Nonlinear Theory of Space-Charge Wave in Moving, Interacting Electron Beams with Application to Solar Radio Noise

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A consideration of the complete equations of interaction (without the linear approximation) of two moving electron beams of given densities (with sufficient numbers of ions to make the charges macroscopically neutral) shows that propagation of steady-state space-charge waves is possible in such a medium. The period of the space-charge wave is a function of its amplitude and phase velocity. For small amplitudes the oscillation is simple harmonic, and the characteristic dispersion equation of the first order theory is obtained. For a given phase velocity of the wave, the oscillation becomes increasingly anharmonic with increase of amplitude. Beyond a particular value of the amplitude (which is a function of the phase velocity of the wave), the wave form of the oscillation becomes discontinuous.

The above theory is applied to estimate the relative intensity of the second harmonic component in solar radio outbursts recently discovered by Australian workers.

A theoretical analysis based on the antenna theory of electromagnetic radiation from oscillating plasma gives a radio flux of the order of magnitude of that observed.

1. INTRODUCTION

SPACE-CHARGE wave amplification in moving interacting charged beams has been used in electron wave tubes[1] and to explain the abnormal intensity of solar radio outbursts.[2] The theory so far has been confined to small, sinusoidal oscillations. The large amplifications obtained, however, clearly indicate the need for a nonlinear theory. Such a theory has received added interest from the recent interesting discovery by Australian workers[3] of the second harmonic component in solar radio-noise outbursts, which exhibits the characteristic frequency drift of the fundamental and is comparable with it in intensity.

We have considered in this paper the complete equations of interaction of moving electron beams (with sufficient numbers of ions to make the charges macroscopically neutral), and find that propagation of steady-state space-charge waves is possible in such a medium. The theory indicates the nature of the asymptotic nonlinear steady-state oscillation. Work is needed to trace the buildup of the amplitude under conditions in which the linear (i.e., small-signal) theory predicts wave growth.

We find that the period of the space-charge wave, on the nonlinear theory, is a function of its amplitude and phase velocity. The oscillation becomes increasingly anharmonic with increase of amplitude, and discontinuous beyond a limiting amplitude. An estimate is obtained, from the theory, of the relative intensity of the second harmonic component in solar radio outbursts, and compared with observation.[3] It is also shown that a rapid current buildup will enable the plasma to radiate the observed radio flux.

[1] J. R. Pierce, J. Appl. Phys. **19**, 231 (1948); Proc. Inst. Radio Engrs. **37**, 980 (1949); A. V. Haeff, Proc. Inst. Radio Engrs. **37**, 4 (1949).

[2] A. V. Haeff, Phys. Rev. **75**, 1546 (1949); J. Feinstein and H. K. Sen, Phys. Rev. **83**, 405 (1951).

[3] Wild, Murray, and Rowe, Nature **172**, 533 (1953).

2. THE NONLINEAR DISPERSION RELATION

We shall derive the nonlinear dispersion relation for space-charge waves in moving electron beams (with sufficient numbers of ions to make the charges macroscopically neutral) injected into a common space. For simplicity of treatment, we shall confine our analysis to one dimension and to two electron beams.[4] We shall use the following notation: n = number of charged particles per unit volume; v = velocity of charged particle; e = charge of electron or ion (minus for electron); m = mass of electron; ϕ = potential of space charge; ϵ_0 = dielectric constant of free space; V = phase velocity of space-charge wave.

The suffixes 1 and 2 will refer to the two electron beams. The suffix + will refer to the ions and 0 to the origin of the wave system of coordinates. (See later on.)

We shall neglect the oscillatory motion of the ions on account of their heavier mass relative to the electrons. We also neglect collision effects, which produce a small damping of oscillations in most plasmas (see Bohm and Gross, reference 4). The equation of motion for either electron beam is

$$\frac{\partial v_s}{\partial t}+v_s\frac{\partial v_s}{\partial x}=\frac{e}{m}\frac{\partial \phi}{\partial x}, \tag{1}$$

where $s=1$ or 2. The equation of continuity for either beam is

$$\frac{\partial n_s}{\partial t}+\frac{\partial}{\partial x}(n_s v_s)=0, \tag{2}$$

[4] The analysis can be extended to n beams and to continuous velocity distributions. A. I. Ahiezer and G. Y. Lubarskiy, Compt. rend. **80**, (2), 193 (1951) have considered the nonlinear theory of oscillations of electronic plasma. D. Bohm and E. P. Gross, Phys. Rev. **75**, 1853 (1949) have derived the nonlinear dispersion relation for a continuous velocity distribution. The author has given in Appendix I a derivation of the nonlinear dispersion relation by the Boltzmann equation.

where $s=1$ or 2. The Poisson's equation is

$$\epsilon_0 \frac{\partial^2\phi}{\partial x^2} = 4\pi e(n_1+n_2-n_+). \quad (3)$$

As we are interested in the steady-state space-charge wave propagation in the medium, we shall suppose all oscillating variables to be functions of the variable[5]

$$x' = x - Vt. \quad (4)$$

Equation (4) represents a transformation to the coordinate system stationary with respect to the wave. We choose the origin of this coordinate system (i.e., $x'=0$) at the point where $\phi=0$. We shall denote the physical parameters (particle velocity and concentration) at this point by the suffix zero.

With the substitution of (4), Eqs. (1), (2), and (3) reduce to

$$(v_s-V)\frac{dv_s}{dx'} = \frac{e}{m}\frac{d\phi}{dx'}, \quad (5)$$

$$-V\frac{dn_s}{dx'} + \frac{d}{dx'}(n_s v_s) = 0, \quad (6)$$

and

$$\epsilon_0 \frac{d^2\phi}{dx'^2} = 4\pi e(n_1+n_2-n_+). \quad (7)$$

Equations (5) and (6) yield, on integration:

$$(V-v_s)^2 = \frac{2e}{m}\phi + (V-v_{s0})^2, \quad (8)$$

and

$$n_s(V-v_s) = n_{s0}(V-v_{s0}). \quad (9)$$

Equations (7), (8), and (9), on elimination of n_s and v_s, give the nonlinear dispersion relation:

$$\epsilon_0 \frac{d^2\phi}{dx'^2} = 4\pi e\left[n_{10}\frac{|V-v_{10}|}{[(2e/m)\phi+(V-v_{10})^2]^{\frac{1}{2}}} + n_{20}\frac{|V-v_{20}|}{[(2e/m)\phi+(V-v_{20})^2]^{\frac{1}{2}}} - n_+\right]. \quad (10)$$

3. THE LINEAR DISPERSION RELATION

On the linear (small-signal) theory we suppose that all oscillating quantities vary as $\exp i(\Gamma x - \omega t)$, where $\omega/\Gamma = V$, ω denoting the angular frequency and Γ the wave number. If we expand the right-hand side of Eq. (10) to the first power of ϕ, we shall obtain the usual linear dispersion relation[6]:

$$\frac{\omega_1^2}{(\omega-\Gamma v_{10})^2} + \frac{\omega_2^2}{(\omega-\Gamma v_{20})^2} = 1, \quad (11)$$

where ω_s^2 is the plasma frequency:

$$\omega_s^2 = 4\pi n_s e^2/m\epsilon_0. \quad (12)$$

4. STEADY-STATE NONLINEAR OSCILLATION

Integrating (10), we find that the electric field $E=-d\phi/dx'$ is given by

$$\epsilon_0\left(\frac{d\phi}{dx'}\right)^2 = 8\pi m\left[n_{10}|V-v_{10}|\left(\frac{2e}{m}\phi+(V-v_{10})^2\right)^{\frac{1}{2}} + n_{20}|V-v_{20}|\left(\frac{2e}{m}\phi+(V-v_{20})^2\right)^{\frac{1}{2}} - \frac{e\phi}{m}n_+\right] + C, \quad (13)$$

where C is a constant.

E reaches a maximum (or a minimum) at $\phi=0$, where dE/dx' given by (10) vanishes. Denoting the maximum (or minimum) of E by E^*, we can fix the constant C in (13). Further, changing to the following dimensionless variables and parameters:

$$\nu_{10}=n_{10}/n_+, \quad \nu_{20}=n_{20}/n_+, \quad u_{10}=v_{10}/V, \quad u_{20}=v_{20}/V,$$
$$\psi=(2e/mV^2)\phi, \quad \xi=(2\omega_0/V)x', \quad (14)$$

we obtain from (13) the following nonlinear equation for the oscillation of the electric potential:

$$(d\psi/d\xi)^2 = 2\nu_{10}|1-u_{10}|[\psi+(1-u_{10})^2]^{\frac{1}{2}} + 2\nu_{20}|1-u_{20}|[\psi+(1-u_{20})^2]^{\frac{1}{2}} - \psi - 2\nu_{10}(1-u_{10})^2 - 2\nu_{20}(1-u_{20})^2 + \frac{\epsilon_0}{4\pi m V^2}\frac{E^{*2}}{n_+}. \quad (15)$$

ω_0 in (14) is given by

$$\omega_0^2 = (4\pi e^2/m\epsilon_0)n_+. \quad (16)$$

5. NATURE OF THE OSCILLATION CURVES

A graphical analysis of Eq. (15) is given in Appendix II. We shall merely quote here the results. The oscillations of the electric potential, $\psi(\xi)$, begin with $E^*=0$ at F_1 in Fig. 2. Near F_1, E^* is small, and the oscillations are simple harmonic. With increase of E^*, the oscillations become increasingly anharmonic. Beyond F_0 in Fig. 2, there is a discontinuity in the tangent to $\psi(\xi)$, and hence in the electric field. This is similar to the phenomenon of jump well known in nonlinear oscillation

[5] V in (4) is a constant and a free parameter of the solution. V is, in fact, the velocity of propagation of the progressive space-charge wave. We assume that such a wave is the limiting form of the growing initial perturbation. A proof of this assumption would be desirable.

[6] Note that $n_+=n_{10}+n_{20}$, on account of the assumption of macroscopic neutrality of charge.

theory. We shall call F_0 the *critical point* of the oscillation.

We shall suppose that at the critical point, the maximum amplitude is reached of the steady state nonlinear oscillation.[7] This maximum amplitude is a function of the phase velocity V of the space-charge wave, and is given by the following relation:

$$\frac{\epsilon_0}{4\pi m V^2}\frac{E^{*2}}{n_+}=2\nu_{10}(1-u_{10})^2+(2\nu_{20}-1)(1-u_{20})^2 -2\nu_{10}|1-u_{10}|[(u_{10}-u_{20}) \times(u_{10}+u_{20}-2)]^{\frac{1}{2}}. \quad (17)$$

Equation (17) holds provided that $(1-u_{20})^2 < (1-u_{10})^2$.

6. ILLUSTRATIVE CASES

We shall illustrate the qualitative description in Sec. 5 by numerical integration of Eq. (15), both at and beyond the critical point of the oscillation curves, for the following set of values of the parameters: $n_{10}=9\times10^8$ cm^{-3}, $n_{20}=10^8$ cm^{-3}, $v_{10}=4\times10^7$ cm/sec, $v_{20}=0$, $V=5\times10^7$ cm/sec.

Figure 1, Curve A, gives the oscillation of the electric field at the critical point ($f=f_0=-0.236$, see Appendix II). From (14), the angular frequency ω is given by

$$\omega=4\pi\omega_0/\xi_\lambda, \quad (18)$$

where $\xi_\lambda=2.76$ is the wavelength of the oscillation in Fig. 1A.

The nonlinear frequency ν is $\simeq$1292 Mc/sec. On the small signal theory [Eq. (11)], ν is $\simeq$1349 Mc/sec. We find a decrease in frequency with increase of amplitude.

The Curve A in Fig. 1 is anharmonic with a steep rise and a slower decline. Its analysis into a Fourier series gives for the ratios of the successive sine coefficients (the cosine coefficients are small)[8]:

$$1:0.39:0.21:0.14:0.09.$$

Figure 1, Curve B, shows the oscillation curve for $f=-0.1$, which is beyond the critical point (i.e., in the region of discontinuity or jump).

$\xi_\lambda=3.36$, and the nonlinear frequency ν is $\simeq$1061 Mc/sec. We find a further decrease in frequency with increase of amplitude.

Harmonic analysis of Fig. 1B gives the following ratios of the successive (sine) coefficients:

$$1:0.46:0.29:0.22:0.17.$$

The anharmonicity has increased with increase of amplitude.

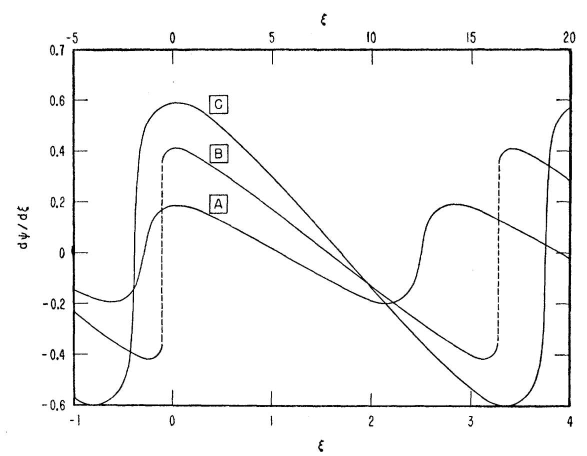

FIG. 1. The steady-state oscillation curves of the electric field. The lower scale of the abscissa refers to Curves A and B, the upper refers to Curve C.

7. APPLICATION TO SOLAR RADIO NOISE

We shall apply the above theory to solar material moving through the ionized plasma of the static corona, which is supposed to account for the abnormal radio noise received from the sun.[2] The following parameters that give space-charge wave amplification in small-signal theory are taken from a paper by the author[9]: $n_{10}=1.68\times10^8$ cm^{-3}, $n_{20}=0.89\times10^8$ cm^{-3}, $v_{10}=5\times10^7$ cm/sec, $v_{20}=0$, $V=5\times10^6$ cm/sec.

Figure 1, Curve C, gives the anharmonic oscillation of the electric field at the critical point ($\xi_\lambda=20.8$). The corresponding nonlinear oscillation frequency is $\nu\simeq87$ Mc/sec. The successive Fourier (sine) coefficients are in the following ratio:

$$1:0.45:0.23:0.15:0.12.$$

8. ABSORPTION OF THE FUNDAMENTAL AND THE SECOND HARMONIC AT THE PLASMA FREQUENCY LEVEL

Australian workers[3] have recently detected the second harmonic component in solar radio outbursts, and invoked the selective absorption of the fundamental at the plasma frequency to explain the fact that the intensity of the second harmonic was found to be comparable with, or greater than, that of the fundamental. We shall examine this hypothesis quantitatively on the nonlinear theory.

The absorption per unit path length of *intensity* of radiation of angular frequency ω in an ionized gas (as the solar corona) is given by[10]

$$k=\frac{4\pi n e^2\nu}{m\omega^2 c\mu}, \quad (19)$$

[7] This assumption is founded on our interest in regular oscillations and not in "jumps."

[8] We should in fact give the ratio of the total amplitudes, i.e., $\surd(\sin^2+\cos^2)$. Since, however, we are interested only in orders of magnitude, we have omitted the cosine terms which are very small.

[9] Hari K. Sen, Australian J. Phys. **6**, 67 (1953).

[10] S. K. Mitra, *The Upper Atmosphere* (The Royal Asiatic Society of Bengal, Calcutta, 1952), p. 184.

where

$$\mu=\left(1-\frac{4\pi ne^2}{m\omega^2}\right)^{\frac{1}{2}}, \tag{20}$$

n is the electron concentration, ν is the collisional frequency, and c is the velocity of light. The formula (19) holds when $\nu\ll\omega$.

We suppose that the radiation originates at the level of plasma frequency[11] where the electron concentration is n_0 and the collisional frequency ν_0. As we are interested in an estimate only, we shall also suppose that the electron concentration, n, and the collisional frequency, ν, follow the exponential laws:

$$n=n_0e^{-\beta h}, \quad \nu=\nu_0e^{-\beta h}, \tag{21}$$

where $1/\beta$ is the scale height of the corona ($\simeq 10^{10}$ cm).[12] Then, for the absorption of the fundamental (which is equal to the plasma frequency), the optical depth τ_f is given by

$$\tau_f=\int_0^\infty k_f dh=\frac{\nu_0}{c}\int_0^\infty \frac{e^{-2\beta h}dh}{(1-e^{-\beta h})^{\frac{1}{2}}}$$
$$=(4/3)(\nu_0/c\beta)=1.78, \tag{22}$$

if one takes $\nu_0\simeq 4$ sec^{-1} at the level ($\simeq 10^{10}$ cm above photosphere) of plasma frequency ($\simeq 87$ Mc/sec), from Smerd's[13] data. For the absorption of the second harmonic, the optical depth τ_s is given by

$$\tau_s=\int_0^\infty k_s dh=\frac{1}{2}\frac{\nu_0}{c}\int_0^\infty \frac{e^{-2\beta h}}{(4-e^{-\beta h})^{\frac{1}{2}}}dh$$
$$=\frac{\nu_0}{c\beta}\left(\frac{16}{3}-3\sqrt{3}\right)=0.18. \tag{23}$$

If we assume a constant conversion factor for space charge into electromagnetic wave energy for the fundamental and the second harmonic, we have, from Sec. 7, the relative intensity at the origin (plasma frequency level) given by

$$I_{10}/I_{20}=1/(0.45)^2. \tag{24}$$

From Eqs. (22)–(24), we have the relative received intensity given by

$$I_1/I_2=I_{10}e^{-(\tau_f-\tau_s)}/I_{20}\simeq 1. \tag{25}$$

If we take Hagen's[14] value of $\nu_0\simeq 6$ sec^{-1}, I_1 will be even less than I_2.

We thus see that the nonlinear theory as outlined above confirms the observational fact of the fundamental radio frequency being comparable with the second harmonic in intensity.[15]

9. ESTIMATE OF THE RADIO INTENSITY

From the observed radio flux we can form an estimate of the efficiency factor, α, for the conversion of space charge into electromagnetic wave energy.

$$\text{Poynting flux}=\alpha(c/8\pi)E^{*2}, \tag{26}$$

where E^* is given by (17). With our solar parameters (Sec. 7),

$$\text{Poynting flux}=3\alpha\times 10^4 \text{ ergs cm}^{-2}\text{ sec}^{-1}. \tag{27}$$

Assuming that the active area is one hundred millionths (10^{-4}) of the solar surface, and that the effective bandwidth of radiation is 50 Mc/sec, we get:

Radiation flux at the earth

$$=\frac{3\alpha}{5}\left(\frac{\text{sun's radius}}{\text{earth-sun distance}}\right)^2\times 10^{-7}$$
$$\times \text{ergs cm}^{-2}\text{ sec}^{-1}\text{ (cps)}^{-1}$$
$$\simeq 10^{-15}\alpha \text{ watt m}^{-2}\text{ (cps)}^{-1}. \tag{28}$$

$$\text{Observed}^3 \text{ flux} \simeq 10^{-19} \text{ to } 10^{-20} \text{ watt m}^{-2}\text{ (cps)}^{-1}. \tag{29}$$

Hence the efficiency factor is

$$\alpha\simeq 10^{-4} \text{ to } 10^{-5}. \tag{30}$$

The efficiency factor α given by (30) is of the same order as that obtained[9] on the assumption that the source of the available energy is the initial difference of energy between the interacting streams.

The extremely low value of α obtained in (30) indicates that the physical mechanism of conversion of space-charge wave into electromagnetic radiation energy need not be efficient at all. Even a very weak coupling between longitudinal and transverse waves, such as is provided by a transverse magnetic field or mass velocity[12] or nonlinear effects, may suffice for such conversion. In the following section, we shall consider one such mechanism.

10. ANTENNA THEORY OF ELECTROMAGNETIC RADIATION FROM OSCILLATING PLASMA

J. Feinstein and the author[16] had previously suggested that the very rapid growth of space-charge waves found on the small signal theory was responsible for conversion of the longitudinal plasma wave energy into transverse electromagnetic radiation. This suggestion was followed up by Feinstein in a letter[17] in which,

[11] J. C. Jaeger and K. C. Westfold, Australian J. Sci. Research **A3**, 376 (1950).

[12] H. C. van de Hulst, *A Course in Radio Astronomy* (Leiden, 1951), p. 62.

[13] S. F. Smerd, Proc. Inst. Elec. Engrs. **97**, 448 (1950).

[14] John P. Hagen, Astrophys. J. **113**, 557 (1951).

[15] The results of this section should indicate orders of magnitude only, as we have employed the linear damping theory for the damping of the nonlinear oscillation. We note also that the nonlinear theory gives an appreciable third harmonic, which should be observable.

[16] See Feinstein and Sen, reference 2, p. 411, Sec. 6.

[17] J. Feinstein, Phys. Rev. **85**, 145 (1952).

extending the conventional analysis of the oscillating current in a radiating antenna, he showed that the rapid spatial changes in the current distribution of the plasma effected the coupling of energy from the oscillating plasma into an electromagnetic radiation field in free space. This section contains a further development of the same idea, which leads to an estimate of the efficiency factor, α, of Sec. 9.

We shall suppose that the oscillating current $I(x)$ in the plasma (of length L) quickly builds up to saturation in a length l($\simeq$wavelength $\lambda=2\pi/\Gamma$)[12] and then slowly decays:

$$\left.\begin{aligned} I(x)&=I_0e^{(b+i\Gamma)x}, \quad 0\leqslant x\leqslant l;\\ &=I_0e^{(\alpha+b)l+(i\Gamma-\alpha)x}, \quad l\leqslant x\leqslant L.\end{aligned}\right\} \tag{31}$$

In (31) $b\simeq\Gamma$, $\alpha\ll\Gamma$ (decay is small in one wavelength), and $\alpha L\gg 1$. Also $\Gamma/k=c/V\simeq c/v$, where v, the stream speed, is of the order of V, the phase velocity of space-charge wave, for effective interchange of energy between wave and stream.[18] For the solar corpuscular streams, $v\ll c$. Hence we may take $\Gamma\gg k$.

At distances far (compared to the plasma dimensions) from the oscillating current in the plasma, the electromagnetic field is transverse to the direction of propagation. At a point (R, θ, ϕ in polar coordinates), the electric field, E_θ, and magnetic field, H_ϕ, are given by[19]:

$$E_\theta=\left(\frac{\mu_0}{\epsilon_0}\right)^{\frac{1}{2}}H_\phi=-\frac{i\omega\mu_0}{4\pi}\frac{\sin\theta}{R}e^{ikR-i\omega t}\times\int_0^\infty e^{-ikx\cos\theta}I(x)dx, \tag{32}$$

where ϵ_0 is the dielectric constant, μ_0 the permeability, and k the wave number of propagation, in free space, and $I(x)$ is given by (31).

The rate of energy radiated is obtained by integrating over the sphere of radius R the real part of the complex Poynting vector,

$$S^*=\tfrac{1}{2}E_\theta H_\phi^*, \tag{33}$$

where H_ϕ^* is the complex conjugate of H_ϕ. We thus obtain, with the indicated approximations, that the oscillating current $I(x)$ in (31) radiates electromagnetic energy at a distant point at the following rate:

$$(c\mu_0/24\pi)I_0^2e^{2bl}(k^2/\Gamma^2). \tag{34}$$

For electromagnetic wave growth, Γ in (34) must be replaced by k, and the corresponding rate of energy radiated will be

$$(c\mu_0/24\pi)I_0^2e^{2bl}. \tag{35}$$

From (34) and (35), we see that the efficiency factor, α, is given by[20]

$$\alpha=(k/\Gamma)^2\simeq(v/c)^2. \tag{36}$$

The time delay in the arrival of magnetic storms[21] indicates the speed of the solar corpuscles to be of the order of 1500 km/sec. Hence we obtain from (36) the following estimate for the efficiency factor:

$$\alpha=2.5\times10^{-5}, \tag{37}$$

which is of the same order as that in (30).

The author is indebted to Mr. James W. Lowry for the graphical analysis in Appendix II and the computational work.

APPENDIX I. DERIVATION OF THE NONLINEAR DISPERSION RELATION BY THE BOLTZMANN EQUATION

As the direction of wave propagation provides a convenient axis of reference, it will be sufficient to restrict our consideration to one dimension.[22] Let $f(v,x,t)$ be the normalized velocity-distribution function for the electrons satisfying the Boltzmann equation[23]

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+\frac{e}{m}\frac{\partial\phi}{\partial x}\frac{\partial f}{\partial v}=0, \tag{1}$$

where ϕ is the electric potential due to the space charge. We neglect the oscillatory motion of the ions, on account of their larger mass. We suppose that the ions are present in sufficient numbers to cancel the static negative charge of the electrons. We also consider the plasma to be rarefied enough to justify the neglect of collisions. In most plasmas, collisions produce a small damping of oscillations, which can usually be neglected.[18]

We assume a traveling wave solution of Eq. (1), so that all oscillating quantities are functions of x', where

$$x'=x-Vt, \tag{2}$$

V being the phase velocity of the wave. With the substitution (2), Eq. (1) transforms into

$$(v-V)\frac{\partial f}{\partial x'}+\frac{e}{m}\frac{d\phi}{dx'}\frac{\partial f}{\partial v}=0. \tag{3}$$

[18] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851, 1864 (1949).
[19] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 440. Following Stratton, we have employed the rationalized mks system of units in Sec. 10.

[20] We note that our case is not strictly the standard antenna one in that a linear current flows in a plasma, and not in a vacuum. The presence of the plasma is responsible for the ratio of the two different propagation constants, Γ and k, of space-charge and free space waves respectively, which appears in formula (36). The author is indebted to the referee of this paper for drawing his attention to the fact that a rigorous discussion should take into account the gradually varying properties of the plasma. Such a discussion would indeed be valuable. As the plasma density decreases outward, however, we do not believe that a more refined treatment would give a greatly diminished radiation yield.
[21] See reference 7, p. 452.
[22] The analysis can be generalized to three dimensions.
[23] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, England, 1952), p. 46.

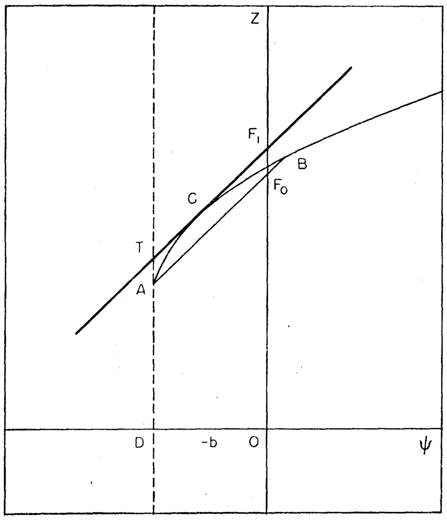

FIG. 2. Graphical analysis of Eq. (15).

The equations of the characteristics of (3) are

$$\frac{dx'}{v-V}=\frac{dv}{(e/m)(d\phi/dx')}=\frac{df}{0}. \tag{4}$$

Independent integrals of (4) are

$$(e/m)\phi-\tfrac{1}{2}(v-V)^2=c_1,\qquad f=c_2, \tag{5}$$

where c_1, c_2 are arbitrary constants. Hence $f\left((e/m)\phi-\tfrac{1}{2}(v-V)^2\right)$ is the solution of (3), where f is an arbitrary function.

The Poisson equation is

$$\epsilon_0\frac{d^2\phi}{dx'^2}=-4\pi en_++4\pi en_-\int f(v,x')dv, \tag{6}$$

where $n_+=n_-$ is the ion or electron concentration. Substituting for f in (6), we obtain the following integro-differential equation for ϕ:

$$\epsilon_0\frac{d^2\phi}{dx'^2}=-4\pi en_++4\pi en_-\int f\left(\frac{e}{m}\phi-\frac{(v-V)^2}{2}\right)dv. \tag{7}$$

We assume a steady-state solution, oscillatory in the laboratory system and static in the wave system of coordinates, so that the number of particles of any given velocity has become constant at each point in space. To specify the arbitrary function f in (7), we shall suppose[24] that the final (steady-state) distribution of velocities at a given point, say, at $\phi=0$, is $f_0(v)$, i.e.,

$$f\left(-\tfrac{1}{2}(v-V)^2\right)=f_0(v). \tag{8}$$

To express the arbitrary function f in (7), in terms of f_0, we make the following substitution:

$$(e/m)\phi-\tfrac{1}{2}(v-V)^2=-\tfrac{1}{2}(v'-V)^2. \tag{9}$$

[24] See reference 14, p. 1853.

The integro-differential Eq. (7) then transforms into:

$$\epsilon_0\frac{d^2\phi}{dx^2}=-4\pi en_++4\pi en_-\int\frac{f_0(v)dv}{\left(1+\dfrac{2e\phi(x)}{m(v-V)^2}\right)^{\frac{1}{2}}}, \tag{10}$$

where we have dropped the primes in (10).

Equation (10) is the nonlinear dispersion relation for electrons with a continuous velocity distribution (see Bohm and Gross).[4] To derive the dispersion relation for s discrete beams, we shall suppose that the normalized velocity distribution function $f_{s0}(v)$ for the sth beam (of concentration n_{s0}) is given by the delta-function:

$$f_{s0}(v)=\delta(v-v_{s0}). \tag{11}$$

Substitution of (11) in (10) gives the following nonlinear dispersion relation for s discrete beams:

$$\epsilon_0\frac{d^2\phi}{dx^2}=4\pi e\left[\sum_1^s\frac{n_{s0}|V-v_{s0}|}{[(2e/m)\phi+(V-v_{s0})^2]^{\frac{1}{2}}}-n_+\right]. \tag{12}$$

Equation (12) is the generalization of the relation (10) in Sec. 2.

APPENDIX II. GRAPHICAL ANALYSIS OF EQ. (15), SEC. 4

Equation (15) in Sec. 4 may be written in the abbreviated form:

$$(d\psi/d\xi)^2=a(\psi+b)^{\frac{1}{2}}+c(\psi+d)^{\frac{1}{2}}-\psi-f, \tag{1}$$

where a, b, c, d, and f are constants. The quantities a, b, c, and d are positive, and we may suppose without loss of generality that $b<d$.

The maxima and the minima of $\psi(\xi)$ are at the zeros of (1), which are obtained from the intersection of the curve (ACB in Fig. 2)

$$z=a(\psi+b)^{\frac{1}{2}}+c(\psi+d)^{\frac{1}{2}}, \tag{2}$$

with the straight line

$$z=\psi+f. \tag{3}$$

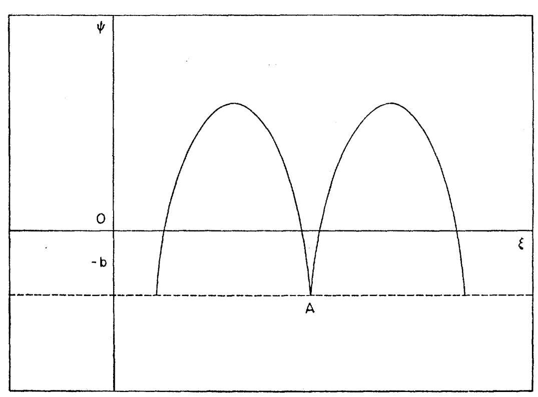

FIG. 3. Oscillation curve of the electric potential beyond the critical point.

In order that ψ may have both a maximum and a minimum, the intercept, f, of the straight line (3) on the z-axis must lie between $f_0=0F_0$ and $f_1=0F_1$. The tangent TCF_1 and the line AF_0B in Fig. 2 mark the extreme positions of (3). From the condition of tangency of line (3) to curve (2), it may be shown that for the position TCF_1 of (3), the amplitude E^* of the electric field in Eq. (15), Sec. 4, vanishes.

When the line (3) lies between TCF_1 and AF_0B, the maxima and minima of $\psi(\xi)$ occur at different values of ψ. So long as (3) lies near TCF_1 (small E^*), the difference between $\psi_{\max}$ and $\psi_{\min}$ is small, and the oscillation may be considered simple harmonic. With increasing E^*, however, the maxima and minima get progressively further apart, and the oscillation becomes increasingly anharmonic.

The point F_0 is obtained from the intersection of (3) with (2) at $\psi=-b$. This gives

$$f_0=0F_0=b+c(d-b)^{\frac{1}{2}}. \tag{4}$$

When the line (3) intercepts the z-axis below $0F_0$, ψ has a maximum but does not reach the minimum given by the vanishing of (1), as $d\psi/d\xi$ in (1) becomes imaginary when $\psi<-b$. The oscillation curve of ψ has the form shown in Fig. 3. There is a discontinuity in $d\psi/d\xi$ and hence in the electric field at the minima of ψ (A in Fig. 3).

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Transmission of Slow Neutrons by Liquid Helium*

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We have determined the total scattering cross section of liquid helium for neutrons of wavelengths between 3 and 16 angstroms and at six helium temperatures between 1.25 and 4.6°K. The neutrons were obtained from the thermal beam of a reactor and were monochromatized by a low resolution velocity selector. The total scattering cross section decreases with temperature at all wavelengths studied. At the shortest wavelengths the cross section approximates the free atom value; it exhibits a rapid drop with increasing wavelength. For the 4.6° liquid it passes through a shallow minimum at about 10 A. At lower helium temperatures, the fall from the free atom cross section is steeper; the existence of a minimum has not been established. The results are discussed qualitatively on the basis of several models of liquid helium. The only one predicting the overall features of the change of cross section with wavelength and temperature is the solid model.

1. INTRODUCTION

IN general, experimental investigations of the properties of liquid helium[1] use methods whereby gross or average properties are studied. Even in the investigations of the saturated and unsaturated helium film,[2] and in studies of first and second sound,[3] the thickness of the film or the wavelength of the periodic motions has been many atomic distances. Hence, these experiments involve sufficiently large numbers of atoms to permit a first treatment by thermodynamics.

An exception to this generalization is the studies of scattering of x-rays and slow neutrons by liquid helium. When the wavelength of the incident radiation is comparable with the atomic spacing, the observed phenomena are governed by the local or atomic rather than the bulk properties. Such studies help to reveal the atomic properties of the liquid, the spatial arrangement of the atoms, the mean forces between them, their mean velocity, and kinetic energy. They might even yield the laws of their velocity distribution.

Neutron scattering studies are of interest, not primarily as an independent check or improvement on x-ray results, but to complement them. Aside from electronic excitation, the vanishing rest mass of the photons inhibits the exchange of energy with the liquid. In addition, the velocities of photon and molecule are so different that the internal motion of the scatterer is unimportant. Because of these facts, the x-ray diffraction pattern is, essentially, the momentum space Fourier transform of the static radial distribution of the atoms around one chosen arbitrarily, as first shown by Zernike and Prins.[4] Many workers have used this transformation to derive the pair distribution from the scattering diagrams.[5] In the case of the scattering of slow neutrons, however, the masses and velocities of incident particle and scatterer may be comparable. As discussed later

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[1] W. H. Keesom, *Helium* (Elsevier, Amsterdam, 1942).

[2] See the review of E. A. Long and L. Meyer, Phil. Mag. Supplement 2, 1 (1953).

[3] For a recent survey see K. R. Atkins, Phil. Mag. Supplement 1, 169 (1952).

[4] F. Zernike and J. A. Prins, Z. Physik. 41, 184 (1927).

[5] See the review of N. S. Gingrich, Revs. Modern Phys. 15, 90 (1943).