Goldstein's Eigenvalue Problem

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(Received August 26, 1954)

A mathematical error is found in Goldstein's solution of the Salpeter-Bethe equation with zero total energy and a cut-off factor. When this error is corrected, no solution remains of the eigenvalue problem.

I. INTRODUCTION

ALL solutions of the Salpeter-Bethe equation so far obtained for the two-nucleon system have been based on rather drastic approximations whose quantitative effect is not easy to see. Klein[1] has examined rather exhaustively the limitations of the "ladder approximation"; but he allowed two other common approximations, the neglect of the time of propagation of the virtual mesons, and the omission of pair contributions, to go unchecked. Arnowitt and Gasiorowicz[2] have recently discussed pair effects, but the effect of the "static potential" approximation is still largely unknown, and most likely could only appear from a completely covariant solution of the S-B equation.

For this reason, the covariant treatment of the problem by Goldstein[3] was of great interest, even though it was limited to the ladder approximation and a highly unrealistic value (zero) of the total energy. Goldstein showed that the S-B equation as it stands, at least in the special case he considered, has a solution for any value of the coupling constant. He suggested that the only way to obtain an eigenvalue of the coupling constant, consistent with the occurrence of a bound state of the two nucleons, was to introduce a cut-off factor in the kernel of the S-B equation. Unfortunately, as will appear below, his solution of the resulting eigenvalue problem has a mathematical flaw, the elimination of which invalidates his conclusion.

Solutions of the S-B equation exist of a more general type than, but for the same total energy as, that which Goldstein considered; but all solutions investigated disappear when a cutoff is introduced. The reason is probably related to the extreme value of the binding energy assumed. So Goldstein's essential idea—that the bound-state problem should be solved by a limiting process involving the use of a convergence factor—may very well be right in general. For independent reasons stated elsewhere,[4] the author does in fact agree with Goldstein on the matter. It is hoped that approximate covariant solutions of the S-B equations for general values of the binding energy will soon be available to afford a more conclusive test of the idea.

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[2] R. Arnowitt and S. Gasiorowicz, Phys. Rev. **94**, 1057 (1954).
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[4] I. E. McCarthy and H. S. Green, Proc. Phys. Soc. (London) **A67**, 719 (1954).

II. SOLUTION OF THE S-B EQUATION

The integral equation whose solution is required is

$$(p-m)\varphi(p)(p+m)=\frac{\lambda}{i\pi^2}\int\frac{c(k^2,\Omega)\gamma_5\varphi(k)\gamma_5 d^4k}{(p-k)^2}, \quad (1)$$

where $c(k^2,\Omega)$ is the cut-off factor which has a value near unity for $k^2\ll\Omega$ and near zero for $k^2\gg\Omega$. Goldstein proposed to solve it by reducing it to a differential equation; this can be achieved, without making any special assumptions about the form of $\varphi(p)$, by applying the operator $\square=\partial^2/\partial p^\mu\partial p_\mu$ to both sides of (1). Since

$$\square\{(p-k)^2\}^{-1}=-(2\pi)^2 i\delta(p-k), \quad (2)$$

the result is

$$\square\psi(p)+4\lambda c(p^2,\Omega)\gamma_5\varphi(p)\gamma_5,$$
$$\psi(p)=(p-m)\varphi(p)(p+m). \quad (3)$$

This equation has to be supplemented by boundary conditions, which can be obtained by inspection from (1); these are

$$\psi(p)=\text{finite constant},\quad p=0; \quad (4)$$

$$p^2\psi(p)=\text{finite constant},\quad p^2\gg\Omega. \quad (5)$$

If now one assumes with Goldstein that $\varphi(p)$ has the form

$$\varphi(p)=\Phi(p^2/m^2), \quad (6)$$

Eq. (3) reduces to

$$X''(s)+\lambda C(s,\Omega)\Phi(s),$$
$$X(s)=s(s-1)\Phi(s), \quad (7)$$

with $s=p^2/m^2$. If one takes $C(s,\Omega)=1$ for $s<\Omega$ and $C(s,\Omega)=0$ for $s>\Omega$, the conditions (4) and (5) require that

$$\Phi(0)=\text{finite constant} \quad (8)$$

and

$$X'(\Omega)=0, \quad (9)$$

which are readily seen to be equivalent to Goldstein's conditions (16a) and (16b) (the latter with Ω substituted for ∞).

Goldstein gave

$$\Phi(s)=\text{const}\times F(1+\alpha,\, 2-\alpha;\, 2;\, s) \quad (10)$$

as that solution of (7) which satisfies (8), where the constant α is a root of the equation

$$\alpha(\alpha-1)+\lambda=0 \quad (11)$$

and is to be determined from the boundary condition (9). The "hypergeometric function" is, however, easily expressed in terms of Legendre functions, as one can see by writing

$$z=2s-1, \tag{12}$$

whereupon (7) becomes, for $z<\Omega$,

$$(z^2-1)X''(z)+\alpha(\alpha-1)X(z)=0. \tag{13}$$

Since

$$\frac{d}{dz}\left\{(z^2-1)\frac{dP_{\alpha-1}(z)}{dz}\right\}+\alpha(\alpha-1)P_{\alpha-1}(z)=0,$$

one sees that the general solution of (13) is

$$X(z)=\int_{-z}^{1}P_{\alpha-1}(z)dz+A\int_{1}^{z}P_{\alpha-1}(z)dz. \tag{14}$$

Here one must set $A=0$ to satisfy (8), and (9) reduces to

$$P_{\alpha-1}(1-2\Omega)=0. \tag{15}$$

For real positive values of λ, α is restricted to real values between 0 and 1, and complex values of the form $\frac{1}{2}(1+i\eta)$, where η is real. In the real domain of α, it is well known[5] that (15) can only be satisfied by values of Ω between 0 and 1. In the complex domain, (15) can only be satisfied by values of Ω less than 0. Hence (15) has no solution for values of Ω greater than unity.

To see how Goldstein obtained a different result, notice that the asymptotic form of (15) is

$$\frac{\Gamma(2\alpha-1)(-\Omega)^{-\alpha}}{\Gamma(\alpha-1)\Gamma(1-\alpha)}+\frac{\Gamma(1-2\alpha)(-\Omega)^{\alpha-1}}{\Gamma(\alpha)\Gamma(-\alpha)}=0, \tag{16}$$

or, since $\Gamma(z+1)=z\Gamma(z)$ and $\sin(\pi z)\Gamma(z)\Gamma(1-z)=\pi$,

$$\frac{\sin(\pi\alpha)}{\pi(2\alpha-1)}\{(\alpha-1)\Gamma(2\alpha)(-\Omega)^{-\alpha}+\alpha\Gamma(2-2\alpha)(-\Omega)^{\alpha-1}\}=0. \tag{17}$$

This agrees with the condition Goldstein at first obtains; but he proceeds to cancel a factor and write the condition in a form [Eq. (40)]:

$$\frac{(-1)^{2\alpha}\Gamma(2\alpha)}{\Gamma(2-2\alpha)}=\Omega^{2\alpha-1}\frac{\alpha}{\alpha-1},$$

which is satisfied by $\alpha=\frac{1}{2}$. The factor $(2\alpha-1)^{-1}$ in (17), however, may clearly *not* be removed for $\alpha=\frac{1}{2}$ without introducing a spurious solution; and the left-hand side of (17) in fact approaches a nonzero value as $\alpha\rightarrow\frac{1}{2}$.

III. OTHER TRIAL SOLUTIONS

One possible reason which suggests itself for the absence of a solution to Goldstein's eigenvalue problem, is that the bound-state solution might not be of the type assumed in (6). If one assumes instead that

$$\psi(p)=\Psi(p^2/m^2)p, \tag{18}$$

one obtains

$$(s-1)s\Psi''(s)+3(s-1)\Psi'(s)-\lambda\Psi(s). \tag{19}$$

The conditions (4) and (5) now require that $\Psi(0)$ should be finite and that the derivative of $s^2\Psi(s)$ should vanish for $s=\Omega$. [The latter is also the condition that $\Psi(s)$ and $\Psi'(s)$ should be continuous at $s=\Omega$.] The solution of (19) which is finite for $s=0$ is

$$\Psi(s)=F(a,b;3;s), \tag{20}$$

where

$$a=1+(1+\lambda)^{\frac{1}{2}},\quad b=1-(1+\lambda)^{\frac{1}{2}}, \tag{21}$$

and one has also

$$\frac{d}{ds}\{s^2\Psi(s)\}=2sF(a,b;2;s), \tag{22}$$

so that to satisfy the boundary condition at $s=\Omega$, $F(a,b;2;\Omega)$ must vanish. Now

$$F(a,b;2;\Omega)=\frac{\Gamma(a-b)(-\Omega)^{-a}}{\Gamma(b)\Gamma(a-2)}F(a,a-1;1+a-b;\Omega^{-1})+\frac{\Gamma(b-a)(-\Omega)^{-b}}{\Gamma(a)\Gamma(b-2)}F(b,b-1;1+b-a;\Omega^{-1}); \tag{23}$$

and since, for large values of Ω, Ω^a and Ω^b will be of different orders of magnitude, $F(a,b;2;\Omega)$ cannot vanish except possibly in the degenerate case when $a-b$ is an integer. Then $F(a,b;2,s)$ reduces to one of the Jacobi polynomials $F_n(2,2,s)$; but the zeros of these polynomials are known to lie in the interval $0<s<1$, so they cannot provide a solution of the problem either.

This investigation has shown that there are no covariant solutions of the integral equation (1) with real positive values of λ. There might possibly be noncovariant solutions, and since the derivation of Eq. (1) assumes a relativistic frame in which the mass-center of the two nucleons is at rest, it is fairly plausible that the bound-state solution, if any, should depend on p_4 and γ_4 as well as the covariant variables p^2 and p. But a full investigation shows that the independent solutions of (3), which are functions of $s=p^2$ and $z=p_4(p^2)^{-\frac{1}{2}}$, and are finite for $s=0$, are of the form $(s-1)^{-1}s^{\frac{1}{2}n}F(\frac{1}{2}n+\alpha,\frac{1}{2}n+1-\alpha;2+n;s)C_n^1(z)$, where α is the greater root of the equation

$$\alpha(1-\alpha)+\tfrac{1}{2}n(\tfrac{1}{2}n+1)=\lambda,$$

n is any integer, and $C_n^1(z)$ is the corresponding Gegenbauer polynomial. The boundary condition requires that the derivative $s^nF(\frac{1}{2}n+\alpha,\frac{1}{2}n+1-\alpha;n+1;s)$ of $s^{n+1}F(\frac{1}{2}n+\alpha,\frac{1}{2}n+1-\alpha;2+n;s)$ should vanish for $s=\Omega$,

[5] See W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1949).

and this is impossible for any positive real value of λ. Adding a factor p to the trial solution does not improve the situation.

Thus even the possibility of noncovariant solutions of the integral equation has to be excluded.

IV. DISCUSSION

The foregoing conclusions might be interpreted as casting doubt on the ability of the S-B equation to predict bound states. It might appear that the solutions of this equation obtained by various authors are merely a feature of the noncovariant approximations which they used. The present author, however, would prefer to take the view that the extreme value of the binding energy assumed by Goldstein is responsible for his failure to obtain a valid discrete value of the coupling constant. If one gave the binding energy of the ground state its maximum value (infinity) in the nonrelativistic approximation, one would not get a solution there either. One may thus maintain that to give a binding energy equal to the total rest-energy of the two nucleons, the coupling constant would have to be infinite, and that the possibility of a discrete finite value for any other binding energy is not excluded.

Goldstein stated that an expansion of the solution in powers of the total energy appeared to be singular, and if that is so it rather supports such a conclusion. But what is really needed is independent evidence of the nature of the solutions of the S-B equation for general values of the binding energy. Such evidence should not be founded on a noncovariant approximation, as covariance is clearly the crux of the matter. The author hopes to present a completely covariant treatment of the S-B equation in the near future.

PHYSICAL REVIEW VOLUME 97, NUMBER 2 JANUARY 15, 1955

Coulomb Functions for Large Charges and Small Velocities*

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(Received July 30, 1954)

Expansions in powers of $\eta^{-\frac{1}{3}}$, where η is defined in the introduction below, for the Coulomb wave functions $F_L(\rho)$ and $G_L(\rho)$ and their derivatives are given for special values of $\rho=2\eta$ and $\rho=\rho_L=\eta+[\eta^2+L(L+1)]^{\frac{1}{2}}$, the classical turning points for $L=0$ and any L, respectively. Expansions applicable in the vicinity of the turning point are given as a series involving Bessel functions of order $\pm n/3$ with the expansion parameter $\rho_L^{-\frac{1}{3}}$. Approximations valid for large values of η are given and discussed.

I. INTRODUCTION

NUCLEAR reactions involving "heavy" charged particles[1,2] and the inelastic scattering of charged particles by nuclei[3,4] have recently been the object of several investigations, both theoretical and experimental. In both cases, the Coulomb interaction can be expected to play a dominant role, and the Coulomb wave functions are necessary for discussions of nuclear interactions of this type. It is evident, that for the parameter

$$\eta=ZZ'e^2/\hbar v,$$

which, together with $\rho=kr$ and L, characterize the Coulomb function,[5] the values of interest will be fairly large; η, for example, lies in the range 5−15. Tabulations in this particular range of parameters are either unavailable or incomplete[6] and the present work was undertaken to fill this need as far as feasible, with particular emphasis on large values of the parameter η. It extends and supplements the earlier work of Breit and his associates,[7] and of Abramowitz and Morse,[8] and in part runs parallel to or overlaps work of Newton,[9]

* Assisted by the Office of Ordnance Research, U. S. Army and by the joint program of the U. S. Office of Naval Research and U. S. Atomic Energy Commission.
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