PHYSICAL REVIEW VOLUME 97, NUMBER 3 FEBRUARY 1, 1955

Dynamics of Nuclear Deformation

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(Received September 24, 1954)

The droplet model of collective nuclear motions is ordinarily based on the assumption of classical irrotational fluid flow and has been used to obtain both the rotational properties of nuclei associated with a circulating surface wave and the vibrational properties of a surface deformation, though only the rotational results are confirmed experimentally. As a partial justification of the irrotational assumption, the energy of collective rotation of a closed-shell nucleus has been derived from nucleon wave functions in an earlier paper. The vibrational problem is here similarly treated and it is shown that, in contrast to the case of rotation, the vibrational distortion of a closed-shell nucleus is so rapid that the adiabatic treatment used is not valid. The potential energy of distortion calculated with phenomenological nuclear forces is not very different from that calculated by use of the deformed three-dimensional oscillator potential, which verifies that the latter is a fairly good approximation to a "self-consistent field" for a deformed closed-shell nucleus.

INTRODUCTION

THE collective motion of nucleons associated with a change in nuclear shape may be approximated by introducing a nonspherical distortion into the zero-order wave functions of the shell model, as was done for the case of a rotating distortion in a recent paper.[1] There a perturbation procedure equivalent to the adiabatic approximation was used and it was shown that for the low rotational states of moderately heavy and appreciably nonspherical nuclei the rotation is indeed slow compared to the nucleon motions, as required for the validity of the adiabatic treatment. The rotational energy was found to agree with that obtained by assuming irrotational fluid flow in the droplet model, and Aage Bohr has, by defining a local collective velocity associated with a rotating distortion, discussed in general terms the relation between shell-model closed shells and classical irrotational flow.[2] The ad-hoc assumption of irrotational fluid flow introduced into the usual droplet-model treatment is just as essential for the vibration as for the rotation, although no experimental data are as clearly relevant, and it would be desirable to have a similar derivation starting from nucleon wave functions in the vibrational case. Unfortunately, the collective frequencies for vibration are considerably higher than for the rotation and it is here shown that the adiabatic approximation is no longer valid. This treatment is based explicitly on harmonic-oscillator wave functions, but the result leads to the anticipation that more general treatments of collective surface or volume vibration will encounter trouble with density fluctuations. However, it does not preclude the possible existence of states whose properties are determined partly by modes of collective vibration.

In the binding and vibration problem of a diatomic molecule, one calculates the energy of the system, arising principally from the electron behavior, as a function of internuclear distance R considered fixed, and then because of the obviously adiabatic nature of the motion takes this to represent a potential energy curve, of which a term in $(\Delta R)^2$ determines the frequency of vibration. The electrons are so light compared to the two nuclei that one ordinarily does not consider their contribution to the kinetic energy. In the problem of vibration of shape of a nucleus about an equilibrium shape, the mass is contained entirely in the nucleons and their contribution, in the total energy of the system, to a term in the square of the rate of distortion is to be interpreted similarly as the collective kinetic energy.

ADDITIONAL ENERGY ACCOMPANYING ADIABATIC DISTORTION

The method here introduced for calculating the kinetic energy associated with the rate of deformation of a charge distribution which arises from quantization of a particle in a slowly-varying confining field depends on an application of the adiabatic approximation. In solving the slowly time-dependent wave equation $i\hbar\partial\Psi/\partial t=\mathcal{H}(t)\Psi$, we expand $\Psi(x,t)=\sum a_n u_n \exp[-(i/\hbar)\int^t E_n dt']$ in terms of normalized solutions u_n, supposed known at any time, of $[\mathcal{H}(x,t)-E_n(t)]u_n(x,t)=0$. This substitution, followed by multiplying by $\bar{u}_k$ and integrating, yields[3]

$$\dot{a}_k=-\sum I_{kn}a_n \exp\left[-(i/\hbar)\int^t (E_k-E_n)dt'\right], \qquad (1)$$

or

$$a_k\approx\sum I_{kn}(a_n/i\omega_{kn})[\exp(-i\omega_{kn}t)-C_k], \qquad (2)$$

if in this "adiabatic" integration we consider $(E_n-E_k)=\hbar\omega_{kn}$ and the other factors to vary slowly compared to t. Here

$$I_{kn}=\int_{\mathrm{vol}} \bar{u}_k(\partial u_n/\partial t)d\tau, \qquad (3)$$

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[1] D. R. Inglis, Phys. Rev. **96**, 1059 (1954).

[2] Aage Bohr, *Rotational States of Atomic Nuclei* (Munksgaards Forlag, Copenhagen, 1954), especially, the appendix. See also A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952); A. Bohr and B. R. Mottleson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[3] M. Born and V. Fock, Z. Physik **51**, 165 (1928); L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 209.

an integral which may be further evaluated,[3]

$$I_{kn}=-(k|(\partial\mathcal{H}/\partial t)|n)/\hbar\omega_{kn}, \tag{4}$$

if one wishes to stress the Hamiltonian rather than the wave functions. In this approximation the coefficient a_k experiences no secular increase, since the time-dependent terms in (2) are rapidly oscillatory. Because of these terms, the system cannot remain in one of these states, with $a_n=1$ and all other $a_k=0$ (unless all the I_{kn} vanish), and if this is introduced as an initial condition, as is frequently done with the $C_k=1$, there is inevitably a small temporary occupation of some of the other states k and a corresponding contribution to the average energy of the system. If instead we seek a solution with one a_n large and all other a_k small, we may obtain an energy with no rapid oscillatory time dependence by taking the small a_k from (2) with the initial condition $C_k=0$, neglecting the small terms. Thus we have

$$E=\int\bar{\psi}\mathcal{H}\psi d\tau=E_n+\sum\nolimits_k(E_k-E_n)\bar{a}_k a_k$$

$$=E_n+\sum\nolimits_k\hbar I_{kn}{}^2/\omega_{kn}. \tag{5}$$

Since I_{kn} contains linearly the rate of change of a parameter which may describe the size or shape or position of the system, that is, essentially a collective speed, the last term of (5) contains the square of a speed and is equivalent to a kinetic energy in problems involving only the variation of this parameter, as is explicitly exemplified in the Appendix.

KINETIC ENERGY OF AN EXPANDING ONE-DIMENSIONAL OSCILLATOR

The manner in which the adiabatic admixture of other "quasi-stationary" states contributes to the energy of the system is very simply illustrated by considering the problem of a one-dimensional oscillator. In the familiar equations,

$$[(\hbar^2/2M)\partial^2/\partial x^2+E_n-(K/2)x^2]u_n=0 \quad\text{or}$$

$$[\partial^2/\partial\xi^2+\lambda_n-\xi^2]u_n(\xi)=0, \tag{6}$$

we consider $K=K(t)$ as a slowly-varying externally imposed parameter, and as a consequence the scale of length of the wave functions varies, $d=d(t)$, with $d^2=(x/\xi)^2=\hbar/M\omega$, $\omega^2=K/M$, $\lambda=2E_n/\hbar\omega=2n+1$.

The $u_n(x,t)$ normalized in x are then $d^{-\frac{1}{2}}u_n(\xi)=d^{-\frac{1}{2}}H_n(\xi)\exp(-\xi^2/2)$, where the $u_n(\xi)$ are normalized in ξ-space, and obey the recursion relations (10) of reference 1. From these we obtain

$$\partial u_n(x,t)/\partial t=(\dot{d}/2d)[(n+1)^{\frac{1}{2}}(n+2)^{\frac{1}{2}}u_{n+2}(\xi)$$

$$-(n-1)^{\frac{1}{2}}n^{\frac{1}{2}}u_{n-2}(\xi)]. \tag{7}$$

This in (3) gives

$$I_{n+2,\,n}=\int_{-\infty}^{\infty}\bar{u}_{n+2}(\partial u_n/\partial t)dx=(n+1)^{\frac{1}{2}}(n+2)^{\frac{1}{2}}\dot{d}/2d$$

$$I_{n-2,\,n}=-(n-1)^{\frac{1}{2}}n^{\frac{1}{2}}\dot{d}/2d. \tag{8}$$

All the other I_{kn} vanish, so a state consisting mainly of the ground state $n=0$ or the first excited state $n=1$ can have an admixture by this adiabatic process only of the state $k=n+2$, with $\omega_{kn}=2\omega$, and the added energy in (5) arising from the admixture is

$$E-E_n=(\hbar\dot{d}^2/8\omega d^2)(n+1)(n+2)$$

$$=(M/8)\dot{d}^2(n+1)(n+2). \quad n=0,1. \tag{9}$$

For a state consisting mainly of a higher state the admixture comes from down as well as up by two units, $\Delta n=\pm 2$, and the sum of these two contributions to the average increment of energy is

$$E-E_n=(M/8)\dot{d}^2[(n+1)(n+2)-(n-1)n]$$

$$=(M/2)\dot{d}^2(n+\tfrac{1}{2}). \tag{10}$$

It will be noted that (10) agrees with (9) and thus applies for all n because $(n-1)n$ vanishes for $n=0, 1$.

The kinetic energy of the distribution $H_n{}^2(x/d)\exp(-x^2/d^2)$ considered as an expanding classical fluid is likewise $(M/2)\dot{d}^2(n+\frac{1}{2})$. The adiabatic admixture of other states u_k may thus be understood as an adequate wave-packet description of the expanding motion. Although we calculate an average over the admixed quasi-stationary states of their excitation energy $\hbar\omega_{kn}$ which is half kinetic and half potential energy in the varying impressed field, it all appears as kinetic energy associated with the rate of expansion.

In one dimension there is no distinction between irrotational and vortex flow and for this reason the example here given is a somewhat trivial one, but the fact that the kinetic energy calculated in one dimension is equivalent to that of a classical expansion of a density distribution, when taken together with the separability of the contributions from the several dimensions in the calculation below, does lead to the expectation that the three-dimensional flow in this approximation is irrotational, corresponding to a simple expansion in one dimension and contraction in the others.

With more particles in the same varying oscillator potential, each would make its contribution (10) to this "collective" kinetic energy of distortion. Since the Hamiltonian is a sum of single-particle terms, there are no cross-term contributions, and the only effect of antisymmetry of the total wave function is to suppress the contributions of individual-particle states with $n\leq n_{\max}-2$, if the states are filled up to and including $n_{\max}$.

COLLECTIVE VIBRATIONS IN THE DEFORMED OSCILLATOR MODEL

In the central model with a harmonic oscillator potential we substitute for the actual average interaction energy the fictitious zeroth-order potential $V(r) = \frac{1}{2}\ \mathrm{K}r^2$ so as to have simple oscillator wave functions. If we deform this in such a way that the sphere on which we have $V(r)$ becomes an ellipsoid of semiaxis ar along the z-axis and br along the x- and y-axes, maintaining volume so that $ab^2=1$, then V may be written[4]

$$\begin{aligned} V(xyz) &= \tfrac{1}{2}K[(x^2+y^2)/b^2+z^2/a^2] \\ &= \tfrac{1}{2}K[(1-\gamma)^{-\frac{1}{3}}(x^2+y^2)+(1-\gamma)^{\frac{2}{3}}z^2] \\ &= \tfrac{1}{2}K[(x^2+y^2)(1+\epsilon)^4+z^2(1+\epsilon)^{-8}] \\ &= \tfrac{1}{2}\sum K_i x_i^2. \end{aligned} \tag{11}$$

We shall use the expansion parameter $\epsilon=(a/b)^{1/6}-1$ for a reason that will appear, in preference to $\gamma=1-(b/a)^2$ used by Pfirsh[4] in discussing static deformation. For a nucleon with oscillator quantum numbers n_x, n_y, n_z we have the energy

$$\begin{aligned} E_n &= \hbar\omega[(n_x+n_y+1)(1+\epsilon)^2+(n_z+\tfrac{1}{2})(1+\epsilon)^{-4}] \\ &\approx \hbar\omega[n_x+n_y+n_z+\tfrac{3}{2}+2(n_x+n_y-2n_z)\epsilon \\ &\qquad +(n_x+n_y+10n_z+6)\epsilon^2+\cdots], \end{aligned} \tag{12}$$

with $\omega=(K/M)^{\frac{1}{2}}$. For each particle the oscillator wave function arising from (1) is a product of three factors, one for each dimension,

$$\psi_{ni}(x_i)=H_{ni}(\xi_i)\exp(-\xi_i^2/2), \tag{13}$$

with $\xi_i=\beta_i x_i$ and $\beta_i^2=M\omega_i/\hbar=(MK_i)^{\frac{1}{2}}/\hbar$. In the oscillator representation of O^{16}, for example, each of the following sets of oscillator quantum numbers is characteristic of four nucleons:

n_x	n_y	n_z	
0	0	0	
0	0	1	(14)
0	1	0	
1	0	0	

Because of the closed-shell nature of this nucleus, this representation is equivalent to one displaying the nucleon orbital angular momenta l through the relations $P_1^1(\cos\theta)e^{i\phi}\sim x+iy$, etc., or to the (jj) representation obtained by introducing spin and taking further linear combinations, and the total spin-orbit coupling energy for the closed shell is zero. The first set contains the four nucleons of the s-shell, the next three sets the twelve nucleons of the p-shell, and the single-nucleon excitation energy from one shell to the next is $\hbar\omega_i$. The total oscillator energy (12) for the four nucleons of He^4, described by the first row of (14), is

$$E_\alpha=2\hbar\omega[2(1+\epsilon)^2+(1+\epsilon)^{-4}]=6\hbar\omega(1+4\epsilon^2+\cdots), \tag{15}$$

and summing similarly over all of (14) for O^{16}, we obtain $E(O^{16})=6E_\alpha$. The term in ϵ^2 arises equally from kinetic and potential energy of the nucleons, a characteristic property of a harmonic oscillator. If we assume that ϵ varies so slowly that the deformation of the oscillators may be considered adiabatic, we may consider (15) to be the equivalent potential energy of the system as a function of ϵ.

In the oscillator approximation the explicit time dependence of the Hamiltonian lies in the elastic constants $K_i(t)$:

$$\partial\mathcal{H}/\partial t=(1/2)\sum_{\text{nucleons}}\sum_i \dot{K}_i x_i^2 \tag{16}$$

x_i being summed over x, y, and z. One sees from (11) that $K_x=K_y=[1+\epsilon(t)]^4K$ and $K_z(t)=[1+\epsilon(t)]^{-8}K$, so $\dot{K}_x=\dot{K}_y=4K(1+\epsilon)^3\dot{\epsilon}$ and $K_z=-8K(1+\epsilon)^{-9}\dot{\epsilon}$. Since the terms in K_i occur in (11) for the individual oscillators, with no cross terms, there exist matrix elements only between states of the system differing by the excitation of a single oscillator, and these matrix elements are simply the individual-oscillator matrix elements for the x_i^2, which obey the selection rule $\Delta n_i=0, \pm 2$. Thus, we have

$$\begin{aligned} (n_i+2|\partial\mathcal{H}/\partial t|n_i) &= (\dot{K}_i/2)(n_i+2|x_i^2|n_i) \\ &= (\dot{K}_i/4\beta_i^2)[(n_i+1)(n_i+2)]^{\frac{1}{2}}. \end{aligned} \tag{17}$$

There is a state k of the system admixed by the distortion process for an excitation by two units of each oscillator, according to (2) and (4). For He^4, for example, one has eight oscillators of the type n_x (or n_y) $=0$ and four of the type $n_z=0$ and from (5) with $\omega_{km}=2\omega_i$, $\omega_x=\omega_y=(1+\epsilon)^2\omega$, $\omega_z=(1+\epsilon)^{-4}\omega$, one obtains

$$\begin{aligned} E_\alpha(\dot{\epsilon}) &= [4(1+\epsilon)^2+2(1+\epsilon)^{-4}](\hbar/\omega)\dot{\epsilon}^2 \\ &= 6(1+4\epsilon^2+\cdots)(\hbar/\omega)\dot{\epsilon}^2. \end{aligned} \tag{18}$$

Similarly for O^{16}, with four times the number of n_i's indicated in (14), we have an effective kinetic energy again just six times as great. The term in the first power of ϵ drops out in this expression because of the choice of the expansion parameter [with the particular powers 4 and -8 in (11)], and we neglect the $\epsilon^2\dot{\epsilon}^2$ term, as is justified for small values of ϵ, in comparison with the leading term in $\dot{\epsilon}^2$. After this explanation of the approximation, the factors $(1+\epsilon)$ may be dropped from the expressions for $\dot{K}_i$ and ω_i in this part of the calculation. If we thus neglect higher powers of ϵ, the sum of (15) and (18) is of the form $(\text{constant}+\frac{1}{2}k\epsilon^2+\frac{1}{2}\mu\dot{\epsilon}^2)$ and in the more general case of any closed-shell nucleus we have

$$\tfrac{1}{2}k=\hbar\omega\sum(n_x+n_y+10n_z+6) \tag{19}$$

$$\tfrac{1}{2}\mu=(\hbar/4\omega)\sum{}'(n_i+1)(n_i+2). \tag{20}$$

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[4] D. Pfirsch, Z. Physik **132**, 409 (1952).

The summation in (19) is over all nucleons and the summation $\sum'$ in (20) is over the n_i of the protons in the two outermost proton-filled shells and similarly for the neutrons, that is, over all the n_i which may be excited by two units without making states forbidden by the exclusion principle. Carrying out the summations over the various nucleon states, without multiplying by the number of nucleons per state, up to include the shell with $n_x+n_y+n_z=n$, one finds

$$\sum_{\text{states}} (n_x+n_y+10n_z+6)=\sum_{\text{states}}{}' (n_i+1)(n_i+2)$$
$$=3\sum_{q=0}^{n}(2q+1)(n+2-q)(n+1-q) \qquad (21)$$

having the numerical values 6, 36, 120, 300, 630, 1176, and 2016 for $n=0, 1, \cdots 6$, respectively. The noteworthy equality of the first two members of (21) means that the calculated frequency of the deformation vibration, $\Omega/2\pi$, is the same for any "closed-shell" nucleus such as listed in Table I of (I), whether or not it has a neutron excess and independent of A:

$$\Omega=(k/\mu)^{\frac{1}{2}}=2\omega. \qquad (22)$$

This result unfortunately denies its own validity, for the adiabatic approximation by which it was derived requires that the frequency of deformation is appreciably less than the frequency of the single-nucleon oscillators, that is, $\Omega\ll\omega$.

DISCUSSION

This unsatisfactory result raises some doubt concerning the possible existence of states of collective deformation of this sort. It probably means more than merely that a convenient method of calculation is not applicable. It suggests that without other physical influences there is not time for the nucleons to define a meaningful vibrating deformed surface in terms of their average positions, and leaves the possibility that some other correlations between nucleon positions, not expressed in terms of the surface, may be more important. In the case of O^{16}, for example, the shell model which seems promising for most of the states of the neighboring nuclei does not explain the first-excited state, a 0^+ state, without two-nucleon excitation and the possibility of accounting for it as a spheroidal surface vibration would be of some interest. There are, on the other hand, special reasons why the correlation in position associated with the stability of the alpha groups of four nucleons should be important in this particular nucleus, and the success of Dennison[5] in accounting for most of its many newly-identified excited states indicates that this type of correlation, leading to a tetrahedral symmetry of the charge distribution, seems here to be dominant. In such a case the effective nuclear surface would also show a distortion with tetrahedral symmetry, but this cannot be included in the simple treatment of the distortion given above, which because it neglects configuration interaction contains the approximations of first-order perturbation theory in addition to the adiabatic approximation.

Correlations invoking the alpha stability through the saturation properties of (phenomenological) nuclear forces would be expected to appear in the second order of a deformed oscillator-model treatment, and might show that the stable shape of a nucleus is a deformation with a favorable symmetry, rather than the spherical shape. This would leave the possibility of small vibrations about the deformed shape analogous in a nebulous way to the molecule-like vibrations of the alpha model, and probably not resembling irrotational flow.

In a nucleus with one shell partly filled with protons and another partly filled with neutrons, there may still be a vibration with a frequency determined mainly by the behavior of the nucleons in unfilled shells. It may be in some sense collective among those nucleons, with a suppressed participation of the closed-shell nucleons given perhaps by the present calculation if the frequency is decreased sufficiently for adiabaticity by such additional complexities as the Gallone-Salvetti "spilling over" discussed in the introduction of reference 1. An extension of this calculation to include the extra nucleons, in order to examine the possible equivalence to irrotational flow, might become rather intricate.

IMPROVED EVALUATION OF THE POTENTIAL

The potential energy responsible for the large value (22) of Ω was calculated from the extreme zeroth-order assumption that the average of the interactions between nucleons may be replaced by the oscillator potential which is unrealistic in having infinitely high boundaries to confine the wave functions. It is perhaps demanding too much of the zeroth-order potential energy to expect it to give the term in ϵ^2 after cancellation of the term in ϵ. We shall show, however, that the first-order potential energy, obtained by integrating over reasonable specific nuclear interactions gives a rather similar result which also fails to justify the adiabatic approximation.

For calculations of this sort, it is customary to admit our ignorance of the detailed nature of nuclear interactions by introducing phenomonological exchange interactions between pairs of nucleons, and the partial success of this procedure in a variety of problems has given rise to the impression that we thereby make a fair approximation to some of the principal dynamical features of the actual interactions which may, for example, be many-particle interactions with the two-particle terms predominant. A convenient form for the assumed interaction is

$$B[\exp(-\alpha r_{ij}^2)]$$
$$\times[(1-g-g_1-g_\sigma)P+gPQ+g_1+g_\sigma Q], \qquad (23)$$

where P exchanges space coordinates and Q spin. The

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[5] D. M. Dennison, Phys. Rev. **96**, 378 (1954). See also D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).

parameter $\alpha^{-\frac{1}{2}}$ thus establishes the range of the interaction, assumed Gaussian. We introduce also the parameters σ_i (replacing β_i) to make $\sigma_i^{-\frac{1}{2}}$ give the extension of the wave functions relative to the range of the interaction, the wave function containing a factor $\exp(-\sigma_i\alpha x_i^2/2)$. Thus $\sigma_i=\beta_i^2/\alpha$, $\sigma\alpha=(MK)^{\frac{1}{2}}/\hbar=M\omega/\hbar$, $\sigma_x=\sigma_y=(1+\epsilon)^4$, $\sigma_z=(1+\epsilon)^{-8}\sigma$. The simple integrations encountered in this "Cartesian" representation are conveniently tabulated in an earlier paper.[6] The average of the interaction (23) summed over all pairs of nucleons, calculated with an antisymmetrized product wave function for the ellipsoidally distorted nucleus He^4 is

$$\bar{U}_\alpha=6B(\sigma_x/\tau_x)(\sigma_z/\tau_z)^{\frac{1}{2}}(1-g-g_\sigma)=6B(\sigma/\tau)^{\frac{3}{2}} \times(1-g-g_\sigma)[1-24(\sigma/\tau)(1-\sigma/\tau)\epsilon^2]+\cdots, \tag{24}$$

and for O^{16} it is

$$\begin{aligned}\bar{U}&=24B(\sigma_x/\tau_x)(\sigma_z/\tau_z)^{\frac{1}{2}}\{X+Y(2/\tau_x+1/\tau_z)\\&\qquad+Z(8/\tau_x^2+4/\tau_x\tau_z+3/\tau_z^2)\}\\&=24B(\sigma/\tau)^{\frac{3}{2}}\{[X+3Y/\tau+15Z/\tau^2]\\&\quad\times[1-24(\sigma/\tau)(1-\sigma/\tau)\epsilon^2+\cdots]\\&\quad-48(\sigma/\tau^2)[Y(1-2\sigma/\tau)+(2Z/\tau)(5-12\sigma/\tau)]\\&\qquad\times\epsilon^2+\cdots\},\end{aligned} \tag{24'}$$

where $\tau=\sigma+2$, $X=5g_1+2(g_\sigma-g)$, $Y=\frac{2}{3}(1+g-5g_1-3g_\sigma)$, and $Z=\frac{1}{4}(1-g-g_\sigma)$. With $\epsilon=0$ this reduces to the first-order result for the undistorted O^{16} nucleus given by Kroeger.[7] Thus $\bar{U}$ appears in the form of $a+b\epsilon^2$ and it is the coefficient b which contributes to the "stiffness" of the nucleus. Another contribution is made by the kinetic energy of the internal motion of the nucleons, which for He^4 in the oscillator approximation is

$$\bar{T}_\alpha=(9/4)(\hbar^2/M)\sigma\alpha(1+4\epsilon^2+\cdots), \tag{25}$$

and for O^{16} is

$$\bar{T}=(69/4)(\hbar^2/M)\sigma\alpha(1+4\epsilon^2+\cdots)\equiv c+d\epsilon^2. \tag{25'}$$

This is essentially half of (15), since a harmonic oscillator has half kinetic and half potential energy, but for He^4 there is a factor $\frac{3}{4}$ and for O^{16} an even more trivial factor 23/24 arising from center-of-mass considerations. The "stiffness" against deformation is then determined by $b+d$, which replaces $k/2$ in (22), to give the frequency of distortional oscillation.

There is considerable arbitrariness in the phenomenological representation of nuclear interactions and it is not immediately clear in a calculation such as this how much the result may depend on this arbitrariness, so we give in Table I the computed numerical result for several fairly reasonable choices of the various parameters involved. The next-to-last row gives the distortional frequency, replacing (22) for He^4, the last row for O^{16}. The first column uses the parameters of reference 7, $\sigma=0.5$ being the result of minimizing the first-order energy of O^{16}. The 1938 range parameter $\alpha\hbar^2/mMc^2=22$ had moved toward shorter ranges from the old long-range value 16 used in 1937, but recent analysis of scattering measurements[8] indicates that the even shorter-range value 26.56 is better. This value, which is used in the rest of Table I, is obtained from the data given in reference 8, pages 56 and 201, and from the unit-of-length conversion factor $\hbar/(mM)^{\frac{1}{2}}c=9.012\times10^{-13}$ cm, thus: $\alpha\hbar^2/mMc^2=(1.4345\times9.012/2.51)^2=26.56$. The values[8] for well depth $-B$ and for $-B(1-g-g_\sigma)$ (that is, "singlet depth"), are then $103.3mc^2$ and $64.5mc^2$, implying $g+g_\sigma=0.1875$. In order to keep the rounded-off value $g+g_\sigma=0.2$, without changing the average of the singlet and triplet depths, we take $B=-105mc^2$. The g's in the second column of Table I are a simplified set which have been used frequently and roughly satisfy the various saturation and other requirements, and the g's in the fifth column correspond[9] to the symmetrical form $\tau_i\cdot\tau_j(0.1+0.23\sigma_i\cdot\sigma_j)$.

TABLE I. Collective frequency of distortion for various choices of the parameters.

$\alpha\hbar^2/Mmc^2$	22	26.56			26.56		
$-B/mc^2$	92	105			105		
g	0	0			−0.26		
g_1	−0.126	0			−0.13		
g_σ	0.22	0.2			0.46		
σ	0.5	0.7	1.0	1.3	0.7	1.0	1.3
Ω_α/ω	1.87	2.12	2.37	2.48	2.12	2.37	2.48
$\Omega(\mathrm{O}^{16})/\omega$	2.40	2.32	2.29	2.20	2.31	2.31	2.26

The choice of nuclear size parameter σ should be made by minimizing the energy. The inadequate available calculations of the total energy serve only to suggest within what range of values it is expected to lie. The first-order energy is given by Eqs. (13) plus (14) with $\epsilon=0$, or by Eq. (8) of reference 7. The g's of column 1 used in reference 7 are less favorable than the others for this purpose. With α and B given by the newer numbers 26.56 and 105, the g's of column 2 give a minimum first-order energy $-50mc^2$ at $\sigma=0.7$, the g's of column 5 give $-89mc^2$ at $\sigma=0.9$, whereas the g's of column 1 give only $-9mc^2$ at $\sigma=0.5$. As an indication of the less pronounced variation with α and B, the latter energy is to be compared with $-19mc^2$ at $\sigma=0.5$ given in reference 7 for the α and B of column 1. The experimental value of the total energy is $-249mc^2$. The improvement available from a more refined calculation is indicated by Kroeger's result[7] that the minimum second-order energy with those parameters is about $-79mc^2$ at the minimizing value $\sigma=0.7$. It is significant that the better calculation not only improves the

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[6] D. R. Inglis, Phys. Rev. **51**, 531 (1937).

[7] W. J. Kroeger, Phys. Rev. **54**, 1048 (1938). Erratum: In Eq. (8), the first term in g_σ should have a negative sign.

[8] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

[9] L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, New York, 1948).

calculated energy but also shifts the minimum from $\sigma=0.5$ to 0.7, indicating a more compact nucleus. It is clear from the improved first-order energy that the parameters of column 5 would give a much more satisfactory second-order energy, though probably still considerably higher than the experimental value, and that the minimizing σ would be somewhat greater than 0.9. A good calculation with parameters capable of giving the experimental energy would be expected to give σ considerably greater than unity, perhaps as high as $\sigma=1.3$. Thus the range $\sigma=0.7$ to 1.3 explored in Table I is about sufficient to cover the reasonable possibilities.

The frequency of collective vibration is thus in these simple examples even higher than in (22), and this more careful treatment of the potential energy fails to rescue the validity of the adiabatic approximation. However, the values of Ω/ω in Table I are only slightly larger than the value 2 of (22), which shows that the oscillator potential is a fairly good approximation to a "self consistent" field for a deformed nucleus, and this is relevant to the use[1] of the oscillator approximation as a criterion for the validity of the assumption of irrotational fluid flow in treating rotating surface waves in the droplet model.

Helpful comments of Aage Bohr have been much appreciated.

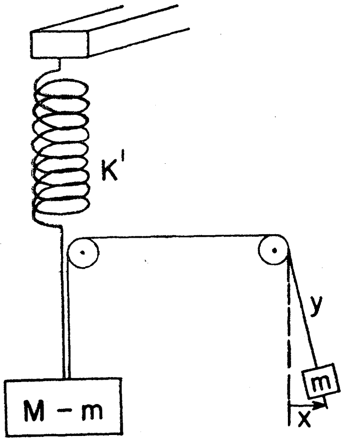

FIG. 1. A simple mechanical system in which the adiabatically varying quantized motion of a small part affects the classical motion of the rest.

APPENDIX

In a problem of this sort it is perhaps instructive to illustrate the predominant physical effect in a very simple case. Consider in the harmonic approximation a simple pendulum of mass m tethered by way of small pulleys to a body of large mass $M-m$ suspended on a spring of stiffness K', as in Fig. 1. The potential and kinetic energies are

$$\begin{aligned} V&=\tfrac{1}{2}K'(y-y_0)^2-mgy(1-x^2/2y^2),\\ T&=\tfrac{1}{2}M\dot{y}^2+\tfrac{1}{2}m\dot{x}^2, \end{aligned} \tag{26}$$

where the length of the pendulum, y, describes the motion of the large mass and where x is the horizontal displacement of the pendulum. If y varies slowly compared to x, we may treat the pendulum swing separately, its wave equation being (6) (with $M\rightarrow m$), where $K=mg/y=m\omega^2=\hbar^2/md^4$. The average value of the terms in (26) containing x and $\dot{x}$ is given by E of Eq. (10) and the total energy E_y associated with the y-motion is thus

$$\begin{aligned} E_y=\tfrac{1}{2}[M+(\hbar/16g^{\frac{1}{2}}y^{\frac{3}{2}})(n+\tfrac{1}{2})]\dot{y}^2\\ +\tfrac{1}{2}K'(y-y_0)^2-mgy+(n+\tfrac{1}{2})\hbar(g/y)^{\frac{1}{2}}. \end{aligned} \tag{27}$$

This is not in Hamiltonian form, since the momentum conjugate to y has not been defined. With the pendulum excited to the quantum number n, there is an equilibrium position y_n, and in terms of the small coordinate $z=y-y_n$ we may write the total energy:

$$\begin{aligned} E_y&=\tfrac{1}{2}\mu\dot{z}^2+\tfrac{1}{2}\kappa z^2+\text{consts},\\ \mu&=M+(\hbar/16g^{\frac{1}{2}}y_n^{\frac{3}{2}})(n+\tfrac{1}{2}),\\ \kappa&=K'+(3\hbar g^{\frac{1}{2}}/4y_n^{5/2})(n+\tfrac{1}{2}). \end{aligned} \tag{28}$$

Energy is conserved with a simple solution $z=A\cos\Omega t$ if

$$\Omega=(\kappa/\mu)^{\frac{1}{2}}=\Omega_0\{1+[3-(\Omega_0/2\omega)^2](n+\tfrac{1}{2})\hbar/8K'y_n^2+\cdots\}$$

with $\Omega_0=(K'/M)^{\frac{1}{2}}$ and $\omega=(g/y_n)^{\frac{1}{2}}$. Thus the frequency of the slow classical motion is influenced by the quantization of the small rapid part of the system. The term containing the factor 3 arises from expansion of the quantum energy $(n+\frac{1}{2})\hbar(g/y)^{\frac{1}{2}}$, and may be obtained as well from the constancy of the action variable in the classical adiabatic discussion. The last term arises from the mixing of the quasi-stationary states.