above data we estimate that $\mu^+ \to e^+ + \gamma$ occurs less than $6\times 10^{-4}$ as frequently as the process $\mu^+ \to e^+ + 2\nu$. This means that the decay $\mu^+ \to e^+ + \gamma$ has a lifetime longer than $10^{-3}$ second. This value of the lifetime is about 100 times longer than the lower limits previously given in the literature.[4,6]

It might be mentioned that if the same strength of interaction were responsible for both reactions (A) and (B) that the result obtained on reaction (B) is more significant than the result on reaction (A) by many orders of magnitude in setting a limit on the strength of the interaction.

[6] Since these measurements were performed, we have been informed by J. Steinberger that he and S. Lokanathan have found that the lifetime for the decay $\mu \to e + \gamma$ is appreciably longer than the value we obtained. S. Lokanathan and J. Steinberger, Abstract for the Chicago, Illinois meeting of the American Physical Society, November, 1954 [Bull. Am. Phys. Soc. 29, No. 7, 25 (1954)].

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# Biquadratic Spinor Identities

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The representation-independent biquadratic identities which Pauli has proved to hold between a Dirac wave function and its adjoint are shown to be generalizable in several ways. To obtain these generalizations it is first shown how the symmetrical Kronecker product of two spinor representations of the orthogonal group in $n$ dimensions decomposes. Then a method is given to express the tensors formed in a particular way from two covariant and two contravariant spinors in terms of those formed in any other way. These results are applied to write explicitly all biquadratic scalar and pseudoscalar identities in $2\nu$ dimensions and all scalar identities in $2\nu+1$ dimensions. The way to obtain more general tensor identities is indicated.

## BIQUADRATIC SPINOR IDENTITIES

Soon after the discovery of the Dirac equation it was noted by Fock and Darwin[1] that there are some quadratic identities which hold between the scalars obtained from the five covariants which can be formed with a Dirac wave function and its adjoint. The proof consisted of writing down these scalars with a particular choice of the matrices and comparing terms. Additional identities involving the pseudoscalars, vectors, and pseudovectors which can be formed from the square of a wave function and the square of its adjoint were given by Uhlenbeck and Laporte.[2] Again a special representation was used.

In 1936 Pauli[3] gave a representation-independent proof of the scalar and pseudoscalar identities. A proof from a different point of view was given by Harish-Chandra,[4] who also proved some of the other identities given in reference 2. In addition Harish-Chandra obtained some relations which hold between tensors formed with two arbitrary wave functions and their adjoints. To a certain extent these relations are similar to results of Fierz.[5] Fierz was interested in expressing the scalars occurring in beta decay interaction terms formed from one arrangement of wave functions in terms of those with another arrangement.

In the following a systematic method is described which gives directly all identities of the desired type. It is hoped that the general mathematical structure of these relations will become particularly clear. The origin of the identities can be stated, group theoretically, quite succinctly. Certain of the irreducible representations of the orthogonal group which occur in the direct product of the spinor representation with itself do not occur in the symmetrized product.

The theory of spinors in $n$ dimensions has been shown by Brauer and Weyl[6] to be extremely similar to the Dirac (four-dimensional) case. Since the essence of the argument to be used is independent of the number of dimensions and since it is desired to shed light on the mathematical basis, we show how to obtain directly all the biquadratic identities existing between $n$-dimensional spinors. As an essential step, the decomposition of the symmetrical Kronecker square of the spin representation of the orthogonal group is obtained.

Briefly, the method is the following. First we show that certain of the covariants formed with two covariant (or with two contravariant) spinors vanish when the spinors are identical. Second, the work of Fierz is generalized. The biquadratic covariants, formed with two covariant and two contravariant spinors by combining the first and second sets separately into tensors and then combining the two sets of resulting tensors, are expressed in terms of the covariants obtained by first combining a covariant with a contravariant spinor in the more usual manner. Using these two groups of results the desired identities are obtained. It will be

[1] V. Fock, Z. Physik 57, 261 (1929); C. G. Darwin, Proc. Roy. Soc. (London) 120, 621 (1928).
[2] G. E. Uhlenbeck and O. Laporte, Phys. Rev. 37, 1552 (1931).
[3] W. Pauli, Ann. inst. Henri Poincaré 6, 109, (1936).
[4] Harish-Chandra, Proc. Indian Acad. Sci. 22, 30 (1945).
[5] M. Fierz, Z. Physik 104, 553 (1937).
[6] R. Brauer and H. Weyl, Am. J. Math. 57, 425 (1935).

readily apparent that *all* identities of any given type are found and that all the identities given are independent.

In Sec. II a short summary of the relevant results of Brauer and Weyl[6] are given in a form similar to that used by Pauli.[3] The representation of the matrices is not specified. Section III discusses the connection between covariant and contravariant spinors by means of a generalization of an argument of Pauli and Haantjes.[3] The results of III are applied in IV to obtain the decomposition of the symmetrical product of two spin representations. The differences between even and odd dimensions require separate discussions for the identities. These are given in V and VI. Some physical applications, primarily to beta decay, are indicated. Most computational details are relegated to Appendices, as are some of the arguments which differ in only minor ways from those given in the major part of the text.

## II. SUMMARY OF SPINOR PROPERTIES

For brevity we confine the discussion to the spin representations of the complex orthogonal group. No essential modifications are necessary for the groups obtained on restriction to real variables except that the definitions of some tensors formed with the spinors might be more conveniently given by prefixing various factors of $i$.

By following Brauer and Weyl,[6] it is simplest to discuss the cases of even and odd dimensions separately.

### $n=2\nu$

We consider a set of $2\nu$ quantities $\Gamma(i)$ satisfying

$$\Gamma(i)\Gamma(j)+\Gamma(j)\Gamma(i)=2\delta_{ij}. \tag{1}$$

These quantities together with a unit element (1) and its negative generate a group with elements:

$$\begin{array}{ll} 1 & -1 \\ \Gamma^{(1)} & -\Gamma^{(1)} \\ \Gamma^{(2)} & -\Gamma^{(2)} \\ \cdots & \cdots \\ \Gamma^{(2\nu)} & -\Gamma^{(2\nu)}, \end{array} \tag{2}$$

where $\Gamma^{(f)}$ is the set of $\binom{2\nu}{f}$ quantities[7]:

$$\Gamma^{(f)}(i_1i_2\cdots i_f)=i^{[f/2]}\Gamma(i_1)\Gamma(i_2)\cdots\Gamma(i_f),\qquad i_1<i_2<\cdots<i_f. \tag{3}$$

The power of $i$ in (3) is chosen so that for any element $\Gamma_A$ of the form

$$\Gamma_A=\Gamma^{(f)}(i_1i_2\cdots i_f),$$

we have

$$(\Gamma_A)^2=1. \tag{4}$$

[7] $[a]$ denotes the largest integer contained in $a$.

The group (2) has one faithful irreducible representation. From now on the symbols $\Gamma$ will denote the matrices of this representation. Their degree is $2^\nu$. Since, for any element of (2) which is not (1) or $(-1)$, both $\Gamma_A$ and $-\Gamma_A$ are members of the same class, we have

$$\mathrm{tr}\Gamma_A=-\mathrm{tr}\Gamma_A=0. \tag{5}$$

As a consequence the $2^{2\nu}$ matrices,

$$1,\ \Gamma^{(1)}(i_1)\cdots,\ \Gamma^{(2)}(i_1i_2)\cdots,\ \Gamma^{(2\nu)}(i_1\cdots i_{2\nu}), \tag{6}$$

are linearly independent.

**Proof**

Let

$$a^{(f)}\cdot b^{(f)}=\sum_{i_1<i_2<\cdots<i_f} a^{(f)}(i_1i_2\cdots i_f)b^{(f)}(i_1i_2\cdots i_f). \tag{7}$$

If the quantities (6) are not linearly independent there exists a nontrivial relation of the form

$$\sum_{f=0}^{2\nu} a^{(f)}\cdot\Gamma^{(f)}=0. \tag{8}$$

Multiplying (8) by $\Gamma^{(f')}(j_1j_2\cdots j_{f'})$, noting that

$$\Gamma_A\Gamma_B\neq 1\quad (A\neq B), \tag{9}$$

and taking the trace yields

$$a^{(f')}(j_1j_2\cdots j_{f'})=0\quad (\text{all } a^{(f')}). \tag{10}$$

Hence there is no relation of the form (8).

It may be remarked that since the group (2) is finite we can (and always will) take the matrices $\Gamma_A$ to be unitary. Since the square of any $\Gamma_A$ is 1 the matrices will also be Hermitian.

The matrices $\Gamma_A$ of (6), being linearly independent and $2^{2\nu}$ in number, provide a basis by means of which all $2^\nu\times 2^\nu$ matrices can be linearly expressed.

The spin representation of the orthogonal group is obtained by noting that corresponding to an orthogonal transformation,

$$x_i'=\theta_{ik}x_k, \tag{11}$$

we can find a new set of matrices $\Gamma'(i)$,

$$\Gamma'(i)=\theta_{ki}\Gamma(k), \tag{12}$$

which also satisfy (1). These new matrices then give another representation of the group (2). By Schur's lemma the two representations are equivalent and

$$\Gamma'(i)=S(\theta)\Gamma(i)S^{-1}(\theta)\quad (\text{all } i). \tag{13}$$

A set of $2^\nu$ quantities which transform as

$$\psi_A'=\textstyle\sum_B S_{AB}(\theta)\psi_B(\theta) \tag{14a}$$

or

$$\psi'=S(\theta)\psi \tag{14b}$$

are called covariant spinors. Quantities which transform as[8]

$$\psi^{\dagger\prime}=S^{t-1}\psi^{\dagger}=\psi^{\dagger}S^{-1} \tag{15}$$

are contravariant spinors.

The connection between covariant and contravariant spinors is obtained by noting that if the $\Gamma(i)$ are a fixed set of matrices giving a representation of (2), the matrices $\Gamma^t(i)$ also satisfy (1). Hence these matrices give an equivalent representation and

$$\Gamma^t(i)=C\Gamma(i)C^{-1}, \tag{16}$$

with some nonsingular matrix $C$. The matrices $S(\theta)$ are defined by (13) only up to a multiplicative constant. In reference 6 it is shown that this constant can be chosen so that

$$(S^t)^{-1}(\theta)=CS(\theta)C^{-1}. \tag{17}$$

Using this matrix $C$ we can associate a contravariant spinor $\psi$ with a covariant spinor $\psi$ by means of the relation

$$\psi^{\dagger}=C\psi. \tag{18}$$

Further properties of $C$ are discussed below.

### $n+2\nu+1$

Again we consider quantities $\Gamma(i)$ $(i=1,2,\cdots,2\nu+1)$, $2\nu+1$ in number, satisfying (1). Forming the quantities

$$\begin{array}{ll} 1 & -1 \\ \Gamma^{(1)} & -\Gamma^{(1)} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \Gamma^{(2\nu)} & -\Gamma^{(2\nu)} \\ U=i^\nu\Gamma(1)\Gamma(2)\cdots\Gamma(2\nu+1) & -U, \end{array} \tag{19}$$

we obtain a group of $2^{2\nu+2}$ elements. The only irreducible representations of dimension greater than one are two inequivalent $2^\nu$ dimensional representations. Since $U$ commutes with all the elements, we obtain by Schur's lemma:

$$U=b1. \tag{20}$$

But

$$U^2=1, \tag{21}$$

and thus

$$b=\pm 1. \tag{22}$$

The two inequivalent representations can then be characterized by

$$U=1 \quad\text{or}\quad U=-1. \tag{23}$$

That these representations are inequivalent is readily seen by noting that the character of $U$ is $2^\nu$ or $-2^\nu$, respectively.

The argument given previously shows that all traces except those of $1, -1, U, -U$ are zero. Applying the same method as gave (1) shows that the even rank quantities ($\Gamma^{(0)}\equiv 1$, $\Gamma^{(2)}$, $\Gamma^{(4)}$, $\cdots$) are linearly independent. In number they are

$$N=\sum_{i=0}^{\nu}\binom{2\nu+1}{2i}=2^{2\nu}, \tag{24}$$

i.e., the even-rank quantities are just sufficient to provide a basis for all $2^\nu\times 2^\nu$ matrices.

The connection with the orthogonal group proceeds as for even dimensions. We restrict ourselves to the pure rotation group. (Representations of the full orthogonal group can be obtained from the one below by adding the commuting operation of reflection.) Let

$$\Gamma(i) \quad i=1,2,\cdots,2\nu \tag{25}$$

be the matrices of the irreducible representation for the case $n=2\nu$. Choosing

$$\Gamma(2\nu+1)=i^\nu\Gamma(1)\Gamma(2)\cdots\Gamma(2\nu) \tag{26}$$

gives a representation of (19) with $U=1$.

Associated with a rotation,

$$x_i'=\theta_{ik}x_k, \tag{27}$$

we have a new set of matrices $\Gamma'(i)$,

$$\Gamma'(i)=\theta_{ki}\Gamma(k), \tag{28}$$

which also satisfy (1) and thus give a $2^\nu$ dimensional representation of (19). Since

$$U'=(\det\theta)U=U=1, \tag{29}$$

we see this new representation is equivalent to the original representation. Hence there is a nonsingular $S(\theta)$ such that

$$\Gamma'(i)=S(\theta)\Gamma(i)S^{-1}(\theta). \tag{30}$$

The covariant-contravariant connection is also similar to that for $n=2\nu$. Consider the matrices $\Gamma(i)$ $(i=1,2,\cdots,2\nu+1)$, given by (25) and (26). The matrices $\Gamma^t(i)$ satisfy the same relations except, perhaps, for the condition $U=1$. Direct use of the relations (1) yield

$$i^\nu\Gamma^t(1)\Gamma^{(t)}(2)\cdots\Gamma^t(2\nu+1)=(-1)^\nu, \tag{31a}$$

or

$$U'=(-1)^\nu. \tag{31b}$$

Hence if $\nu$ is even, the matrices $\Gamma^t(i)$ give an equivalent representation and thus

$$\Gamma^t(i)=C\Gamma(i)C^{-1}. \tag{32a}$$

Indeed, if $C_e$ be the matrix occurring in (16) for the case $n=2\nu$, we see we can take $C=C_e$, or

$$\Gamma^{(t)}(i)=C_e\Gamma(i)C_e^{-1}. \tag{32b}$$

If $\nu$ be odd, consider the quantities $-\Gamma(i)$ $(i=1,2,\cdots,2\nu+1)$. These also satisfy the relations (1) and give

$$\bar{U}=i^\nu[-\Gamma(1)][-\Gamma(2)]\cdots[-\Gamma(2\nu+1)] =(-1)^{2\nu+1}U=-1. \tag{33}$$

[8] $t$ denotes transposed, * complex conjugate, † Hermitian conjugate.

Hence, the matrices $-\Gamma(i)$ give the other, inequivalent, representation. For $\nu$ odd, the representation by means of $\Gamma^t(i)$ give a representation equivalent to this and so there is a $C$ such that

$$C\Gamma(i)C^{-1}=-\Gamma^t(i). \tag{34}$$

A matrix satisfying this is

$$C=C_e\Gamma(2\nu+1). \tag{35}$$

The equivalence of covariant and contravariant representations is then seen, since in either case we can write

$$C\Gamma(i)C^{-1}=(-1)^\nu\Gamma^t(i), \tag{36}$$

which implies that

$$[S^t]^{-1}=CS(\theta)C^{-1}. \tag{37}$$

## III. PROPERTIES OF THE MATRICES $C$

### $n=2\nu$

$C$ can be chosen unitary.

**Proof**

$$\Gamma^t(i)=C\Gamma(i)C^{-1}. \tag{16}$$

Take the Hermitian conjugate,

$$\Gamma^t(i)^\dagger=C^{-1\dagger}\Gamma^\dagger(i)C^\dagger, \tag{38}$$

or, since we are taking the $\Gamma(i)$ to be Hermitian,

$$\Gamma^t(i)=C^{-1\dagger}\Gamma(i)C^\dagger. \tag{39}$$

Then, inserting (16) in (39), we obtain

$$C^\dagger C\Gamma(i)=\Gamma(i)C^\dagger C. \tag{40}$$

By Schur's lemma,

$$C^\dagger C=\epsilon 1, \tag{41}$$

or

$$\epsilon=\textstyle\sum_j C^\dagger{}_{ij}C_{ji}=\sum_j|C_{ij}|^2. \tag{42}$$

Thus $\epsilon$ is real and positive. Dividing $C$ by $\epsilon^{1/2}$ we obtain

$$C^{-1}=C^\dagger; \tag{43}$$

i.e., $C$ is unitary.

We can obtain the symmetry properties of $C$ in a similar manner. From (16),

$$\Gamma^{tt}(i)=\Gamma(i)=[C^t]^{-1}\Gamma^{(t)}(i)C^t=[C^t]^{-1}C\Gamma(i)C^{-1}C^t, \tag{44}$$

or

$$[C^t]^{-1}C\Gamma(i)=\Gamma(i)[C^t]^{-1}C. \tag{45}$$

Hence

$$[C^t]^{-1}C=a1, \tag{46}$$

and

$$C=aC^t. \tag{47}$$

Thus

$$C^t=aC=C/a, \tag{48}$$

or

$$a^2=1,\quad a=\pm 1; \tag{49}$$

i.e., $C$ is either symmetric or antisymmetric. The determination of which proceeds by an argument of Pauli and Haantjes. Fortunately, this also yields much useful additional information.

Let us suppose first that $a=+1$. Then $C=C^t$. By repeated use of (16) and (1), we find:

$$C\Gamma^{(f)}C^{-1}=(-1)^{[f/2]}[\Gamma^{(f)}]^t, \tag{50}$$

or

$$C\Gamma^{(f)}=(-1)^{[f/2]}[C\Gamma^{(f)}]^t. \tag{51}$$

Hence

$$\begin{aligned}&C\Gamma^{(f)}\text{ is symmetric,} && f=0,1\pmod 4,\\ &C\Gamma^{(f)}\text{ is antisymmetric,} && f=2,3\pmod 4.\end{aligned} \tag{52}$$

We now ask how many matrices ($N_1$) there are of the form

$C\Gamma^{(f)}$, $f=0, 1 \pmod 4$,

$$N_1=\binom{2\nu}{0}+\binom{2\nu}{1}+\binom{2\nu}{4}+\binom{2\nu}{5}+\cdots. \tag{53}$$

It is readily shown that

$$N_1=(2^{2\nu}+2^\nu)/2,\quad \nu=0,1\pmod 4, \tag{54a}$$

$$N_1=(2^{2\nu}-2^\nu)/2,\quad \nu=2,3\pmod 4. \tag{54b}$$

The number of matrices ($N_2$) of form

$$C\Gamma^{(f)}\quad f=2,3\pmod 4$$

is

$$N_2=2^{2\nu}-N_1$$

$$=(2^{2\nu}-2^\nu)/2,\quad \nu=0,1\pmod 4 \tag{55a}$$

$$N_2=(2^{2\nu}+2^\nu)/2,\quad \nu=2,3\pmod 4. \tag{55b}$$

These matrices $C\Gamma^{(f)}$ are all linearly independent since the $\Gamma^{(f)}$ are all linearly independent and $C$ is unitary.

The number of linearly independent symmetric ($N_s$) or antisymmetric matrices ($N_{as}$) of degree $2^\nu$ are:

$$N_s=[2^\nu(2^\nu-1)/2]+2^\nu=(2^{2\nu}+2^\nu)/2, \tag{56a}$$

$$N_{as}=2^\nu(2^\nu-1)/2=(2^{2\nu}-2^\nu)/2. \tag{56b}$$

Comparing (54a) with (56a) we see the assumption $a=+1$ is correct for $\nu=0$, 1 (mod 4) but incorrect for $\nu=2$, 3 (mod 4). In the latter case we must have $a=-1$. Indeed from (52), (54), and (56) we can conclude further that:

If $\nu=0$, 1 (mod 4),

$$C\Gamma^{(f)}\text{ is symmetric,}\quad f=0,1\pmod 4; \tag{57a}$$

$$C\Gamma^{(f)}\text{ is antisymmetric,}\quad f=2,3\pmod 4. \tag{57b}$$

If $\nu=2$, 3 (mod 4),

$$C\Gamma^{(f)}\text{ is antisymmetric,}\quad f=0,1\pmod 4; \tag{58a}$$

$$C\Gamma^{(f)}\text{ is symmetric,}\quad f=2,3\pmod 4. \tag{58b}$$

TABLE I. Symmetry properties of $C\Gamma^{(f)}$ for $n=2\nu+1$.

| $\nu$ (mod 4) | $C\Gamma^{(f)}$ symmetric | $C\Gamma^{(f)}$ antisymmetric |
|---|---|---|
| 0 | $f=0, 1$ (mod 4) | $f=2, 3$ (mod 4) |
| 1 | $f=0, 3$ (mod 4) | $f=1, 2$ (mod 4) |
| 2 | $f=2, 3$ (mod 4) | $f=0, 1$ (mod 4) |
| 3 | $f=0, 3$ (mod 4) | $f=1, 2$ (mod 4) |

### $n=2\nu+1$

The unitarity of the matrix $C$ occurring in (36) follows directly from (43). For $\nu$ even, $C=C_e$ (32b), which is unitary. For $\nu$ odd:

$$\begin{aligned} C&=C_e\Gamma(2\nu+1),\\ C^\dagger&=\Gamma^\dagger(2\nu+1)C_e^\dagger=\Gamma^{-1}(2\nu+1)C_e^{-1}\\ &=C_e\Gamma^{-1}(2\nu+1)=C^{-1}. \end{aligned} \tag{35}$$

The symmetry properties of $C$ (and $C\Gamma^{(f)}$) follow directly from those for $C_e$. The results are given in Table I.

## IV. DECOMPOSITION OF THE SYMMETRICAL DIRECT PRODUCT

### $n=2\nu$

Consider two covariant spinors $\phi$, $\psi$. The $2^{2\nu}$ products $\phi_A\psi_B$ form a $2^\nu\times 2^\nu$ matrix which from the results of II can be expanded in terms of the matrices,

$$\Gamma^{(f)}(i_1i_2\cdots i_f)\equiv\Gamma^{(f)}(f_i). \tag{59}$$

Alternately, since $C$ is unitary we can expand using the quantities $\Gamma^{(f)}(f_i)C^{-1}$ as a basis. Thus:

$$\phi_A\psi_B=\sum_f a^{(f)}\cdot(\Gamma^{(f)}C^{-1})_{BA}. \tag{60}$$

To determine the coefficients $a^{(f)}(f_i)$, we multiply by

$$[C\Gamma^{(f')}(f'_i)]_{AB}, \tag{61}$$

and sum over $A$ and $B$:

$$\begin{aligned} (\phi,C\Gamma^{(f')}(f'_i)\psi)&=\sum_f a^{(f)}\cdot\sum_{AB} C\Gamma^{(f')}(f'_i)\Gamma^{(f)}C^{-1}\\ &=\sum_f a^{(f)}\cdot\mathrm{tr}C\Gamma^{(f')}(f'_i)\Gamma^{(f)}C^{-1}. \end{aligned} \tag{62}$$

From (9) and (5) we find:

$$\mathrm{tr}C\Gamma^{(f')}(f'_i)\Gamma^{(f)}C^{-1}=2^\nu\delta(i_1,i'_1)\delta(i_2,i'_2)\cdots. \tag{63}$$

Hence,

$$a^{(f)}=(\phi,C\Gamma^{(f)}\psi)/2^\nu, \tag{64}$$

or

$$\phi_A\psi_B=\sum_{f=0}^{2\nu}\frac{(\phi,C\Gamma^{(f)}\psi)\cdot(\Gamma^{(f)}C^{-1})_{BA}}{2^\nu}. \tag{65a}$$

The results of (57) and (58) for $C$ can be summarized as

$$C^t=(-1)^{[\nu/2]}C. \tag{66}$$

Thus (65a) can also be written:

$$\phi_A\psi_B=\sum_{f=0}^{2\nu}\frac{(-1)^{[\nu/2]}(C\phi,\Gamma^{(f)}\psi)\cdot(\Gamma^{(f)}C^{-1})_{BA}}{2^\nu}. \tag{65b}$$

Under an orthogonal transformation the spinors transform [(14b), (18), (15)] as:

$$\psi'=S(\theta)\psi,\quad \phi'=S(\theta)\phi,\quad C\phi'=(C\phi)S^{-1}(\theta).$$

Then (65b) becomes:

$$\phi_A'\psi_B'=\sum_{f=0}^{2\nu}\frac{(-1)^{[\nu/2]}(C\phi,S^{-1}\Gamma^{(f)}S\psi)\cdot(\Gamma^{(f)}C^{-1})_{BA}}{2^\nu}. \tag{67}$$

From (12) and (13) we then obtain the well-known result that the direct product $(S\times S)$ decomposes into the representations

$$S\times S\sim\sum_{f=0}^{2\nu}\{1^f\}, \tag{68}$$

where $\{1^f\}$ denotes the representation whose basis are the antisymmetric tensors of rank $f$.

Returning to (65a), we obtain on identifying $\phi$ with $\psi$:

$$\psi_A\psi_B=\sum_{f=0}^{2\nu}\frac{(\psi,C\Gamma^{(f)}\psi)\cdot(\Gamma^{(f)}C^{-1})_{BA}}{2^\nu}. \tag{69}$$

From (57) and (58) we know that if $\nu=0, 1$ $(=2, 3)$ mod 4 then $C\Gamma^{(f)}$ is antisymmetric for $f=2, 3$ $(=0, 1)$ mod 4. These terms in (69) must then vanish. Thus corresponding to the decomposition (68) we obtain for the decomposition of the symmetrical direct product $(S^{[2]})$:

$$S^{[2]}\sim{\sum_{f=0}^{2\nu}}'\{1^f\}, \tag{70}$$

$$f\neq 2, 3\quad\text{for}\quad \nu=0, 1\quad(\text{mod } 4)$$
$$f\neq 0, 1\quad\text{for}\quad \nu=2, 3\quad(\text{mod } 4).$$

As examples we have the familiar results:

$n=2$ $(\nu=1)$

$$\begin{aligned} &S\times S\sim\text{scalar}+\text{vector}+\text{pseudoscalar},\\ &S^{[2]}\sim\text{scalar}+\text{vector}; \end{aligned} \tag{71}$$

$n=4$ $(\nu=2)$

$$\begin{aligned} S\times S\sim&\text{scalar}+\text{vector}+\text{second rank tensor}\\ &+\text{pseudovector}+\text{pseudoscalar}, \end{aligned} \tag{72}$$

$$S^{[2]}\sim\text{second rank tensor}+\text{pseudovector};$$

$n=6$ $(\nu=3)$

$$\begin{aligned} S\times S\sim&\text{scalar}+\text{vector}+\text{second rank tensor}\\ &+\text{third rank tensor}+\text{fourth rank tensor}\\ &+\text{pseudovector}+\text{pseudoscalar}, \end{aligned} \tag{73}$$

$$\begin{aligned} S^{[2]}\sim&\text{second rank tensor}+\text{third rank tensor}\\ &+\text{pseudoscalar}. \end{aligned}$$

**$n=2\nu+1$**

The procedure is as above with the exception that now only the even rank quantities $\Gamma^{(2\lambda)}C^{-1}$ are used as a basis. For the product of two spinors we obtain:

$$\phi_A\psi_B=\sum_{\lambda=0}^{\nu}\frac{(\phi,C\Gamma^{(2\lambda)}\psi)\cdot(\Gamma^{(2\lambda)}C^{-1})_{BA}}{2^\nu}. \quad (74a)$$

From Table I

$$C^t=(-1)^{\alpha_\nu}C, \quad (75)$$

where

$$\begin{aligned}\alpha_\nu&=0, \quad \nu=0,\,1,\,3 \pmod 4\\ &=1, \quad \nu=2 \pmod 4).\end{aligned} \quad (76)$$

Hence (74a) can also be written:

$$\phi_A\psi_B=\sum_{\lambda=0}^{\nu}\frac{(-1)^{\alpha_\nu}(C\phi,\Gamma^{(2\lambda)}\psi)\cdot(\Gamma^{(2\lambda)}C^{-1})_{BA}}{2^\nu}. \quad (74b)$$

The same argument as leads to (68) gives the well known result:

$$S\times S\sim\sum_{\lambda=0}^{\nu}\{1^{2\lambda}\}. \quad (77)$$

Since the terms with $C\Gamma^{(2\lambda)}$ antisymmetric in (74a) vanish when $\phi$ and $\psi$ are identified we obtain from Table I the decomposition of the symmetrical product:

$$S^{[2]}\sim{\sum_{\lambda=0}^{\nu}}'\{1^{2\lambda}\}, \quad (78)$$

where the sum is over

$$\begin{aligned}&\lambda \text{ even}, \quad \nu=0,\,1,\,3 \pmod 4\\ &\lambda \text{ odd}, \quad \nu=2 \pmod 4).\end{aligned}$$

Examples are:

$n=3$ $(\nu=1)$:

$$\begin{aligned}S\times S&\sim\text{scalar}+\text{pseudovector},\\ S^{[2]}&\sim\text{scalar};\end{aligned} \quad (79)$$

$n=5$ $(\nu=2)$

$$\begin{aligned}&S\times S\sim\text{scalar}+\text{second rank tensor}+\text{pseudovector},\\ &S^{[2]}\sim\text{second rank tensor}.\end{aligned} \quad (80)$$

### V. IDENTITIES FOR $n=2\nu$

As a preliminary to obtaining the desired identities it is convenient to consider the relations between covariant quantities formed from two covariant and two contravariant spinors by combining the spinors in various arrangements. First we prove the following:

**Theorem**

Let $\psi$, $\phi(\psi^\dagger,\phi^\dagger)$ be two covariant (contravariant) spinors. (No relation is assumed between the two sets.) Then, if $\theta$, $\theta'$ are two arbitrary $2^\nu$ dimensional matrices,

$$(\psi^\dagger,\theta\phi)(\phi^\dagger,\theta'\psi)=\sum_{f=0}^{2\nu}\frac{(\psi^\dagger,\Gamma^{(f)}\psi)\cdot(\phi^\dagger,\theta'\Gamma^{(f)}\theta\phi)}{2^\nu}. \quad (81)$$

**Proof**

We can expand the matrix $\psi^\dagger{}_A\psi_B$ as:

$$\psi^\dagger{}_A\psi_B=\sum_{f'=0}^{2\nu}a^{(f')}\cdot(\Gamma^{(f')})_{BA}. \quad (82)$$

As was previously done we determine the coefficients $a^{(f)}$ by multiplying with $(\Gamma^{(f)})_{AB}$ and summing over $A$ and $B$. This gives

$$(\psi^\dagger,\Gamma^{(f)}(f_i)\psi)=\sum_{f'=0}^{2\nu}a^{(f')}\cdot\mathrm{tr}\Gamma^{(f)}(f_i)\Gamma^{(f')}=a^{(f)}(f_i)2^\nu, \quad (83)$$

and hence:

$$\psi^\dagger{}_A\psi_B=\sum_{f=0}^{2\nu}\frac{(\psi^\dagger,\Gamma^{(f)}\psi)\cdot(\Gamma^{(f)})_{BA}}{2^\nu}. \quad (84)$$

Multiplying (84) by $\theta_{AC}\phi_C\phi_D{}^\dagger\theta'_{DB}$ and summing over $ABCD$ gives (81).

To illustrate the utility of this result we consider the problem of expressing the scalars formed by first combining $\phi$ and $\psi^\dagger$, $\psi$ and $\phi^\dagger$ into tensors and then contracting in terms of the scalars formed by first combining $\psi^\dagger$ and $\psi$, $\phi^\dagger$ and $\phi$. From (68) it is clear that with two covariant and two contravariant spinors we can form $2\nu+1$ scalars. As a basis for all such scalars we can use the $2\nu+1$ scalars:

$$S^{(0)}{}_f(\psi^\dagger,\psi;\phi^\dagger,\phi)\equiv(\psi^\dagger,\Gamma^{(f)}\psi)\cdot(\phi^\dagger,\Gamma^{(f)}\phi). \quad (85)$$

That these are linearly independent is proved by assuming the converse. Then there exists a nontrivial relation of the form

$$\sum_{f=0}^{2\nu}A_f(\psi^\dagger,\Gamma^{(f)}\psi)\cdot(\phi^\dagger,\Gamma^{(f)}\phi)=0, \quad (86)$$

for all $\phi^\dagger$, $\phi$. But this implies

$$\sum_{f=0}^{2\nu}A_f(\psi^\dagger,\Gamma^{(f)}\psi)\cdot(\Gamma^{(f)})_{BA}=0, \quad \text{all } A,\,B, \quad (87)$$

i.e., the $\Gamma^{(f)}$ would not be linearly independent, contrary to a theorem proved previously. Hence, there exists a $2\nu+1$ dimensional matrix $(a_{\lambda f})$ by means of which the scalars obtained by grouping $\psi^\dagger$, $\phi$ and $\phi^\dagger$, $\psi$ in (85) can be linearly expressed in terms of the $S^{(0)}{}_f$. Thus if we define

$$S^{(0)\prime}{}_\lambda\equiv S_\lambda{}^{(0)}(\psi^\dagger,\phi;\phi^\dagger,\psi)\equiv(\psi^\dagger,\Gamma^{(\lambda)}\phi)\cdot(\phi^\dagger,\Gamma^{(\lambda)}\psi), \quad (88)$$

we have

$$S^{(0')}{}_\lambda=a_{\lambda f}S_f{}^{(0)}. \quad (89)$$

Substitution of

$$\theta=\theta'=\Gamma^{(\lambda)}(i_1i_2\cdots i_\lambda)\equiv\Gamma^{(\lambda)}(\lambda_i) \tag{90}$$

in Eq. (81) and summing over all components of $\Gamma^{(\lambda)}$ gives:

$$(\psi^\dagger,\Gamma^{(\lambda)}\phi)\cdot(\phi^\dagger,\Gamma^{(\lambda)}\psi)=\sum_{f=0}^{2\nu}\frac{(\psi^\dagger,\Gamma^{(f)}\psi)}{2^\nu}\cdot\sum_{\lambda_i}(\phi^\dagger,\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}\Gamma^{(\lambda)}(\lambda_i)\phi). \tag{91}$$

Now all $\Gamma^{(\lambda)}(\lambda_i)$ and $\Gamma^{(f)}(f_j)$ either commute or anticommute. Moreover,

$$[\Gamma^{(\lambda)}(\lambda_i)]^2=1.$$

Hence

$$\sum_{\lambda_i}\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}(f_j)\Gamma^{(\lambda)}(\lambda_i)=(\text{constant})\Gamma^{(f)}(f_j). \tag{92}$$

Since we are summing over all components of $\Gamma^{(\lambda)}$ it is apparent that the constant is independent of which component of $\Gamma^{(f)}$ is involved. Thus, with a suitable normalization, we have

$$\sum_{\lambda_i}\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}\Gamma^{(\lambda)}(\lambda_i)=\binom{2\nu}{\lambda}d_{\lambda f}\Gamma^{(f)}. \tag{93}$$

Comparing (93), (91) and (89) we have:

$$a_{\lambda f}=\binom{2\nu}{\lambda}d_{\lambda f}/2^\nu, \tag{94}$$

where

$$\binom{2\nu}{\lambda}d_{\lambda f}\Gamma^{(f)}=\sum_{\lambda_i}\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}\Gamma^{(\lambda)}(\lambda_i). \tag{95a}$$

Multiplying the $f_j$ component of this equation by $\Gamma^{(f)}(f_j)$ and summing over $f_j$ gives:

$$d_{\lambda f}\binom{2\nu}{f}\binom{2\nu}{\lambda}1=\sum_{\lambda_i f_j}\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}(f_j)\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}(f_j). \tag{95b}$$

Taking the trace and dividing by factors on the left yields the very convenient expression

$$d_{\lambda f}=\frac{1}{\binom{2\nu}{f}\binom{2\nu}{\lambda}2^\nu}\times\text{tr}\sum_{\lambda_i f_j}\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}(f_j)\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}(f_j). \tag{95c}$$

Properties of $d_{\lambda f}$ and special values are given in Appendix (A). The most generally useful form of the results obtained are:

$$d_{\lambda f}=\frac{1}{\binom{2\nu}{\lambda}}(-1)^{f\lambda}\times\{\text{coefficient of } x^\lambda \text{ in } (1+x)^{2\nu-f}(1-x)^f\}. \tag{95d}$$

Hence,

$$a_{\lambda f}=(-1)^{f\lambda}\times\{\text{coefficient of } x^\lambda \text{ in } (1+x)^{2\nu-f}(1-x)^f\}. \tag{96}$$

As a well-known illustration we consider the $\beta$-decay interaction. Here we have the four spinors $\Psi_n^\dagger$, $\Psi_p$, $\phi_\nu^\dagger$, $\phi_e$ representing neutron, proton, neutrino, and electron wave functions, respectively. The scalars formed by grouping neutron-electron, neutrino-proton are then expressed in terms of the usual grouping neutron-proton, neutrino-electron as:

$$S_\lambda^{(0)\prime}\equiv S_\lambda^{(0)}(\Psi_n^\dagger,\phi_e;\ \phi_\nu^\dagger,\Psi_p)=a_{\lambda f}S_f^{(0)}(\Psi_n^\dagger,\Psi_p;\ \phi_\nu^\dagger,\phi_e). \tag{97}$$

From the results of Appendix A it follows trivially that

$$(a_{\lambda f})=\begin{pmatrix}\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ 1 & -\frac{1}{2} & 0 & \frac{1}{2} & -1\\ \frac{3}{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{2}\\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1\\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4}\end{pmatrix}. \tag{98}$$

Hence[9]:

$$\begin{aligned}S_0'&=\tfrac{1}{4}\{S_0+S_1+S_2+S_3+S_4\},\\ S_1'&=S_0-\tfrac{1}{2}S_1+\tfrac{1}{2}S_3-S_4,\\ S_2'&=\tfrac{3}{2}S_0-\tfrac{1}{2}S_2+\tfrac{3}{2}S_4,\\ S_3'&=S_0+\tfrac{1}{2}S_1-\tfrac{1}{2}S_3-S_4,\\ S_4'&=\tfrac{1}{4}\{S_0-S_1+S_2-S_3+S_4\}.\end{aligned} \tag{99}$$

The inversion of the relation (89) is particularly easy. Since the prime merely denotes the interchange $\phi\rightleftarrows\psi$, a double prime brings the expression back to its original form:

$$S_\lambda^{(0)\prime\prime}=S_\lambda^{(0)}=a_{\lambda f}S_f^{(0)\prime}=(a^2)_{\lambda f}S_f^{(0)}, \tag{100}$$

i.e.,

$$a^2=1, \tag{101a}$$

or

$$a^{-1}=a. \tag{101b}$$

A direct proof of this using the formula (96) is given in Appendix B.

Similar considerations are obviously applicable to the other tensors formed with four spinors. A particularly simple case is that of the pseudoscalars.

While it is intuitively clear that there are $2\nu+1$ independent pseudoscalars it may be worth while to give a simple proof. Restricting ourselves (temporarily)

[9] This is essentially Eq. (1.4) of reference 5.

to the real orthogonal group will not change this number and permits the use of group integration. Let $h$ denote the volume of the pure rotations, $dS$ denote the volume element for pure rotations, $dU$ for rotation-reflections, and $dT$ a general volume element. Then if $\chi$ be the character of the spinor representation, $\chi^4$ is the character of the representation given by four spinors. The number ($N_s$) of scalars contained in this representation is:

$$N_s=(1/2h)\int\chi^4dT$$
$$=(1/2h)\left\{\int\chi^4dS+\int\chi^4dU\right\}, \tag{102a}$$

while the number of pseudoscalars ($N_{ps}$) is:

$$N_{ps}=(1/2h)\left\{\int\chi^4dS-\int\chi^4dU\right\}. \tag{102b}$$

Since $\chi$ vanishes for reflections:

$$N_{ps}=N_s=2\nu+1. \tag{103}$$

For our independent pseudoscalars we may choose those obtained by inserting a factor $\Gamma^{(2\nu)}$ in one of the two factors in our scalars. Thus a basis is formed by:

$$S_f^{(2\nu)}\equiv S_f^{(2\nu)}(\psi^\dagger,\psi;\phi^\dagger,\phi)$$
$$=(\psi^\dagger,\Gamma^{(f)}\psi)\cdot(\phi^\dagger,\Gamma^{(2\nu)}\Gamma^{(f)}\phi). \tag{104}$$

The quantities $S_\lambda^{(2\nu)\prime}$ obtained by interchanging $\psi$ and $\phi$ may be expressed in terms of these as:

$$S_\lambda^{(2\nu)\prime}=a_{\lambda f}^{(2\nu)}S_f^{(2\nu)}. \tag{105}$$

Inserting

$$\theta=\Gamma^{(\lambda)}(\lambda_i),\quad \theta'=\Gamma^{(2\nu)}\Gamma^{(\lambda)}(\lambda_i),$$

in (81) and summing over $\lambda_i$ gives:

$$(\psi^\dagger,\Gamma^{(\lambda)}\phi)\cdot(\phi^\dagger,\Gamma^{(2\nu)}\Gamma^{(\lambda)}\psi)=\sum_{f=0}^{2\nu}(1/2^\nu)(\psi^\dagger,\Gamma^{(f)}\psi)$$
$$\cdot\sum_{\lambda_i}(\phi^\dagger,\Gamma^{(2\nu)}\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}\Gamma^{(\lambda)}(\lambda_i)\phi). \tag{106}$$

But we have seen that

$$(1/2^\nu)\sum_{\lambda_i}\Gamma^{(\lambda)}(\lambda_i)\Gamma^{(f)}\Gamma^{(\lambda)}(\lambda_i)=a_{\lambda f}\Gamma^{(f)}. \tag{107}$$

Comparing (107), (106), and (105) gives:

$$a_{\lambda f}^{(2\nu)}=a_{\lambda f}. \tag{108}$$

Of paramount importance for the identities is the problem of expressing the covariants formed by combining the two covariant and the two contravariant spinors together first in terms of the combinations as in (85). For this the relevant starting point is obtained from (84) by multiplying by

$$(\theta C^{-1})_{AC}\phi_C{}^\dagger\phi_D(C^t\theta')_{DB}$$

and summing over $A$, $B$, $C$, and $D$. This gives:

$$(\psi^\dagger,\theta C^{-1}\phi^\dagger)(\phi C^t\theta'\psi)$$
$$=\textstyle\sum_f(1/2^\nu)(\psi^\dagger,\Gamma^{(f)}\psi)\cdot(\phi,C^t\theta'\Gamma^{(f)}\theta C^{-1}\phi^\dagger). \tag{109}$$

From (109) we obtain the scalars,

$$T_\lambda{}^{(0)}=(\psi^\dagger,\Gamma^{(\lambda)}C^{-1}\phi^\dagger)\cdot(\phi,C^t\Gamma^{(\lambda)}\psi), \tag{110}$$

in terms of the $S_f{}^{(0)}$ of (85) by substituting

$$\theta=\theta'=\Gamma^{(\lambda)}(\lambda_i)$$

and summing over the components $\lambda_i$. Remembering

$$\Gamma^t(i)=C\Gamma(i)C^{-1},\quad [C^t]^{-1}=(-1)^{[\nu/2]}C^{-1}, \tag{111}$$

and (107) gives:

$$T_\lambda{}^{(0)}=g_{\lambda f}S_f{}^{(0)}, \tag{112}$$

where

$$g_{\lambda f}=(-1)^{([\nu/2]+[f/2])}a_{\lambda f}. \tag{113}$$

Since it has been seen that $(a)^2=1$ the inverse of $(g)$ is obviously:

$$(g^{-1})_{\lambda f}=(-1)^{([\nu/2]+[\lambda/2])}a_{\lambda f}. \tag{114}$$

The application of (112) to $\beta$ decay is of some importance. From (113) and (98) we obtain

$$(g)=\begin{pmatrix}-\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4}\\ -1 & \frac{1}{2} & 0 & \frac{1}{2} & 1\\ -\frac{3}{2} & 0 & -\frac{1}{2} & 0 & -\frac{3}{2}\\ -1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1\\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}\end{pmatrix}, \tag{115}$$

giving (on omitting superscripts for convenience)

$$\begin{aligned}T_0&=\tfrac{1}{4}\{-S_0-S_1+S_2+S_3-S_4\}\\ T_1&=\tfrac{1}{2}\{-2S_0+S_1+S_3+2S_4\},\\ T_2&=\tfrac{1}{2}\{-3S_0+S_2-3S_4\},\\ T_3&=\tfrac{1}{2}\{-2S_0-S_1-S_3+2S_4\},\\ T_4&=\tfrac{1}{4}\{-S_0+S_1+S_2-S_3-S_4\}.\end{aligned} \tag{116}$$

To obtain the scalar biquadratic identities we need only combine (112) with (57) and (58). Thus if we identify $\psi=\phi$, $\psi^\dagger=\phi^\dagger$ those $T_\lambda{}^{(0)}$ will vanish for which $C\Gamma^{(\lambda)}$ is antisymmetric. This gives as the scalar identities:

(a) If $\nu=0$, 1 (mod 4)

$$0=g_{\lambda f}S_f{}^{(0)}(\psi^\dagger,\psi;\psi^\dagger,\psi)\quad \lambda=2,3\quad (\text{mod } 4). \tag{117}$$

Hence there are $\nu$ scalar identities in this case.

(b) If $\nu=2$, 3 (mod 4)

$$0=g_{\lambda f}S_f{}^{(0)}(\psi^\dagger,\psi;\psi^\dagger,\psi)\quad \lambda=0,1\quad (\text{mod } 4). \tag{118}$$

These are $\nu+1$ identities.

Since $(g_{\lambda f})$ is nonsingular (113), we see that these identities are all independent. Moreover, since only one of the two factors of $T_\lambda{}^{(0)}$ need vanish, (117) and (118) are correct provided only $\psi^\dagger=\phi^\dagger$ *or* $\psi=\phi$.

For $n=2$ ($\nu=1$), (117) yields the single identity:

$$g_{2f}S_f^{(0)}=0. \qquad (119)$$

Since the matrix $(g)$ is readily found to be

$$(g)=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \qquad (120)$$

we obtain

$$S_0-S_1-S_2=0; \qquad (121a)$$

or, expressing Γ's as Pauli matrices:

$$(\psi^*,\psi)^2=\sum_{i=1}^{3}(\psi^*,\sigma_i\psi)^2. \qquad (121b)$$

For $n=4$ ($\nu=2$) we have, using (118) and (115), the three identities:

$$0=\tfrac{1}{4}\{-S_0-S_1+S_2+S_3-S_4\}, \qquad (122a)$$

$$0=\tfrac{1}{2}\{-2S_0+S_1+S_3+2S_4\}, \qquad (122b)$$

$$0=\tfrac{1}{4}\{-S_0+S_1+S_2-S_3-S_4\}. \qquad (122c)$$

Adding (a) and (c) gives

$$S_0+S_4-S_2=0. \qquad (123a)$$

Subtracting (c) from (a) gives

$$S_1=S_3. \qquad (124)$$

Substituting this in (122b) gives

$$S_0-S_4-S_1=0, \qquad (123b)$$

and

$$S_0-S_4-S_3=0. \qquad (123c)$$

Identities (123b, a, c) are just Pauli's[3] Eqs. ($34,_{1,2,3}$).

Pseudoscalar identities may be obtained in a similar manner. Thus by an argument paralleling that leading from (81) to (112) we obtain

$$T_\lambda^{(2\nu)}(\psi^\dagger,\phi^\dagger;\,\phi,\psi)=h_{\lambda f}S_f^{(2\nu)}(\psi^\dagger,\psi;\,\phi^\dagger,\phi), \qquad (125)$$

where

$$T_\lambda^{(2\nu)}(\psi^\dagger,\phi^\dagger;\,\phi,\psi)=(\psi^\dagger,\Gamma^{(\lambda)}C^{-1}\phi^\dagger)\cdot(\phi C^t\Gamma^{(2\nu)}\Gamma^{(\lambda)}\psi), \qquad (126)$$

and

$$h_{\lambda f}=(-1)^{([\nu/2]+[f/2]+\nu+f)}a_{\lambda f}. \qquad (127)$$

$(h)$ has the inverse

$$(h^{-1})_{\lambda f}=(-1)^{([\nu/2]+[\lambda/2]+\nu+\lambda)}a_{\lambda f}. \qquad (128)$$

From (57b) and (58b) we have:

If $\nu=0$, 1 (mod 4),

$$(\psi^\dagger,\Gamma^{(\lambda)}C^{-1}\psi^\dagger)=0, \quad \lambda=2,\,3 \quad (\text{mod } 4). \qquad (129)$$

If $\nu=2$, 3 (mod 4),

$$(\psi^\dagger,\Gamma^{(\lambda)}C^{-1}\psi^\dagger)=0, \quad \lambda=0,\,1 \quad (\text{mod } 4). \qquad (130)$$

Thus, irrespective of the relation between $\phi$ and $\psi$, we have for $\phi^\dagger=\psi^\dagger$ the identities,

$$0=h_{\lambda f}S_f^{(2\nu)}, \qquad (131)$$

for the values of $\lambda$ in (129) and (130) corresponding to the respective values of $\nu$.

Similarly, from (57b), (58b), and (1) it can be shown that (all equations for $\nu$ and $\lambda$ holding mod 4):

$$(\psi,C^t\Gamma^{(2\nu)}\Gamma^{(\lambda)}\psi)=0\begin{cases}\text{if } \nu=0, & \text{for } \lambda=1,\,2 \quad (132a)\\ \text{if } \nu=1, & \text{for } \lambda=0,\,3 \quad (132b)\\ \text{if } \nu=2, & \text{for } \lambda=0,\,3 \quad (132c)\\ \text{if } \nu=3, & \text{for } \lambda=1,\,2. \quad (132d)\end{cases}$$

Hence, for arbitrary $\phi^\dagger$ and $\psi^\dagger$, we obtain for $\phi=\psi$ the identities (131) holding for the $\lambda$ corresponding to the $\nu$ of (132).

The most interesting identities are those in which we have simultaneously $\psi^\dagger=\phi^\dagger$, $\psi=\phi$. Combining the two sets of identities given shows that

$$h_{\lambda f}S_f^{(2\nu)}(\psi^\dagger,\psi;\,\psi^\dagger,\psi)=0, \quad \lambda\neq\nu \quad (\text{mod } 4). \qquad (133)$$

While (133) gives *all* the pseudoscalar identities it is unnecessarily complicated for calculation since some of the identities contained therein are rather trivial and better obtained by other methods. This is seen by two examples.

For $\nu=1$, we have

$$(h)=\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad (134)$$

and the identities:

$$\sum_{f=0}^{2} h_{\lambda f}S_f^{(2)}=0, \quad \lambda=0,\,2; \qquad (135)$$

i.e.,

$$0=\tfrac{1}{2}\{-S_0^{(2)}+S_1^{(2)}+S_2^{(2)}\}, \qquad (136a)$$

$$0=\tfrac{1}{2}\{-S_0^{(2)}-S_1^{(2)}+S_2^{(2)}\}. \qquad (136b)$$

Adding, we get the identity:

$$S_0^{(2)}=S_2^{(2)}, \qquad (137a)$$

which, on expressing the $S_\lambda^{(2)}$ in terms of Pauli matrices and wave functions, is seen to be the trivial statement that

$$(\psi^\dagger,\psi)(\psi^*,\sigma_3\psi)=(\psi^*,\sigma_3\psi)(\psi^*,\psi). \qquad (137b)$$

For obvious reasons this will be called a "relabeling identity."

Inserting (137a) in (136) gives as the other identity,

$$S_1^{(2)}=0. \qquad (138a)$$

In terms of the Pauli matrices this is the trivial result;

$$(\psi^*,\sigma_1\psi)(\psi^*,\sigma_3\sigma_1\psi)+(\psi^*,\sigma_2\psi)(\psi^*,\sigma_3\sigma_2\psi)$$
$$=i(\psi^*,\sigma_1\psi)(\psi^*,\sigma_2\psi)-i(\psi^*,\sigma_2\psi)(\psi^*,\sigma_1\psi)=0, \qquad (138b)$$

which we will also call a "relabeling identity."

For $\nu=2$ we have:

$$(h)=\begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ -\frac{3}{2} & 0 & -\frac{1}{2} & 0 & -\frac{3}{2} \\ -1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix},$$

and the identities:

$$h_{\lambda f}S_f^{(4)}=0,\quad \lambda=0,\,1,\,3,\,4. \tag{139}$$

Explicitly these are (omitting the superscript 4)

$$0=-S_0+S_1+S_2-S_3-S_4, \tag{140a}$$

$$0=-S_0-S_1/2-S_3/2+S_4, \tag{140b}$$

$$0=-S_0+S_1/2+S_3/2+S_4, \tag{140c}$$

$$0=-S_0-S_1+S_2+S_3-S_4. \tag{140d}$$

Subtracting (d) from (a) and adding (b) to (c) gives the two trivial relabelling identities:

$$S_1=S_3, \tag{141a}$$

$$S_0=S_4. \tag{141b}$$

Inserting (141) in (140a) and (140b) gives the two nontrivial identities:

$$S_2=2S_0, \tag{142a}$$

$$S_1=0. \tag{142b}$$

These are Pauli's[3] Eqs. $(34_4)$ and $(34_5)$, respectively.

The nature of the "relabeling identities" is shown from the result of Appendix C that

$$S_{\nu+i}^{(2\nu)}=(-1)^{\nu+i}S_{\nu-i}^{(2\nu)}. \tag{143}$$

This gives an identity for each of the $\nu$ possibilities $i=1,\,2,\,\cdots,\,\nu$. Moreover if $\nu$ is odd Eq. (143) for $i=0$ shows that

$$S_\nu^{(2\nu)}=0. \tag{144}$$

Thus for even $\nu$, we have $\nu$ of these trivial statements, while for odd $\nu$ we have $\nu+1$ relabeling identities. Table II shows the number of pseudoscalar identities ($N_{ps}$) given by (133), the number of trivial relabeling identities ($N_l$), and the remaining number of nontrivial identities $N_n$.

From (139) and (143) explicit forms for the nontrivial identities can be obtained. However, in practice it is just as convenient to use (139) as it stands merely remembering (143).

### VI. IDENTITIES FOR $n=2\nu+1$

All calculations proceed the same as for even dimensions except that now we need only use the even quantities $\Gamma^{(2\lambda)}$ as a basis for expansion of $2^\nu\times 2^\nu$ matrices and we use the properties of Table I. For brevity we will give only the results for scalars formed with 2 covariant and 2 contravariant spinors. The proof of the statements is outlined in Appendix D where we also give various properties of the matrices introduced.

TABLE II. The number of pseudoscalar identities $N_{ps}$, trivial relabeling identities $N_l$, and nontrivial identities $N_n$.

| | Pseudoscalar identities | | |
|---|---|---|---|
| $\nu$ (mod 4) | $N_{ps}$ | $N_l$ | $N_n$ |
| 0 | $\frac{3}{2}\nu$ | $\nu$ | $\nu/2$ |
| 1 | $\frac{3}{2}\nu+\frac{1}{2}$ | $\nu+1$ | $\nu/2-\frac{1}{2}$ |
| 2 | $\frac{3}{2}\nu+1$ | $\nu$ | $\nu/2+1$ |
| 3 | $\frac{3}{2}\nu+\frac{3}{2}$ | $\nu+1$ | $\nu/2+\frac{3}{2}$ |

The $\nu+1$ scalars obtained by first combining the spinors as $\psi^\dagger$ with $\phi$, $\phi^\dagger$ with $\psi$ are expressed in terms of those with the grouping $\psi^\dagger$ with $\psi$, $\phi^\dagger$ with $\phi$ as:

$$S_{2\lambda}'(\psi^\dagger,\phi;\,\phi^\dagger,\psi)=\sum_{f=0}^{\nu}\alpha_{\lambda f}S_{2f}(\psi^\dagger,\psi;\,\phi^\dagger,\phi),\qquad \lambda=0,\,1,\,\cdots,\,\nu, \tag{145}$$

where

$$S_{2\lambda}'(\psi^\dagger,\phi;\,\phi^\dagger,\psi)=(\psi^\dagger,\Gamma^{(2\lambda)}\phi)\cdot(\phi^\dagger,\Gamma^{(2\lambda)}\psi), \tag{146}$$

$$S_{2f}(\psi^\dagger,\psi;\,\phi^\dagger,\phi)=(\psi^\dagger,\Gamma^{(2\lambda)}\psi)\cdot(\phi^\dagger,\Gamma^{(2\lambda)}\phi), \tag{147}$$

$$\alpha_{\lambda f}=(1/2^\nu)\{\text{coefficient of } x^{2\lambda} \text{ in } (1+x)^{2\nu+1-2f}(1-x)^{2f}\}. \tag{148}$$

As examples we have:

*For* $n=3$ ($\nu=1$) ($\lambda,f=0,1$)

$$S_0'=\tfrac{1}{2}(S_0+S_2), \tag{149a}$$

$$S_2'=\tfrac{1}{2}(3S_0-S_2). \tag{149b}$$

*For* $n=5$ ($\nu=2$) ($\lambda,f=0,2,4$)

$$S_0'=\tfrac{1}{4}\{S_0+S_2+S_4\}, \tag{150a}$$

$$S_2'=\tfrac{1}{2}\{5S_0-S_2+S_4\}, \tag{150b}$$

$$S_4'=\tfrac{1}{4}\{5S_0+S_2-3S_4\}. \tag{150c}$$

The scalars formed by combining first covariant with covariant, contravariant with contravariant spinors can be expressed in terms of the $S_{2f}$ of (147) as

$$T_{2\lambda}(\psi^\dagger,\phi^\dagger;\,\psi,\phi)=\sum_{f=0}^{\nu}\gamma_{\lambda f}S_{2f}\quad(\lambda=0,\,1,\,\cdots\nu), \tag{151}$$

where

$$T_{2\lambda}(\psi^\dagger,\phi^\dagger;\,\psi,\phi)=(\psi^\dagger,\Gamma^{(2\lambda)}C^{-1}\phi^\dagger)\cdot(\phi,C^t\Gamma^{(2\lambda)}\psi), \tag{152}$$

$$\gamma_{\lambda f}=(-1)^{(f+\nu+[\nu/2])}\alpha_{\lambda f}. \tag{153}$$

For example we have, for $\nu=1$,

$$T_0=\tfrac{1}{2}(-S_0+S_2), \tag{154a}$$

$$T_2=\tfrac{1}{2}(-3S_0-S_2); \tag{154b}$$

while, for $\nu=2$,

$$T_0=\tfrac{1}{4}(-S_0+S_2-S_4), \tag{155a}$$

$$T_2=-\tfrac{1}{2}(5S_0+S_2+S_4), \tag{155b}$$

$$T_3=-\tfrac{1}{4}(-5S_0+S_2+3S_4). \tag{155c}$$

As in Sec. V, we obtain nontrivial identities between the $S_{2f}$ for $\psi^\dagger=\phi^\dagger$ and (or) $\phi=\psi$ for those values of $\lambda$ in (151) for which these assumptions make $T_{2\lambda}$ vanish. That is, we obtain identities for those $\lambda$ such that $C\Gamma^{(2\lambda)}$ is antisymmetric. Thus from Table I we obtain the identities:

$$\sum_{f=0}^{\nu}\gamma_{\lambda f}S_{2f}=0\begin{cases}\lambda \text{ odd}, & \nu=0 \pmod 4\\ \lambda \text{ even}, & \nu=1 \pmod 4\\ \lambda \text{ even}, & \nu=2 \pmod 4\\ \lambda \text{ odd}, & \nu=3 \pmod 4.\end{cases} \tag{156}$$

For $n=3$ ($\nu=1$) we have only the equation for $\lambda=0$:

$$\gamma_{0f}S_{2f}=0, \tag{157a}$$

or

$$S_0=S_2. \tag{157b}$$

It may be noted that (157b) is just the identity (121b) obtained for $n=2$.

For $n=5$ ($\nu=2$), we obtain identities (156) for $\lambda=0, 2$. These are

$$-S_0+S_2-S_4=0, \tag{158a}$$

$$-5S_0+S_2+3S_4=0, \tag{158b}$$

or

$$S_0=S_4, \tag{159a}$$

$$S_2=2S_0. \tag{159b}$$

In Table III we give the number ($N_I$) of scalar identities given by (156). Since there are $\nu+1$ scalars formed from two arbitrary covariant and two arbitrary contravariant spinors, the number ($N_s$) of independent scalars when spinors are identified is:

$$N_s=\nu+1-N_I \tag{160}$$

which is also given in Table III.

## VII. CONCLUSION

The decomposition of the representation of the orthogonal group in $n$ dimensions given by the symmetrical product of two covariant or two contravariant spinors has been obtained. It has been shown that using the decomposition all identities holding between covariants formed with four such spinors when some of them are identical are readily found. Explicit formulas for the scalar and pseudoscalar identities in $n=2\nu$ dimensions and for the scalar identities in $n=2\nu+1$ dimensions are given.

TABLE III. Numbers $N_s$ of scalars formed from 2 sets of identical spinors. $n=2\nu+1$. $N_I$ is the number of scalar identities.

| $\nu$ (mod 4) | $N_I$ | $N_s$ |
|---|---|---|
| 0 | $\nu/2$ | $(\nu/2)+1$ |
| 1 | $(\nu/2)+\frac{1}{2}$ | $(\nu/2)+\frac{1}{2}$ |
| 2 | $(\nu/2)+1$ | $\nu/2$ |
| 3 | $(\nu/2)+\frac{1}{2}$ | $(\nu/2)+\frac{1}{2}$ |

## APPENDIX A. PROPERTIES OF THE MATRIX $a_{\lambda f}$

In the text we obtained the result

$$a_{\lambda f}=\binom{2\nu}{\lambda}d_{\lambda f}/2^\nu, \tag{A1}$$

where

$$d_{\lambda f}=\left\{\binom{2\nu}{\lambda}\binom{2\nu}{f}2^\nu\right\}^{-1} \times \mathrm{tr}\sum_{\substack{i_1<i_2<\cdots i_\lambda\\ j_1<j_2<\cdots j_f}}[\Gamma^{(\lambda)}(i_1i_2\cdots i_\lambda)\Gamma^{(f)}(j_1j_2\cdots j_f)]^2. \tag{A2}$$

Since (A2) is symmetrical in $\lambda$ and $f$, we have

$$d_{\lambda f}=d_{f\lambda}. \tag{A3}$$

To calculate $d_{\lambda f}$ we will assume $\lambda\leq f$. (From (A3) we see this is no significant restriction.) Since it is clear that each term in the sum over $j_1, j_2, \cdots j_f$ gives the same contribution we can restrict ourselves to the single term $\Gamma^{(f)}(12\cdots f)$ on multiplying by $\binom{2\nu}{f}$. Thus

$$d_{\lambda f}=\left\{\binom{2\nu}{\lambda}2^\nu\right\}^{-1}\mathrm{tr}\sum_{i_1<i_2<\cdots<i_\lambda}\Gamma^{(\lambda)}(i_1i_2\cdots i_\lambda) \times\Gamma^{(f)}(12\cdots f)\Gamma^{(\lambda)}(i_1i_2\cdots i_\lambda)\Gamma^{(f)}(12\cdots f). \tag{A4}$$

Consider a fixed term in the sum. Since $\Gamma^{(\lambda)}(i_1i_2\cdots)$ either commutes or anticommutes with $\Gamma^{(f)}(12\cdots f)$ and the squares of both are unity we have:

$$(2^\nu)^{-1}\,\mathrm{tr}\Gamma^{(\lambda)}(i_1i_2\cdots i_\lambda)\Gamma^{(f)}(12\cdots f) \times\Gamma^{(\lambda)}(i_1i_2\cdots i_\lambda)\Gamma^{(f)}(12\cdots f)=\pm1. \tag{A5}$$

If $k$ of the indices $i_1, i_2, \cdots i_\lambda$ are greater than $f$, the sign is

$$[(-1)^{f-1}]^{\lambda-k}[(-1)^f]^k. \tag{A6}$$

This occurs in the sum over $i_1, \cdots, i_\lambda$ a number of times equal to

$$\binom{f}{\lambda-k}\binom{2\nu-f}{k}.$$

Hence

$$d_{\lambda f}=\left[\binom{2\nu}{\lambda}\right]^{-1}(-1)^{f\lambda} \times\sum_{k=0}^{\lambda}(-1)^{\lambda-k}\binom{f}{\lambda-k}\binom{2\nu-f}{k}. \tag{A7}$$

Comparing with the power series expansion,

$$(1+x)^m(1-x)^n=\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}(-1)^jx^{i+j}, \tag{A8}$$

we see that

$$d_{\lambda f}=\left[\binom{2\nu}{\lambda}\right]^{-1}(-1)^{f\lambda}$$
$$\times\{\text{coefficient of } x^{\lambda} \text{ in } (1+x)^{2\nu-f}(1-x)^{f}\}. \quad \text{(A9)}$$

Using (A1), we obtain the result (96).

A further symmetry property is obtained by changing the sign of $x$ in (A9). Thus

$$d_{\lambda f}=\left[\binom{2\nu}{\lambda}\right]^{-1}(-1)^{f\lambda}(-1)^{\lambda}$$
$$\times\{\text{coefficient of } x^{\lambda} \text{ in } (1-x)^{2\nu-f}(1+x)^{f}\}. \quad \text{(A10)}$$

However, (A9) also says that

$$d_{\lambda,\,2\nu-f}=\left[\binom{2\nu}{\lambda}\right]^{-1}(-1)^{f\lambda}$$
$$\times\{\text{coefficient of } x^{\lambda} \text{ in } (1+x)^{f}(1-x)^{2\nu-f}\}. \quad \text{(A11)}$$

Comparing, we see

$$d_{\lambda,\,2\nu-f}=(-1)^{\lambda}d_{\lambda f}. \quad \text{(A12)}$$

From the symmetry properties (A3) and (A12), we need the coefficients $d_{\lambda f}$ only for $\lambda\leq f\leq\nu$. This is only $(\nu+1)(\nu+2)/2$ coefficients.

For $\lambda$ small, Eq. (A7) is particularly easy to use. Thus, by direct calculation we readily obtain:

$$d_{0f}=1, \quad \text{(A13)}$$

$$d_{1f}=(-1)^{f}(\nu-f)/\nu, \quad \text{(A14)}$$

$$d_{2f}=[2(\nu-f)^{2}-\nu]/\nu(2\nu-1). \quad \text{(A15)}$$

A particularly simple result when one of the indices is $\nu$ is obtained from (A10).

$$d_{\nu f}=d_{f\nu}=\left[\binom{2\nu}{f}\right]^{-1}(-1)^{\nu f}$$
$$\times\{\text{coefficient of } x^{f} \text{ in } (1+x)^{\nu}(1-x)^{\nu}\}. \quad \text{(A16)}$$

But

$$(1+x)^{\nu}(1-x)^{\nu}=(1-x^{2})^{\nu}=\sum_{i=0}^{\nu}(-1)^{i}\binom{\nu}{i}x^{2i}. \quad \text{(A17)}$$

Thus:

$$d_{2i+1,\,\nu}=0, \quad \text{(A18)}$$

$$d_{2i,\,\nu}=\left[\binom{2\nu}{2i}\right]^{-1}(-1)^{i}\binom{\nu}{i}. \quad \text{(A19)}$$

The simple formulas (A13–19) give all coefficients of the matrices $(d)$, $(a)$, $(g)$, $(h)$ for $\nu$ up to and including $3(n=6)$.

## APPENDIX B. PROOF THAT $a^2=1$

A direct proof is obtained using the generating function of (96). This relation may be rewritten

$$a_{\mu\sigma}=2^{-\nu}(-1)^{\mu\sigma}(2\pi i)^{-1}$$
$$\times\oint_{C}(1+x)^{2\nu-\sigma}(1-x)^{\sigma}dx/x^{\mu+1}, \quad \text{(B1)}$$

where the contour $C$ encloses the origin. Similarly, using the generating function (A16) for $(d)$, the symmetry property (A3) and the relation (A1) between $(a)$ and $(d)$ we obtain:

$$a_{\sigma\gamma}=2^{-\nu}(-1)^{\gamma\sigma}\binom{2\nu}{\sigma}\binom{2\nu}{\gamma}^{-1}(2\pi i)^{-1}$$
$$\times\oint(1+y)^{2\nu-\sigma}(1-y)^{\sigma}dy/y^{\gamma+1}. \quad \text{(B2)}$$

Hence:

$$\sum_{\sigma=0}^{2\nu}a_{\mu\sigma}a_{\sigma\gamma}=2^{-2\nu}\binom{2\nu}{\gamma}^{-1}(2\pi i)^{-2}$$
$$\times\oint\oint dxdy\{\quad\}/x^{\mu+1}y^{\gamma+1}, \quad \text{(B3)}$$

where

$$\{\quad\}=\sum_{\sigma=0}^{2\nu}\binom{2\nu}{\sigma}[(-1)^{\mu+\gamma}]^{\sigma}$$
$$\times[(1-x)(1-y)]^{\sigma}[(1+x)(1+y)]^{2\nu-\sigma}$$
$$=\{(-1)^{\mu+\gamma}(1-x)(1-y)+(1+x)(1+y)\}^{2\nu}. \quad \text{(B4)}$$

It is convenient to divide the discussion into two cases.

### Case 1. $\mu+\gamma$ Even

$$\{\quad\}=2^{2\nu}(1+xy)^{2\nu}. \quad \text{(B5)}$$

Inserting this in (B3), expanding by the binomial theorem and evaluating by residues gives:

$$\sum_{\sigma=0}^{2\nu}a_{\mu\sigma}a_{\sigma\gamma}$$
$$=\binom{2\nu}{\gamma}^{-1}(2\pi i)^{-2}\oint\oint dxdy(1+xy)^{2\nu}/x^{\mu+1}y^{\gamma+1}$$
$$=\binom{2\nu}{\gamma}^{-1}(2\pi i)^{-2}\oint\oint dxdy(x^{\mu+1}y^{\gamma+1})^{-1}\sum_{i=0}^{2\nu}\binom{2\nu}{i}x^{i}y^{i}$$
$$=(2\pi i)^{-1}\oint x^{\gamma}dx/x^{\mu+1}=\begin{cases}0, & \gamma\neq\mu\\ 1, & \gamma=\mu.\end{cases} \quad \text{(B6)}$$

**Case 2. $\mu+\gamma$ Odd**

$$\{\quad\}=2^{2\nu}(x+y)^{2\nu}$$

$$\sum_{\sigma=0}^{2\nu} a_{\mu\sigma}a_{\sigma\gamma}$$
$$=\binom{2\nu}{\gamma}^{-1}(2\pi i)^{-2}\oint\oint\frac{dxdy}{x^{\mu+1}y^{\gamma+1}}\sum_{i=0}^{2\nu}\binom{2\nu}{i}x^iy^{2\nu-i}$$
$$=\binom{2\nu}{\mu}\left[\binom{2\nu}{\gamma}\right]^{-1}(2\pi i)^{-1}\oint y^{2\nu-\mu}dy/y^{\gamma+1}$$
$$=0 \quad \text{unless} \quad 2\nu-\mu=\gamma. \tag{B7}$$

Thus in Case 2 we get zero unless

$$\mu+\gamma=2\nu, \tag{B8}$$

which was treated under Case 1. Hence:

$$\sum_{\sigma=0}^{2\nu} a_{\mu\sigma}a_{\sigma\gamma}=0, \quad \mu\neq\gamma,$$
$$=1, \quad \mu=\gamma, \tag{B9}$$

or

$$a^2=1. \tag{B10}$$

## APPENDIX C. THE RELABELING IDENTITIES

From the definition

$$S_\lambda^{(2\nu)}(\psi^\dagger,\psi;\psi^\dagger,\psi)=(\psi^\dagger,\Gamma^{(\lambda)}\psi)\cdot(\psi^\dagger,\Gamma^{(2\nu)}\Gamma^{(\lambda)}\psi)$$
$$=\sum_{i_1<i_2\cdots<i_\lambda}(\psi^\dagger,\Gamma^{(\lambda)}(i_1i_2\cdots i_\lambda)\psi)$$
$$\times(\psi^\dagger,\Gamma^{(2\nu)}\Gamma^{(\lambda)}(i_1i_2\cdots i_\lambda)\psi), \tag{C1}$$

it is clear that

$$S_\lambda^{(2\nu)}=\mathcal{A}S_{2\nu-\lambda}^{(2\nu)}, \tag{C2}$$

with some constant $\mathcal{A}$. To determine $\mathcal{A}$ we need only find the relationship between a fixed term on the left and the corresponding term on the right. For simplicity we consider the term in which $i_1\cdots i_\lambda$ is the sequence 1, 2, $\cdots\lambda$. This term is

$$(\psi^\dagger,i^{[\lambda]}\Gamma(1)\Gamma(2)\cdots\Gamma(\lambda)\psi)i^\nu(\psi^\dagger,\Gamma(1)\cdots$$
$$\times\Gamma(2\nu)i^{[\lambda]}\Gamma(1)\Gamma(2)\cdots\Gamma(\lambda)\psi)$$
$$=(-1)^\lambda(\psi^\dagger,\Gamma(1)\cdots\Gamma(\lambda)\psi)$$
$$\times i^\nu(\psi^\dagger,\Gamma(\lambda+1)\cdots\Gamma(2\nu)\psi). \tag{C3}$$

The corresponding term of $S_{2\nu-f}^{(2\nu)}$ is

$$(\psi^\dagger,i^{[2\nu-\lambda]}\Gamma(\lambda+1)\cdots\Gamma(2\nu)\psi)i^\nu(\psi^\dagger,\Gamma(1)\Gamma(2)\cdots$$
$$\times\Gamma(2\nu)i^{[2\nu-\lambda]}\Gamma(\lambda+1)\cdots\Gamma(2\nu)\psi)$$
$$=(\psi^\dagger,\Gamma(1)\cdots\Gamma(\lambda)\psi)i^\nu(\psi^\dagger,\Gamma(\lambda+1)\cdots\Gamma(2\nu)\psi). \tag{C4}$$

Comparing (C3) and (C4) we see that

$$\mathcal{A}=(-1)^\lambda, \tag{C5}$$

and hence

$$S_\lambda^{(2\nu)}=(-1)^\lambda S_{2f-\lambda}^{(2\nu)}, \tag{C6}$$

or, more symmetrically,

$$S_{\nu+i}^{(2\nu)}=(-1)^{\nu+i}S_{\nu-i}^{(2\nu)}. \tag{C7}$$

## APPENDIX D. DETAILS IN CASE $n=2\nu+1$

To express the covariants obtained by interchanging $\phi$ and $\psi$, we expand

$$\psi^\dagger{}_A\psi_B=\sum_{f=0}^{\nu}A^{(2f)}\cdot(\Gamma^{(2f)})_{BA}. \tag{D1}$$

Taking traces:

$$A^{(2f)}=2^{-\nu}(\psi^\dagger,\Gamma^{(2f)}\psi), \tag{D2}$$

or

$$\psi^\dagger{}_A\psi_B=\sum_{f=0}^{\nu}2^{-\nu}(\psi^\dagger,\Gamma^{(2f)}\psi)\cdot(\Gamma^{(2f)})_{BA}. \tag{D3}$$

Multiplying by $\theta_{AC}\phi_C\phi^\dagger{}_D\theta'{}_{DB}$ and summing gives

$$(\psi^\dagger,\theta\phi)(\phi^\dagger,\theta'\psi)=\sum_{f=0}^{\nu}2^{-\nu}(\psi^\dagger,\Gamma^{(2f)}\psi)(\phi^\dagger,\theta'\Gamma^{(2f)}\theta\phi). \tag{D4}$$

In particular this gives for the scalars:

$$(\psi^\dagger,\Gamma^{(2\lambda)}\phi)\cdot(\phi^\dagger,\Gamma^{(2\lambda)}\psi)$$
$$=\sum_{f=0}^{\nu}2^{-\nu}(\psi^\dagger,\Gamma^{(2f)}\psi)\cdot\sum_{\lambda_i}(\phi^\dagger,\Gamma^{(2\lambda)}(\lambda_i)\Gamma^{(2f)}\Gamma^{(2\lambda)}(\lambda_i)\phi)$$
$$\text{for} \quad \lambda=0,1,\cdots,\nu. \tag{D5}$$

Now

$$2^{-\nu}\sum_{\lambda_i}\Gamma^{(2\lambda)}(\lambda_i)\Gamma^{(2f)}\Gamma^{(2\lambda)}(\lambda_i)=\alpha_{\lambda f}\Gamma^{(2f)}. \tag{D6}$$

Using the same arguments as in the even-dimensional case, we have:

$$\alpha_{\lambda f}=\binom{2\nu+1}{2\lambda}\epsilon_{\lambda f}/2^\nu, \tag{D7}$$

$$\epsilon_{\lambda f}=\left[2^\nu\binom{2\nu+1}{2\lambda}\binom{2\nu+1}{2f}\right]^{-1}\operatorname{tr}\sum_{\lambda_i f_j}[\Gamma^{(2\lambda)}(\lambda_i)\Gamma^{(2f)}(f_j)]^2$$
$$=\epsilon_{f\lambda}. \tag{D8}$$

Performing the sum as in Appendix A, we obtain

$$\epsilon_{\lambda f}=\binom{2\nu+1}{2\lambda}^{-1}\sum_{i=0}^{2\lambda}[(-1)^{2f-1}]^{2\lambda-i}$$
$$\times[(-1)^{2f}]^i\binom{2f}{2\lambda-i}\binom{2\nu+1-2f}{2i}$$
$$=\binom{2\nu+1}{2\lambda}^{-1}\sum_{i=0}^{2\lambda}(-1)^{2\lambda-i}$$
$$\times\binom{2f}{2\lambda-i}\binom{2\nu+1-2f}{i}, \tag{D9}$$

or

$$\epsilon_{\lambda f}=\binom{2\nu+1}{2\lambda}^{-1}\{\text{coefficient of } x^{2\lambda} \text{ in}$$
$$(1+x)^{2\nu+1-2f}(1-x)^{2f}\}. \tag{D10}$$

On changing the sign of $x$ in (D10) we see

$$\epsilon_{\lambda f}=\epsilon_{\lambda,\frac{1}{2}(2\nu+1-f)}. \quad \text{(D11)}$$

Some special values are

$$\epsilon_{0f}=1, \quad \text{(D12)}$$

$$\epsilon_{\lambda\nu}=\binom{2\nu+1}{2\lambda}^{-1}\left\{\binom{2\nu}{2\lambda}-\binom{2\nu}{2\lambda-1}\right\}. \quad \text{(D13)}$$

To obtain the linear combinations formed by combining similar type spinors, we multiply (D3) by $(\theta C^{-1})_{AC}\phi^{\dagger}{}_C\phi_D(C^t\theta')_{DB}$ and sum:

$$(\psi^{\dagger},\theta C^{-1}\phi^{\dagger})(\phi,C^t\theta'\psi)$$
$$=\sum_{f=0}^{\nu}2^{-\nu}(\psi^{\dagger},\Gamma^{(2f)}\psi)\cdot(\phi^{\dagger},C^{t-1}\theta^t\Gamma^{t(2f)}\theta^{t\prime}C\phi). \quad \text{(D14)}$$

Putting $\theta=\theta'=\Gamma^{(2\lambda)}(\lambda_i)$ and summing over components gives:

$$T_{2\lambda}=\sum_f 2^{-\nu}(\psi^{\dagger},\Gamma^{(2f)}\psi)$$
$$\cdot\sum_{\lambda_i}(\phi^{\dagger},C^{t-1}\Gamma^{t(2\lambda)}(\lambda_i)\Gamma^{t(2f)}\Gamma^{t(2\lambda)}(\lambda_i)C\phi). \quad \text{(D15)}$$

Using the symmetry properties of $C$, which can be summarized as

$$C^t=(-1)^{(\nu+[\nu/2])}C, \quad \text{(D16)}$$

and Eqs. (1), (36), and (D6) gives:

$$T_{2\lambda}=\sum_{f=0}^{\nu}(\psi^{\dagger},\Gamma^{(2f)}\psi)\cdot(-1)^{(f+\nu+[\nu/2])}\alpha_{\lambda f}(\phi^{\dagger},\Gamma^{(2f)}\phi), \quad \text{(D17)}$$

or

$$T_{2\lambda}=\gamma_{\lambda f}S_{2f}, \quad \text{(D18)}$$

where

$$\gamma_{\lambda f}=(-1)^{(f+\nu+[\nu/2])}\alpha_{\lambda f}. \quad \text{(D19)}$$

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# Renormalization of a Covariant Approximation Scheme in Field Theory*

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An approximation scheme for the one-nucleon Green's functions previously put forward by the authors is renormalized. The experimental mass and the constants $Z_1$ and $Z_2$ are rigorously expressed as free-particle limits of integrals over the kernels appearing in the scheme. The mass and wave-function renormalization are carried out rigorously; the vertex renormalization is performed by a slight redefinition of the approximation scheme, without greatly altering the physical assumptions peculiar to each approximation. General prescriptions for renormalization are written down, and the first three approximations are explicitly shown to be finite.

## 1. INTRODUCTION

RECENTLY the authors[1] have proposed a covariant approximation scheme for the treatment of the coupled Green's functions equations of meson-nucleon systems. The procedure led to the replacement of the infinite set of coupled equations for the rigorous kernels by a finite set of approximate equations, involving Green's functions which describe processes with no more than a fixed number of external meson lines.

In (I) the question of renormalization was ignored. It is of course not known whether the usual infinities of pseudoscalar meson theory with pseudoscalar coupling are due to the use of the perturbation expansions in which they appear; however, whether the theory is finite or not, a renormalization has to be carried out. In the approximation scheme, whose validity may only be motivated in the low-energy region, it is expected that such high-frequency phenomena as the self-energy, etc., will not be described correctly, and the existence of infinities are a not unexpected feature. Nevertheless the lack of a correct description in the high-energy domain does not prevent one from performing a renormalization. For example, when a subset of perturbation graphs is summed rigorously,[2] the radical difference in the high-energy behavior of the sum and the individual terms of the series does not prevent the renormalization of the latter by perturbation methods.

In this paper a nonperturbation renormalization of the approximation scheme is carried out, i.e., equations involving the renormalized Green's functions, with finite masses and coupling constants, are derived. Although it is of course necessary to solve the resulting

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