independent of the direction of wave propagation.

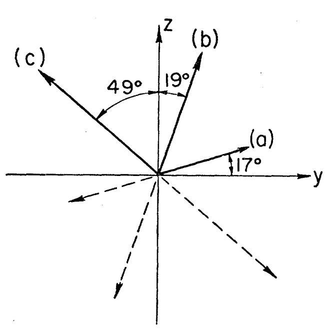

FIG. 3. Directions of longitudinal wave propagation in α quartz. The length of each arrow is proportional to the value of the corresponding stiffness-coefficient q.

7. GROUP OF HEXAGONAL SYMMETRY

The matrix of elastic constants c_{ik} of the hexagonal system is obtained by setting $c_{14}=0$ in the matrix (12) of the trigonal system considered above. The possible directions of longitudinal wave propagation in a crystal of hexagonal symmetry are therefore determined by our former Eqs. (13) to (15) with $c_{14}=0$. Equation (15) is then fulfilled automatically and we are left with the equations

$$l_2l_3\{(l_1^2+l_2^2)(-c_{11}+c_{13}+2c_{44}) + l_3^2(-c_{13}+c_{33}-2c_{44})\}=0, \quad (35)$$

$$l_1l_3\{(l_1^2+l_2^2)(-c_{11}+c_{13}+2c_{44}) + l_3^2(-c_{13}+c_{33}-2c_{44})\}=0. \quad (36)$$

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[9] It will be noted that the directions (a) and (c) according to Fig. 3 are very close to the so-called 18.5° cut and *BT*-cut of α quartz. See, for example, W. G. Cady, reference 1, p. 459.

[10] I. Koga, Reports of Radio Researches and Works in Japan **2**, No. 2, 157 (1932). Koga's values differ slightly from the ones given in Eq. (33), owing to the slight difference between Koga's values for the c_{ik} and the values used here from Eq. (27).

These equations are satisfied by

$$l_3=1, \quad l_1=l_2=0; \tag{37}$$

$$l_3=0, \quad l_1^2+l_2^2=1; \tag{38}$$

$$\frac{l_3^2}{l_1^2+l_2^2}=\frac{c_{11}-c_{13}-2c_{44}}{-c_{13}+c_{33}-2c_{44}}. \tag{39}$$

Equation (37) renders the z-axis, and Eq. (38) any axis in the x, y-plane, as possible directions of longitudinal wave propagation. Equation (39) gives another set of directions, all of which lie on a circular cone, the generating lines of which form an angle θ about the z-axis with

$$\tan\theta=\left[\frac{c_{33}-c_{13}-2c_{44}}{c_{11}-c_{13}-2c_{44}}\right]^{\frac{1}{2}}. \tag{40}$$

All compressional directions show rotational symmetry about the z-axis, that is the hexagonal character of the crystal-structure does not manifest itself with respect to such directions as it was the case in the trigonal classes considered.

As an example of the hexagonal group we consider β quartz, which is stable from 573° to 870°C. The elastic constants, in units of 10^{10} dynes cm^{-2}, at 600°C are taken as[11]

$$c_{11}=118.4, \quad c_{13}=32.0, \quad c_{33}=107.0, \quad c_{44}=35.8. \tag{41}$$

With these values, Eq. (40) gives $\tan\theta=0.479$, or $\theta=25.6°$, as the angle about the z-axis of the direction of propagation.

8. GROUP OF TETRAGONAL SYMMETRY, CLASSES V_d, D_4, C_{4v}, D_{4h}

The elastic constants of this group have the following matrix:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}. \tag{42}$$

From this matrix we compute the Γ_{ik} from Eq. (5); from Eqs. (11) and (9) the following set of equations is then obtained:

$$l_2l_3\{l_1^2(-c_{12}+c_{13}+2c_{44}-2c_{66})+l_2^2(-c_{11}+c_{13}+2c_{44}) +l_3^2(-c_{13}+c_{33}-2c_{44})\}=0, \tag{43}$$

$$l_1l_3\{l_1^2(-c_{11}+c_{13}+2c_{44})+l_2^2(-c_{12}+c_{13}+2c_{44}-2c_{66}) +l_3^2(-c_{13}+c_{33}-2c_{44})\}=0, \tag{44}$$

$$l_1l_2(l_1^2-l_2^2)(c_{11}-c_{12}-2c_{66})=0. \tag{45}$$

[11] W. G. Cady, reference 1, p. 158. The difference between c_{ik}^D and c_{ik}^E can be assumed small enough to be neglected.

Unless $c_{11}-c_{12}-2c_{66}$ happens to be zero, Eq. (45) yields the solutions

$$l_1=0, \quad l_2=0, \quad l_1=\pm l_2. \tag{46}$$

Equations (43) and (44) express the tetragonal symmetry, since they pass into each other by interchanging l_1 and l_2. From Eqs. (43) to (46) it is seen immediately that the x-, y-, and z-axes are compressional directions. Further compressional axes are determined by

$$l_1=\pm l_2$$

and

$$\left[\frac{l_1}{l_3}\right]^2=\left[\frac{l_2}{l_3}\right]^2=\frac{c_{13}-c_{33}+2c_{44}}{-c_{11}-c_{12}+2c_{13}+4c_{44}-2c_{66}}, \tag{47}$$

$$l_1=0 \quad \text{and} \quad \left[\frac{l_2}{l_3}\right]^2=\frac{c_{13}-c_{33}+2c_{44}}{-c_{11}+c_{13}+2c_{44}}, \tag{48}$$

$$l_2=0 \quad \text{and} \quad \left[\frac{l_1}{l_3}\right]^2=\frac{c_{13}-c_{33}+2c_{44}}{-c_{11}+c_{13}+2c_{44}}. \tag{49}$$

Equations (47) to (49) show again the tetragonal symmetry about the z-axis, since l_1 and l_2 can be exchanged.

In the special case where $c_{11}-c_{12}-2c_{66}=0$, Eqs. (47) to (49) reduce themselves to

$$(l_1^2+l_2^2)/l_3^2=(c_{11}-c_{33}+2c_{44})/(-c_{11}+c_{13}+2c_{44}). \tag{50}$$

The compressional direction given by Eq. (50) therefore possesses rotational symmetry about the z-axis.

The specialized Green conditions for the tetragonal group under consideration follow from Eqs. (43) to (45) as

$$c_{11}=c_{33}=c_{13}+2c_{44}=c_{12}+2c_{66}. \tag{51}$$

If the elastic constants were mutually related in this way, Eqs. (43) to (45) would be satisfied for any values of l_1, l_2, l_3 and purely compressional waves therefore could be propagated in any direction in a medium of this tetragonal group.

9. GROUP OF CUBIC SYMMETRY

The matrix of the elastic constants c_{ik} for a medium of cubic symmetry is obtained from the matrix (42) by setting $c_{13}=c_{12}$, $c_{33}=c_{11}$, and $c_{66}=c_{44}$. Unless $c_{11}-c_{12}-2c_{44}=0$, it follows from Eqs. (43) to (45) that again the x-, y-, and z-axes are compressional directions. Another set of such directions is

$$\pm l_1=\pm l_2=\pm l_3=\pm 1/\sqrt{3}. \tag{52}$$

All the compressional directions in the cubic system are independent of the special values of the elastic constants.

If $c_{11}-c_{12}-2c_{44}=0$, the cubic system passes into an isotropic system and compressional waves can be propagated in every direction.