Geons*

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Associated with an electromagnetic disturbance is a mass, the gravitational attraction of which under appropriate circumstances is capable of holding the disturbance together for a time long in comparison with the characteristic periods of the system. Such gravitational-electromagnetic entities, or "geons"; are analyzed via classical relativity theory. They furnish for the first time a completely classical, divergence-free, self-consistent picture of the Newtonian concept of body over the range of masses from $\sim 10^{39}$ g to $\sim 10^{57}$ g. Smaller geons are quantum objects whose analysis would call for the treatment of characteristic new effects. Topics covered in the discussion include: 1. Need for a self-consistent formulation of the concept of "body" in classical physics; geons *vs* free waves; electrical neutrality of geon; size and mass relations; the quantum limit and electron pair phenomena. 2. Orders of magnitude for toroidal geons; first estimates of leakage rates; a "phosphor" model of a geon; attrition and attritivity; energy action relation. 3. Idealized spherical geon; conditions required for symmetry; instability relative to pairing of light rays; time scale of instability long compared to vibration periods; spherical metric; wave equation for electromagnetic potential; evaluation of stress-energy tensor; its position as source of gravitation field; the gravitational field equations; the three equations of the self-consistent geon; simplification by scale transformation; first analysis of the eigenvalue problem; further scale transformation to get behavior of solution in active region of geon; further analysis of eigenvalue dependence; electronic calculator integration of equations of self-consistent geon; mass and radius values. 4. Transformations and interactions of electromagnetic geons; evaluation of refractive index barrier penetration integral for spherical geon; photon-photon collision processes as additional mechanism for escape of energy from system; restatement in language of coupling of characteristic modes; the thermal geon; comparison of gravitation and virtual electron pair phenomena as sources of coupling between modes; gravitational coupling and collective vibrations of geon; fission of a geon; interaction between two geons simple at large distances; orientation dependence and exponential term at intermediate distances; violent transmutation processes in closer encounters. 5. Influence of virtual pairs on geon structure; description in terms of refractive index correction; relation to photon-photon collision picture; more precise formulation via Heisenberg-Euler electrodynamics; corrections to stress-energy tensor and electromagnetic field equations. 6. Neutrino-containing geons; general similarity to electromagnetic geons; specificity of geon-geon interactions; the size subject to simple analysis unexpectedly limited by neutrino-neutrino encounters and the process $\nu+\nu\rightarrow\mu+e$; similarity of size limitation to that for electromagnetic geons; comments on present status of neutrino theory of light. 7. Electricity, Gauss's theorem, and gravitational field fluctuations. 8. Conclusions: the geon completes the scheme of classical physics; one's interest in following geons into quantum domain will depend upon one's view of the relation between very small geons and elementary particles.

1. INTRODUCTION AND SUMMARY

THE position of the concept of "body"[1] in the general theory of relativity[2] has always been interesting. A planet moving as a body along a geodesic is the idealization behind one of the most important predictions of the general theory, and certainly the one that has received the most thorough observational con-

* The word "geon" is used here as an abbreviation for the phrase "gravitational-electromagnetic entity" and in place of the name kugelblitz or ball of light previously used in a survey of the problems of fields and particles: Am. J. Phys. (to be published); for some other aspects of these problems, see also J. phys. rad. (to be published); see also the point of view ascribed by the author to Sugawara-no-Michizane in Proc. Phys. Soc. Japan **9**, 36 (1954). The present article is part V of a study of classical field theory. I is unwritten. In parts II, J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **17**, 157 (1945), III, J. A. Wheeler and R. P. Feynman, Revs. Modern Phys. **21**, 425 (1949), and IV, in preparation in collaboration with Professor Gilbert N. Plass, the emphasis is upon action at a distance, not as a way of understanding charged bodies, but as a way of understanding the fields that act between them. In the present paper the emphasis is on the fields interior to a *classical* body.

[1] The noun "body" is used here to connote an object possessed of a mass and of three position coordinates, and subject both externally and internally to a classical description, in contrast to the notion of a particle, with at least the interior of which quantum properties are associated.

[2] By "general theory of relativity" is meant here that battle-tested system of ideas and equations which Einstein developed in 1915 to describe gravitation and electromagnetism. I am indebted to Professor Einstein for several interesting discussions of the evolution of his ideas and of their relation to Newtonian concepts. Excluded here from the phrase "general theory" is the cosmological term, absent with good reason from the original formulation, introduced only when it was found that the original theory could not account for a static universe, and disowned when the universe was found not to be static. Excluded also are varied modifications of general relativity which attempt to "unify" gravitation and electricity, the latest of which has been shown by Joseph Callaway, Phys. Rev. **92**, 1567 (1953) to predict that a test particle will move as if uncharged, no matter how much charge is loaded upon it. In the accepted formulation of general relativity—to use the metric conventions of Pauli's treatise—the proper distance, ds, or proper interval of cotime (c times time), $d\tau$, between two neighboring events is given by

$$(ds)^2=-(d\tau)^2=g_{\alpha\beta}dx^{\alpha}dx^{\beta}.$$

The state of the space time continuum is specified by giving in addition to the ten gravitational potentials, $g_{ik}=g_{ki}$, the four electromagnetic potentials, A_i. The electromagnetic field, $F_{ik}=\partial A_k/\partial x^i-\partial A_i/\partial x^k$, has six distinct components, whereas gravitational effects are expressed in a covariant way by the twenty distinct components of the curvature tensor R_{abcd}; or by the mixed components of the same tensor,

$$R^h{}_{ikj}=\partial\Gamma_{ij}{}^h/\partial x^k-\partial\Gamma_{ik}{}^h/\partial x^j+\Gamma_{kd}{}^h\Gamma_{ij}{}^d-\Gamma_{jd}{}^h\Gamma_{ik}{}^d,$$

where the typical Γ is an abbreviation for

$$\Gamma_{ij,\,k}=\tfrac{1}{2}(\partial g_{jk}/\partial x^i+\partial g_{ik}/\partial x^j-\partial g_{ij}/\partial x^k).$$

Of the twenty distinct components of R_{abcd}, ten remain in the contracted curvature tensor $R_{ik}=R_{i\alpha k}{}^{\alpha}$; and of these one remains in the curvature invariant $R=g^{\alpha\beta}R_{\alpha\beta}$. The Einstein tensor $G_{ik}=R_{ik}-\frac{1}{2}g_{ik}R$ may be considered to be the translation into the language of general relativity of the notion of d'Alembertian of the gravitational potentials, g_{ik}. The gravitational field equations thus have the form

$$G_{ik}=(8\pi G/c^4)T_{ik},$$

where $G=6.67\times10^{-8}$ cm^3/g sec^2 is the Newtonian constant of gravitation, $c=3.00\times10^{10}$ cm/sec is the speed of light, and T_{ik} is the symmetric stress-momentum-energy tensor of the electro-

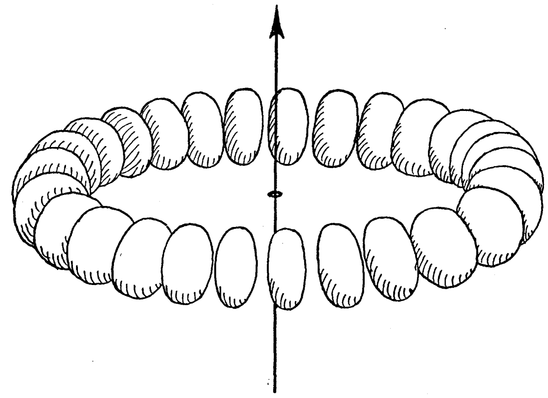

FIG. 1. Regions of strong electric field strength in a simple toroidal geon of zero angular momentum. Two waves of equal strength run around the torus in opposite directions to produce a standing wave, with electric fields strong in the regions indicated, and magnetic fields strong in the region between. The gravitational field created by this disturbance, and required to hold the disturbance together, is representable to a good approximation as static and independent of azimuth.

firmation.[3] In recent years a great advance has been made in the theory. The geodesic equation that determines the motion of the object in a known field, and the field equations that find the metric from the motions of the masses, have been shown not both to be necessary, as they have been in every other formulation of physics. Instead, the equation of motion of the body has been shown to follow as a *consequence* of the field equations.[4] This circumstance, the fact that relativity is the first description of nature that makes geometry a part of physics, the absence of any acceptable alternative of comparable scope, and the battle-tested internal consistency of the theory, all make it is necessary to take seriously general relativity and explore further its consequences and concepts. Of these concepts the notion of body is in an unhappy state. Either one sticks to general relativity as it is, and treats the object as a singularity in the metric, or one postulates that the field is regular everywhere, and counts on quantum theory somehow to explain how this can be so even in the region of the body. Both approaches lead for the time being to an impasse. For this reason it is interesting to discover that there exists a third possibility. On the basis of classical general relativity as it already exists, and without any call on quantum theory, it turns out to be possible to construct an entity that we call geon. This object serves as a classical, singularity free, exemplar of the "bodies" of classical physics. Of such entities there exist in principle a great variety, distinguished from one another by mass, intrinsic angular momentum, and other properties. The simplest variety (Fig. 1) is most easily visualized as a standing electromagnetic wave, or beam of light, bent into a closed circular toroid of high energy concentration. It is held in this form by the gravitational attraction of the mass associated with the field energy itself. It is a self-consistent solution of the problem of coupled electromagnetic and gravitational fields, as defined by the system of equations,

$$G_{ik}=(8\pi G/c^4)T_{ik}, \tag{1}$$

$$T_i^{\ k}=(1/4\pi)(F_{i\alpha}F^{k\alpha}-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta_i^{\ k}), \tag{2}$$

and

$$(-g)^{-\frac{1}{2}}(\partial/\partial x^\alpha)(-g)^{\frac{1}{2}}F^{i\alpha}=0, \tag{3}$$

all derivable from the action principle:

$$\delta\int\int\int\int\{(1/16\pi c)F_{\alpha\beta}F^{\alpha\beta} - (c^3/16\pi G)R\}d(\text{four-volume})=0. \tag{4}$$

An order of magnitude discussion of the properties of these entities appears in Sec. 2. Section 3 presents theory and results of electronic machine solutions of the self-consistent field equations for the particularly simple but rather idealized case of a spherical geon. The existence of geons would seem to impart to classical general relativity theory a comprehensiveness for which one had not dared to hope. This theory turns out not only to account for the fields produced by bodies, and the motions of bodies, but even to explain why there should be bodies. In this sense classical relativity theory would seem to be revealed as a logically self-consistent and completed whole of unexpected comprehensiveness.

Having geons as model for the bodies of classical physics, we can put into a new perspective some parts of general relativity. First, the geodesic equation of motion of a body displays itself clearly as an idealization. To be able to give any meaning to the metric that appears in this equation, we have to be able to speak of the "background field"—the gravitational and electromagnetic magnitudes that would be present in the absence of the geon. This will only be possible when the total fields at some distance from the object vary slowly over a distance comparable to the linear exten-

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magnetic field,

$$T_{ik}=(\partial/\partial g^{ik})(1/8\pi)F_{\alpha\beta}F^{\alpha\beta}.$$

Of the equations for the electromagnetic field itself, half are already automatically satisfied by virtue of the way the fields are expressed in terms of the potentials, A_i. The other half are usually written as inhomogeneous equations,

$$(-g)^{-\frac{1}{2}}(\partial/\partial x^\alpha)(-g)^{\frac{1}{2}}F^{i\alpha}=4\pi s^i,$$

where g is the determinant of the g_{ik}, or where $(-g)^{\frac{1}{2}}$ is the ratio of an element of four volume to the product $dx^1dx^2dx^3dx^4$, and where the four vector s^i describes the density and flow of free electricity. We omit this source term (a) because we do not need it in the considerations of this article (b) because there is no self-consistent classical theory for the existence of electric charge (c) because the considerations of Sec. 1 and Sec. 7 suggest that the appearance of free electricity is a quantum phenomenon. Thus limited to charge free space, general relativity constitutes a well defined, self-consistent, self-contained whole.

[3] G. M. Clemence, Revs. Modern Phys. **19**, 361 (1947); Proc. Am. Phil. Soc. **93**, 532 (1949).

[4] L. Infeld and A. Schild, Revs. Modern Phys. **21**, 408 (1949); see also Einstein, Infeld, and Hoffmann, Ann. Math. **39**, 65 (1938); D. M. Chase, Phys. Rev. **95**, 243 (1954); and a paper by L. Infeld, Acta Phys. Polonica (to be published) for a perusal and discussion of which I am indebted to Professor Infeld.

sion of the geon. In this case the object will accelerate coherently under the action of external forces. The happenings fit into the scheme of physics envisaged by Newton and Maxwell: preexisting bodies generating forces and being acted upon by forces. The appropriate modifications to pre-Einsteinian physics associated with special and general relativity of course add themselves automatically. On the other hand, when the fields outside the geon approach non-uniform values, then this entity does not react as a unit. Its internal degrees of freedom are disturbed. When the external inhomogeneities are strong, the system will break into two or more parts. There is nothing in what we know today of physics that makes unreasonable the failure under such conditions of the idealization of body. Evidently there exist conditions where the idealization of "body" loses all significance, and where there comes into play a new kind of physics of geons—but a physics that is still entirely classical (Sec. 4).

The existence of geon transformation processes makes clear a second point about classical relativity physics, that there exists in principle no sharp distinction between geons as concentrations of electromagnetic energy, capable of break-up and integration processes, and the "free" electromagnetic waves that pass through the space between geons, and experience scattering, absorption, or emission by geons. Legalistically speaking, the state of the universe of classical physics is described by the singularity free electromagnetic and gravitational magnitudes at every point, and by nothing more.

Finally, physics as rounded out by the geon concept forms the only fully comprehensive and self-consistent system of classical physics that we possess. This picture of bodies and fields leaves as little place for free electric charges as for isolated magnetic poles: none at all. The electromagnetic field is as free of singularity as the gravitational field. Lines of forces never begin and never end. At most they form loops that shrink to zero and disappear, or reappear, as time goes on and field strengths change. No natural way is evident to escape the conclusion that the quantum of electricity has to do with the quantum of action and does not belong within the framework of a *completely* classical physics. Considering that our physical world extends at least from 10^{-13} cm to 10^{27} cm, and that in the face of these forty decades or more e^2 differs from $\hbar c$ by only two powers of ten, it would also seem that one has some independent reason to believe that free electricity is a quantum phenomenon. Therefore it appears reasonable to accept as a description of a *classical* body the always electrically neutral geon.

What is the range of sizes of classical geons? These objects obey a simple scaling law. From one self consistent solution of the coupled equations of gravitation and electromagnetism there follows another solution of of $(1/n)$th the mass, by decreasing all distances by a factor n, increasing all electromagnetic field strengths by a factor n, and leaving all the gravitational potentials g_{ik} unaltered in strength. In this change the classical action integral, J, associated with the field disturbance falls to $1/n^2$ of its original value. Such a scaling law must exist because the field equations contain only the gravitation constant and the velocity of light, out of which one can form no quantity with the dimensions either of a length or a mass or a field strength or an action. Consistent with the scaling law, but even more specific, are the order-of-magnitude formulas of Sec. 2 for the simplest circular toroidal geons, with a number of standing waves around the circumference equal to an integer or azimuthal index number, a:

$$\text{mass} \sim a^{\frac{1}{2}}\,(\text{action}\cdot c/G)^{\frac{1}{2}},$$

$$\text{major radius} \sim a^{\frac{1}{2}}\,(\text{action}\cdot G/c^3)^{\frac{1}{2}},$$

$$\text{minor radius} \sim \text{small multiple of } \lambdabar,$$

$$\lambdabar = (\text{wavelength}/2\pi) \sim a^{-\frac{1}{2}}\,(\text{action}\cdot G/c^3)^{\frac{1}{2}},$$

$$\text{frequency} \sim a^{\frac{1}{2}}\,(c^5/G\cdot\text{action})^{\frac{1}{2}},$$

$$\left\{\begin{matrix}\text{mass density in}\\ \text{active region}\end{matrix}\right\} \sim \left\{\begin{matrix}\text{reasonable fraction of}\\ (ac^5/G^2\cdot\text{action})\end{matrix}\right\},$$

$$\left\{\begin{matrix}\text{peak field values}\\ \text{in electrostatic}\\ \text{volts/cm or gauss}\end{matrix}\right\} \sim \left\{\begin{matrix}\text{reasonable fraction of}\\ (ac^7/G^2\cdot\text{action})^{\frac{1}{2}}\end{matrix}\right\}. \quad (5)$$

The upper limit to the size of geons is the linear extension of the universe itself. This limit shows itself in the following way. The self-consistent field equations of the geon possess solutions, the properties of which depend upon the boundary conditions. In simple geon theory we impose the requirement that g_{ik}'s are asymptotic to a flat metric at distances large in comparison with the extension of the geon. This boundary condition has to be modified when the size of the geon is comparable to the radius of curvature of a closed universe. Then the relations (5) undergo characteristic corrections, such that the mass of the geon can never exceed the mass of the universe.

Incidentally it is interesting to notice that a purely classical closed universe consisting of a large number, N, of geons of comparable size must consist mostly of empty space. This conclusion follows from the fact that the proportionality factor between radius and mass,

$$\text{radius} \sim (G/c^2)\,\text{mass} = (0.74\times10^{-28}\,\text{cm/g})\,\text{mass}$$

is the same in order of magnitude for geons as for the universe itself. Consequently a ratio $1/N$ between the mass of the parts and the mass of the whole imples a ratio $\sim N(1/N)^3 = 1/N^2$ between the volume of the parts and the volume of the whole.

To find the lower limit to the size of classical geons, we have to investigate in turn the several physical magnitudes that characterize this object, and see which of these magnitudes, with decreasing geon size, first passes out of the domain of application of classical theory. (1) Is action the critical magnitude? Certainly one will be in the quantum domain when the action is comparable to the quantum of angular momentum, $\hbar = 1.054\times10^{-27}$ g cm^2/sec. Inserting this value into the formulas (5), we find

$$\text{mass}\sim a^{\frac{1}{2}}2.18\times10^{-5}\text{ g},$$
$$\text{major radius}\sim a^{\frac{1}{2}}1.63\times10^{-33}\text{ cm}.$$

We can stop here with the evaluation. We are evidently far below the limit where it might have been right to use classical theory, and *quite outside the domain of application* of Eqs. (5). Electron theory allows no possibility to deal with distances smaller than the reduced Compton wavelength, $\hbar/mc = 3.87\times10^{-11}$ cm, without taking into account complex and specifically quantum mechanical fluctuations in the distribution of pairs and in the electromagnetic field. What about fluctuations in the gravitational field? They appear to be negligible on any ordinary scale of distances, as one sees from the following reasoning. In a region with space and cotime extensions of the order of L the relevant fluctuations in the electromagnetic field[5] are of the order $\Delta\mathcal{E}\sim(\hbar c)^{\frac{1}{2}}/L^2$, and $\Delta A\sim(\hbar c)^{\frac{1}{2}}/L$ in the potentials. The similarity in character of the gravitational and electromagnetic field equations, and the difference in the way they contain the various fundamental constants, indicate that the corresponding formula for the fluctuation in a typical gravitational potential is

$$\Delta g\sim(\hbar G/c^3)^{\frac{1}{2}}/L. \qquad (6)$$

These fluctuations will be inappreciable in comparison to typical average values of the metric, $g\sim1$, so long as the distances, L, under consideration are substantial in comparison with the characteristic length,

$$L^* = (\hbar G/c^3)^{\frac{1}{2}} = 1.63\times10^{-33}\text{ cm}. \qquad (7)$$

This is exactly the distance in comparison with which we have just concluded that classical geons must be enormous.

(2) Is distance the critical magnitude? We set the reduced wavelength in Eqs. (5) equal to $\hbar/mc$ and solve for the action and other physical magnitudes, finding

$$\text{action}\sim a(\hbar c/m^2G)\hbar = a5.72\times10^{44}\hbar,$$
$$\text{mass}\sim a(5.69\times10^{44})^{\frac{1}{2}}2.15\times10^{-5}\text{ g} = a5.12\times10^{17}\text{ g},$$
$$\text{major radius}\sim a3.87\times10^{-11}\text{ cm},$$
$$\text{minor radius}\sim\text{small multiple of }3.87\times10^{-11}\text{ cm},$$
$$(\text{wavelength}/2\pi)\sim3.87\times10^{-11}\text{ cm},$$
$$\text{frequency}\sim mc^2/\hbar,$$

all of which is so far acceptable; but the mass density in the active region,

$$\text{geon density}\sim\text{a reasonable fraction of } (c^2/G)(mc/\hbar)^{\frac{1}{2}} = 0.90\times10^{49}\text{ g/cm}^3$$

is obviously fantastically in excess of the density of nuclear matter,

$$\text{nuclear density}\sim1836m/[(4\pi/3)(e^2/2mc^2)^3] = 1.4\times10^{14}\text{ g/cm}^3. \qquad (8)$$

Consequently we still find ourselves in a domain where the classical geon relations (5) *cannot be applied*.

(3) Is density the limiting magnitude? Taking this time nuclear densities, (8), as the reference point, we find from (5) values of the action, mass, radius, and frequency of the geon which are safely outside the obvious quantum limits; but the field strength in the active region,

$$(c^2 1.4\times10^{14}\text{ g/cm}^3)^{\frac{1}{2}} = 3.6\times10^{17} \text{ (gauss or electrostatic volts/cm)},$$

is then 8000 times greater than the critical value met in the quantum electrodynamics of the vacuum,

$$\mathcal{E}_{\text{crit}} = m^2c^3/e\hbar = 4.42\times10^{13} \text{ (gauss or electrostatic volts/cm)}. \qquad (9)$$

Thus, when an electric field $\mathcal{E}$, working on an elementary charge e over the localizability distance $\hbar/mc$, can impart to an electron an energy of the order mc^2, then this electric field will bring forth from empty space pairs of positive and negative electrons. Under such conditions the geon requires for its description the specification of the state of the pairs as well as the statement of the electromagnetic and gravitational field strengths. Again we find ourselves outside the range of validity of the classical geon equations (5).

(4) We conclude that the *critical field strength*, $\mathcal{E}_{\text{crit}}$, *of pair theory marks the lower limit of classical geons.* Thus, simple *toroidal* electromagnetic geons are only then purely classical entities, when their magnitudes are on the large geon side of the following limits:

$$\text{action}\sim\text{reasonable fraction of } ac^7/G^2\mathcal{E}_{\text{crit}}^2 = a\hbar(e^2/Gm^2)^2(\hbar e/e^2) = 2.38\times10^{87}a\hbar,$$
$$\text{mass}\sim\text{reasonable fraction of } ac^4/G^{\frac{3}{2}}\mathcal{E}_{\text{crit}} = a1.065\times10^{39}\text{ g},$$
$$\text{major radius}\sim\text{reasonable fraction of } ac^2/G^{\frac{1}{2}}\mathcal{E}_{\text{crit}} = a0.791\times10^{11}\text{ cm},$$
$$\text{minor radius}\sim0.791\times10^{11}\text{ cm},$$
$$(\text{wavelength}/2\pi)\sim0.791\times10^{11}\text{ cm},$$
$$\text{frequency}\sim1\text{ vibration}/16.6\text{ sec},$$
$$\text{mass density}\sim\mathcal{E}_{\text{crit}}^2/c^2 = 2.16\times10^6\text{ g/cm}^3,$$
$$\text{peak electric field}\sim\mathcal{E}_{\text{crit}}\text{ of Eq. (9)}. \qquad (10)$$

One has only to compare these critical dimensions with the properties of the sun, mass $=1.97\times10^{33}$ g, radius $=6.95\times10^{10}$ cm, to conclude that even the lightest *classical* geons form entities enormous in comparison to the objects studied in the laboratory. But the critical

[5] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab., Mat-fys. Medd. **12**, No. 8 (1933); Phys. Rev. **78**, 794 (1950).

dimensions are still small compared to the scale of the universe. There is ample room between 10^{11} cm and 10^{28} cm to talk of classical geons and their motions, interactions and transformations. In this sense we continue to regard classical general relativity as a completed self contained subject with a well-defined scope of its own.

Having surveyed the boundaries of the classical theory of geons, we can touch on a few of the implications of quantum theory for these objects. First, as to the general situation, it is clear that the critical magnitudes of Eq. (10) prevent us in no way whatever from considering geons of lower mass. It is only required that we take quantum effects properly into account. It is also clear that these effects will be of various kinds. As we move down in mass, first one effect, previously unimportant, will become decisive, then another effect, and so on. In each region certain idealizations will be appropriate. In each region the formulas connecting mass and other geon properties with the quantity of action will have characteristic forms of their own. Presumably the investigation of each region will present successively greater difficulties. Second, the quantum region earliest encountered (Sec. 5) will lie between the realm of classical electrodynamics and the domain where fields are so strong that substantial numbers of pairs appear. In this intermediate "Region II," the fields vary in space and time fantastically slowly in comparison with the characteristic distances and times of electron theory. Moreover, they are strong enough to produce only virtual pairs. These pairs give rise to charges and currents that make a substantial contribution to the total field, as first analyzed in detail by Euler and Kockel and Heisenberg.[6] In quantum language, if two opposed photons of high frequency can produce a pair, then two opposed photons of low frequency can produce a virtual pair. With the reannihilation of the pair the photons go off in altered directions; hence a scattering of light by light[7] and an effectively nonlinear electrodynamics. The consequence of this nonlinearity for a simple toroidal electromagnetic geon is most simply envisaged as an increase of refractive index in the region of concentration of electromagnetic energy. The effective value of the speed of light is reduced. Geon mass no longer scales in proportion to the square root of the action variable. Instead, as the action of the system diminishes, the mass appears to fall faster than linearly [Eq. (81)]. However, soon we are into Region III, where real pairs first appear in large numbers. Here new studies must be made before any results can be stated.

As third circumstance in the relation of the quantum to the geon, there exist neutrinos: entities apparently coordinate with photons in the description of nature. Their Fermi-Dirac statistics, unlike those of photons, makes it impossible for more than one to be accommodated in a state of definite wave number and polarization. From them no disturbance can be built up of a classical magnitude, describable in correspondence principle terms. They have an inescapably quantum character. Apart from this circumstance they can be used in the construction of geons just as well as photons. In addition to purely electromagnetic geons there consequently also exist in principle (Sec. 6) purely neutrino geons, and geons of mixed type. Thus half integral as well as integral spins are permitted to geons.

As last implication of the quantum for geon physics, the fluctuations in the gravitational metric are inescapable, little as one can say (Sec. 7) about their consequences for the validity of Gausses theorem and for the existence of free electric charge.

Details of the discussion follow.

2. ORDERS OF MAGNITUDE FOR SIMPLE TOROIDAL ELECTROMAGNETIC GEONS

The gravitational deflection of a pencil of light into a torus is no different in principle from the deflection of light by the sun. Apart from factors of the order of two[8] one can estimate the deflection by equating a kinematic acceleration to a gravitational acceleration:

$$c^2/R \sim GM/R^2.$$

Thus the radius R of the torus and the mass M are connected by the relation

$$R \sim (G/c^2)M = (0.741 \times 10^{-28}\ \text{cm/g})M, \qquad (11)$$

This formula is familiar in another connection, for it supplies the well known relation between the mass of an object and its so called gravitational radius, a measure of the distances at which the gravitational potentials depart significantly from the values appropriate to flat space. This circumstance makes it evident that any accurate treatment of geon properties has to be carried out within the framework of general relativity.

Near a section of the torus the gravitational field will resemble approximately the field due to an infinitely long cylindrically symmetrical distribution of mass. It will increase inversely as the first power of the distance, ρ, from the center of the cylinder, until the point of observation moves into the region of strong energy concentration. There the gravitational potential will no longer continue its logarithmic increase.

The narrowness of the pencil of light, or of the region of strong energy concentration, will depend upon the wavelength. Define by a large integer a an azimuthal index number, and let the wavelength constitute the small fraction, $1/a$, of the circumference:

$$(\text{wavelength}/2\pi) \equiv \lambda\!\!\!^{-} \sim R/a.$$

[6] H. Euler and B. Kockel, Naturwiss. **23**, 246 (1935); W. Heisenberg and H. Euler, Physik **98**, 714 (1936).

[7] Low-frequency cross section in reference 6; high-frequency cross section in A. Achieser, Physik Z. Sowjetunion **11**, 263 (1937); forward scattering at all frequencies calculated by R. Karplus and M. Neuman, Phys. Rev. **83**, 776 (1951); derived from dispersion relation by J. Toll (to be published).

[8] See for example R. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, Clarendon Press, 1934).

The pencil of light cannot be concentrated into a region smaller in lateral extension than the order of magnitude of $\lambdabar$ by any type of variation of refractive index with distance. Owing to the logarithmic variation of gravitational potential and effective refractive index near the cylinder, the lateral extension of the disturbance in the case of the torus will be a small multiple of $\lambdabar$, say $F\lambdabar$, where the dimensionless factor F will be approximately represented by a simple power of $\ln(R/\lambdabar)=\ln a$. Even if a equals 10^{10}, this logarithm is only 23.

The circumstance that the minor radius of the torus is significantly larger than $\lambdabar$ implies that the gravitation field can be taken to be static in a reasonable approximation. Thus the source of the gravitation field, in accordance with the principle of equivalence of mass and energy, is the electromagnetic stress-energy tensor. This tensor varies rapidly over a distance of the order of $\lambdabar$. But the elementary contribution to the gravitation field is essentially a $1/r^2$, or long-range, field. The field, even in the most active part of the geon, comes mostly from distances of the order of the minor radius, $F\lambdabar$. Consequently local variations in gravitational source strength are relatively unimportant. What does count is the source strength averaged over a wavelength. In other words, the gravitational field may be regarded as depending upon ρ and z, to use cylindrical polar coordinates, but not on φ and t.

Of course nothing prevents consideration of toroidal geons of small azimuthal index, a, but then it is no longer such a good approximation to treat the gravitational field as static, and the situation is much more complicated to discuss. Moreover, such geons disintegrate rapidly.

Even for the case of large azimuthal index number there is a difference in simplicity between those toroidal geons that carry equal electromagnetic waves going in the positive and negative senses around the ring, and those where the disturbance runs all one way, or there are two disturbances of different magnitudes. In the first case we have to do with a system of zero angular momentum. The electromagnetic disturbance is a simple standing wave. Many components of the metric tensor vanish when expressed in cylindrical polar coordinates.[9] Significantly more distinct components of g_{ik} have to be considered when the system has angular momentum.[10] Particularly interesting is the case when the disturbance is unidirectional. Then the angular momentum of the system is of the order

$$\text{angular momentum}\sim McR\sim(c^3/G)R^2. \qquad (13)$$

In the active region of a simple classical toroidal electromagnetic geon the field strengths will be of the order of magnitude $\mathcal{E}$, where

$$\mathcal{E}^2R(\Delta R)^2\sim Mc^2. \qquad (14)$$

Consequently $\mathcal{E}\Delta R$, the order of magnitude of the typical difference of potential between two sides of the pencil of radiation, will have a universal value,

$$\text{potential difference}\sim(Mc^2/R)^{\frac{1}{2}}\sim c^2/G^{\frac{1}{2}}$$
$$=3.49\times10^{24}(\text{gauss cm or electrostatic volts}), \qquad (15)$$

enough to impart to a particle of electronic charge and mass an energy greater than its rest energy by a factor $(e^2/Gm^2)^{\frac{1}{2}}$. This factor is the root of the ratio of the electric and gravitational forces between two electrons, and has the value $(4.17\times10^{42})^{\frac{1}{2}}=2.04\times10^{21}$.

Through the symmetry axis of the torus pass a typical meridian plane, and note the point in this plane where it intersects the region of maximum field strength in the torus. Then over a circle of radius $\sim\Delta R$ centered on this point the field strength has values smaller than the peak magnitude by a factor of only one or two powers of two. At distances from the most active point several times the magnitude ΔR, the field strengths fall off exponentially with a characteristic decay length of the order of magnitude of $\lambdabar$. This type of decay is familiar from the study of the propagation of light along the length of a long thin solid glass rod. The disturbance in the space surrounding the rod also has a characteristic decrement length of the order of $\lambdabar$. In the case of the geon we deal with a medium whose effective refractive index is nonuniform. The gravitational field has fallen off substantially at distances from the symmetry center of the geon of the order of magnitude of $2R$ and greater. Consequently the electromagnetic disturbance sufficiently far from the geon finds itself once again in an allowed region, where it can propagate normally. The strength of the electromagnetic field outside is an exceedingly small fraction of the strength inside, the approximate value of this fraction being given by an algebraic function of the large number $R/\lambdabar=a$ multiplied by an approximately exponential factor of the form $e(-\text{constant}\cdot a)$.

The geon is thus not in principle an isolated entity. The object in question, and every other classical geon in a classical universe, are nothing but manifestations of the same all pervading electromagnetic field. But the field outside is so extremely small in comparison with the field inside that for most purposes the geon has the character of a well-defined body.

Whether the field outside the geon is an outgoing wave or an incoming wave or a standing wave depends upon the initial conditions. It will be most relevant to consider here, as in most problems of physics, an irreversible dissipation of energy. The external wave then has no incoming component. It describes a continual transport of energy—and mass—out of the geon. We are faced with a purely classical analog to the Gamow-Condon-Gurney theory of radioactive alpha decay.

[9] Frederick J. Ernst, Jr., "Cylindrically Symmetric Fields in General Relativity," Junior Independent Paper, Princeton, 5 May 1954 (unpublished).

[10] For examples of gravitational sources endowed with angular momentum, see for example, G. E. Tauber, Ph.D. thesis, University of Minnesota, 1952 (unpublished).

It seems appropriate to give the same name of radioactive decay to the geon process, though for our present purposes this process is to be regarded as having absolutely no quantum character. Of course, were we temporarily to abandon the ground of classical physics, we would describe the emission process in the language of quanta.

As a model showing the slow decay characteristic of a relatively stable geon, one can consider a sphere of highly transparent glass with a radius of the order of ten wavelengths of light. A high-speed electron which enters the sphere produces ions and excites electromagnetic disturbances including modes of vibration of visible wavelength. Of many of these modes the energy will leak out of the refractive index barrier quickly because of the nearly normal incidence of the equivalent ray system upon the air interface; but those described by spherical harmonics of the highest order, and therefore represented by the most nearly tangential rays, will be endowed with lives as long as 10^{-6} sec. The exponential fall of the relevant modes in the air just outside the sphere "phosphor" will be similar to the behavior in the air above a totally internally reflecting prism. In that case the decay continues indefinitely with distance; but in the spherical case the decay ceases, and free oscillation commences, at a distance comparable to the radius of curvature of the surface.

As the geon slowly loses mass, it shrinks in accordance with the similarity law for these bodies. At the same time the circular frequency, ω, of the emerging electromagnetic disturbances goes up in inverse proportion to the radius and mass. Consequently the ratio of the loss per unit time to what remains at that time is not a constant, as in the usual theory of radioactive decay, but is proportional to the current frequency scale of geon processes:

$$dM/M = dR/R = -d\omega/\omega = -\alpha\omega dt. \tag{16}$$

Here α is a dimensionless constant, the "attrition," or fractional loss of mass per radian of the electromagnetic vibration. Integration of (16) gives

$$-t = 1/\alpha\omega; \tag{17}$$

the mass of the geon decreases linearly with time. The negative sign distinguishes the time at which the geon is observed to have the frequency ω from the time at which it collapses (quantum modifications in the later stages of the decay process being overlooked). Thus the reciprocal of the attrition measures the time to collapse in units of the present time per radian.

What has been said here refers to a geon energized by only a single mode of electromagnetic vibration. When several disturbances are present of different frequencies, they will in general leak out at different rates. Moreover, nonstatic components of the gravitational field will furnish a weak nonlinear coupling between the modes, so that relative amplitudes and frequencies on this account also will change gradually with time. The evolution of the system in time is no longer describable as a simple scale transformation. Now it is appropriate to divide up the output of energy, or mass, dM, in the elementary time interval dt, into elementary parts, δdM, each of which refers to that part of the loss which occurs in a specific interval of circular frequency, $\delta\omega$:

$$(\delta dM)/M = -\beta\delta\omega dt. \tag{18}$$

Here β is again a dimensionless quantity, the "attritivity" (attrition constant) of the geon. The attritivity is of course a function of the frequency in question; or more conveniently, it depends on the dimensionless measure of frequency, $\omega^* = MG\omega/c^3$. The fact that a general geon changes the form of its spectral output as time advances prevents us in no way from considering two geons of quite distinct sizes, one of which can be obtained from the other by a simple scale transformation. Both geons then have the same dimensionless attritivity function, $\beta = \beta(\omega^*)$.

The frequency of the outgoing radiation, even in the case of a simple unimodal geon, can only be defined within a latitude of the order $\Delta\omega \sim \alpha\omega$. An analysis of the external electromagnetic field in time, and an analysis in terms of frequency, stand to each other in principle in a mutually exclusive relation. It is only the smallness of α that allows us to speak approximately of the frequency as a function of time. This we do when we go beyond simple transliteration of the function $\mathcal{E} = \text{constant} \cos[-\alpha^{-1}\ln(-t)]$ to the form $\mathcal{E} = \text{constant} \cos\int\omega dt$, and give $\omega(t) = -1/\alpha t$ the name of "frequency." On this account it is really optional whether the emission spectrum is regarded to be a sharp line of slowly changing frequency, or to be a continuous spectrum concentrated within a region of the order α about a center of gravity of gradually shifting location. In this sense we have a choice whether to describe the decay of a unimodal geon by way of a single number, the attrition, α, or by way of an attritivity function, $\beta(\omega^*)$:

$$\alpha = (\omega^*)^{-1} \int_{\substack{\text{neighborhood} \\ \text{of } \omega^*}} \beta(\omega^*) d\omega^*. \tag{19}$$

In the case of a geon energized by several modes of electromagnetic vibration, the frequencies and amplitudes of which are continually changing by reason of their mutual interaction, the relevant quantity for the description of the slow leakage of energy out of the system is the attritivity.

When a simple monochromatic toroidal geon leaks an amount of radiation energy dE, then its reduced action, $\mathcal{J} = \text{action}/2\pi$, decreases by an amount that satisfies the general relation

$$c^2 dM = dE = \omega d\mathcal{J}. \tag{20}$$

When the decrease becomes substantial in comparison with the original values, then the frequency ω changes substantially. On this account we cannot apply to a geon the special relation, $E=\mathcal{J}\omega$, valid for a harmonic oscillator. Instead, we can note that the circular frequency of the electromagnetic disturbance rises in inverse proportion, $\omega\sim c/\lambda\!\!\!^{-}\sim c/(R/a)$, as the mass and radius of the system diminish:

$$GM/c^2\sim R\sim ac/\omega. \qquad (21)$$

Multiplying together from (20) and (21) the left- and right-hand sides, respectively, we find

$$Gd(M^2)\sim acd\mathcal{J},$$

or

$$M\sim a^{\frac{1}{2}}(c\mathcal{J}/G)^{\frac{1}{2}}. \qquad (22)$$

Thus it is not the mass values themselves, but the squares of the mass values, that are separated in proportion to the intervals of action.

From the mass value (22) follow at once the Eqs. (5) of the introduction for the other physical quantities of the geon as a function of action.

3. THE IDEALIZED SPHERICAL GEON

The simple toroidal geon forms the most elementary object of geon theory much as a simple circular orbit constitutes the first concept of planetary theory. But the simplest physics does not go in the geon case with the simplest mathematics. Toroidal geometry and general relativistic field equations each have their complications, and the mixture requires some time for its analysis.[9] On this account it is natural to look for a geon with spherical symmetry: a rotation-invariant gravitational field, and a spherically symmetric distribution of gravitational source strength, or of stress-energy. Temporarily to adopt a photon point of view, we recognize that each photon orbit is a great circle. Therefor spherical symmetry in the density and flux of photons requires spherical symmetry in the distribution of their angular momentum vectors.

The different elementary disturbances must have different frequencies. If all had the same frequency, they would add coherently to form a single mode of distribution of electromagnetic field strength. But there is no such thing as a nonzero source-free spherically symmetrical electromagnetic field disturbance. Incoherence is essential for sphericity. Let the distribution of field strength be symbolized by the expression

$$\mathcal{E}=\sum E_i(x)\sin(\omega_i t+\delta_i),$$

and the distribution of stress-energy by

$$\mathcal{E}^2=\sum_{i,k}\mathcal{E}_i(x)\mathcal{E}_k(x)\sin(\omega_i t+\delta_i)\sin(\omega_k t+\delta_k).$$

Then distinction of frequencies and randomness of phases is essential to justify the approximation

$$\mathcal{E}^2=\tfrac{1}{2}\textstyle\sum_i \mathcal{E}_i^2(x).$$

The distribution of stress-energy being thus approximately static, it can also be made spherically symmetrical by properly coordinated choices of the elementary solutions $\mathcal{E}_i(x)$.

A geon spherically symmetric in the sense just described is in principle unstable with respect to transformation into a toroidal geon. Tolman showed long ago[8] that two nearly parallel pencils of light attract gravitationally with twice the strength one might have thought when their propagation vectors are oppositely directed, and when similarly directed attract not at all. Consequently a system of randomly oriented circular rays of light will drop to a state of greater stability when half of the angular momentum vectors orient themselves parallel, half antiparallel, to a certain direction in space. Then the number of attractive bonds between orbits will be maximized, and the nullity of the angular momentum will be conserved. Simultaneously there will occur a readjustment in the orbital radii.

The spherical geon, though thus unstable, is in unstable *equilibrium*. We can compare it to a pendulum standing the wrong way up. To envisage such a situation would not be of much use if one had it in mind to discuss the oscillations of the pendulum about its point of support, but is quite acceptable if our aim is to discuss the rigidity of the pendulum rod. Likewise the periods of the various electromagnetic modes of vibration of the geon must be judged very short in comparison with the time of turnover of the system into a toroidal system. In this sense we can talk of the properties of the spherical geon in a reasonably well defined way. It is not necessary to treat all questions of geon stability in order to undertake the problem of the structure of a spherical geon.

The assumed sphericity of the system might appear to be a self-contradictory notion. Spherical symmetry of the static gravitational field, or of the effective refractive index, implies degeneracy of the modes of electromagnetic field oscillation. However, identity of frequency of the various modes, $\mathcal{E}_i(x)$, will rule out the incoherence of the various disturbances so necessary for a spherically uniform mass distribution. Two factors allow us to avoid this degeneracy: slight differences in the scale of the excited modes, and slight departures from spherical symmetry. We consider a disturbance with no radial nodes, and characterize it by the order, l, of the relevant spherical harmonic, and by the azimuthal index number a, of this spherical harmonic. The mode of azimuthal index $a=l$ is concentrated in a toroidal region of major radius $R\sim l\lambda\!\!\!^{-}$ and minor radius $\Delta R\sim l^{\frac{1}{2}}\lambda\!\!\!^{-}$. The angular dependence of the intensity, for example, shows itself simply in the expression

$$\text{constant}\,|P_l^{(l)}(\theta)e^{il\varphi}|^2=\sin^{2l}\theta$$
$$=(1-\cos^2\theta)^l\doteq[1-(\Delta\theta)^2]^l\doteq\exp[-(l^{\frac{1}{2}}\Delta\theta)^2],$$

which falls to $\sim(1/e)$th of its value in a distance

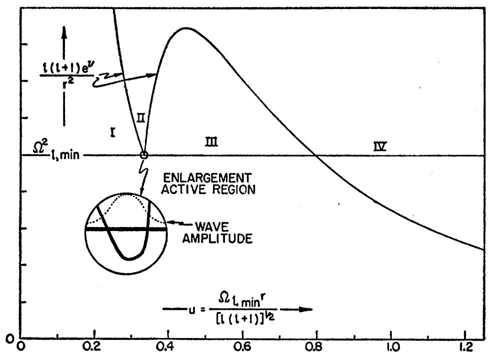

FIG. 2. Case of simple idealized spherical geon; distance dependence of factors in the differential equation for the electromagnetic vector potential,

$$d^2R/dr^{*2}+[\Omega^2-l(l+1)e^\nu/r^2]R=0.$$

The proper solution, R, of lowest circular frequency, Ω/c, for a given high order of spherical harmonic, l, has the qualitative behavior indicated by the dotted line in the diagram. This solution rises exponentially with r in region I, has a single maximum in region II, falls off exponentially in region III, and in region IV oscillates with a very small amplitude (leakage wave). There exist also solutions of the same order, l, but of higher frequency, Ω, corresponding to a higher position for the horizontal line in the figure. The square of the wave number is represented by the distance measured positively upwards from the curve to the horizontal line. An increase of $l^*=[l(l+1)]^{\frac{1}{2}}$ and of Ω by the same factor leaves all turning points unchanged. However, the height of the barrier against leakage is increased as l^{*2}, so that the exponent of the penetration factor is proportional to l^*. The size of the active region is very small on the scale of the diagram, being proportional only to $l^{*\frac{1}{3}}$; hence the enlargement of this region, presented more in detail in Fig. 3. In the present figure the left-hand section of the curve up to $u=b\div\frac{1}{3}$ is given by the expression b^2/u^2; and the right-hand region by $(1/u^2)[1-b(1-b^2/u)]$.

$\Delta R=R\Delta\theta\sim R/l^{\frac{1}{2}}$, measured angularwise. Radialwise let us consider a fixed and pre-existing metric (Fig. 2). Then for each angular order of l there exists a minimum circular frequency, $\omega=c\Omega_{l,\min}$, such that the corresponding mode shows only a single radial maximum in the active region. For different l's the maxima overlap one another. Relative to the coordinate, $r=R$, of this maximum, the quantity $1/\Omega\equiv\lambda\!\!\!^{-}$ has a value of the order of R/l. The width of the region of maximum activity is of the order of $R^{\frac{1}{3}}\lambda\!\!\!^{-\frac{2}{3}}\sim l^{\frac{1}{3}}\lambda\!\!\!^{-}\sim R/l^{\frac{2}{3}}$, as follows by simple JWKB analysis from the circumstance that the square of the wave number changes linearly with departure from the point of maximum wave number with a slope, $d(\text{wave number})^2/dr$, of the order of $\Omega^2/R\sim l^2/R^3$. Thus a sequence of l values yields a family of modes, of which the stress-energy is all concentrated in the same region, $r=R$. To have the spatial spread of energy also nearly the same for all modes, we have only to ask that the range, Δl, of the orders of the activated modes should not exceed some reasonable fraction of l itself. As the other factor freeing the oscillations from degeneracy we must admit some departure of the average gravitational field from exact spherical symmetry. A slight ellipticity will split up into $l+1$ separate frequencies the otherwise $(2l+1)$-fold degenerate vibrations of order l. Counting together both possibilities at our disposal, we have of the order of l^2 modes with distinct frequencies. On the other hand, to obtain a distribution of energy that is completely symmetric in angle it is sufficient to add the squares of the $(2l+1)$ normalized spherical harmonics of order l, according to the well-known completeness theorem of spherical harmonics. Moreover, we do not demand for the geon a completely smooth distribution of the energy over the surface, for the long-range character of the law of gravitation smooths out in the field the minor irregularities that were present in the mass density. Consequently we can actually do with disturbances having considerably fewer than $2l+1$ values of the azimuthal index, a. Comparing $2l+1$ with $\sim l^2$, we conclude that we have more than adequate margin to obtain the postulated uniformity of energy distribution with nondegenerate modes.

For the purposes of calculations it is justifiable to idealize the picture we have just formed. The radial distribution of energy does not differ importantly from one of the excited modes to another; consequently we treat the radial factors in all the distributions as identical to that calculated for a single fixed value of l. The total angular distribution is nearly uniform, so we take it to be exactly uniform; i.e., we take the oscillations of the $2l+1$ values of a belonging to the given order, l, as normalized to identical energy contents. The only consequence of exact spherical symmetry that we do not accept is *coherence* of the $2l+1$ modes. In the actual "spherical" geon, we know that the frequencies are distinct; so in the idealized spherical geon, where legalistically the frequencies come out identical, we disregard this circumstance, and add energy densities according to the law for incoherent disturbances.

To proceed to details, we make Schwarzschild's choice of polar coordinate system, r, θ, φ, such that elementary distances at right angles to the radius vector are represented by the usual geometrical formula. Then dr gives proper distances, ds, in the radial direction only after correction by a certain r-dependent factor. Likewise dt, or rather the cotime, $dT=cdt$, gives correctly intervals of proper cotime, $d\tau$, only after multiplication by another r-dependent factor. Both correction factors are independent of T because the gravitational field is static. One factor is the reciprocal of the other in the well-known Schwarzschild solution for the metric about a point mass, but no such simple relation holds in the interior of an object with the distributed mass of a geon:

$$(ds)^2=-(d\tau)^2=g_{\alpha\beta}dx^\alpha dx^\beta=e^{\lambda(r)}(dr)^2 +r^2[(d\theta)^2+(\sin\theta d\varphi)^2]-e^{\nu(r)}(dT)^2. \quad (23)$$

As solution of the electromagnetic field equations it is reasonable to look for a periodic function of time, multiplied by a function of r, multiplied by a function of angle generated from the spherical harmonic $(\sin\theta)^l\exp(il\varphi)$. Such a disturbance is easily visualized in terms of a toroidal concentration of electromagnetic energy. The other elementary disturbances that we need to give a spherically symmetric distribution of energy are readily generated from this mode of vibration by the operations of the rotation group. This circumstance means that we can consider as starting point one mode as well as another. Mathematically simplest is not the choice just named, with $a=l$, but the spherical harmonic azimuthal index $a=0$, for in this case the dependence upon φ disappears. Thus we look for a solution of the form

$$A_r=A_\theta=A_T=0;$$
$$A_\varphi=(\sin\Omega T)R(r)\sin\theta(d/d\theta)P_l(\cos\theta), \quad (24)$$

where either group theory or the relevant differential equation tells us at once the form of the angular factor. The electric field points always in the φ direction and the magnetic field and the Poynting vector lie always in the meridian plane:

$$\begin{aligned} E_\varphi \sim F^{T\varphi} &= -e^{-\nu}(r\sin\theta)^{-2}\partial A_\varphi/\partial T, \\ -H_\theta \sim F^{r\varphi} &= e^{-\lambda}(r\sin\theta)^{-2}\partial A_\varphi/\partial r, \\ H_r \sim F^{\theta\varphi} &= (r^2\sin\theta)^{-2}\partial A_\varphi/\partial\theta. \end{aligned} \quad (25)$$

In principle we should consider modes which differ from (25) by the interchange of the role of electric and magnetic vectors. To do so would make no difference in the final averaged energy and stress densities, for divergence-free relativity is, despite its appearance, completely symmetric between electricity and magnetism. Consequently we do not need to sum over polarizations. We deal with a standing wave with flow of energy back and forth from north to south pole of the sphere, concentrated in the present case in an active region between R and $R+\Delta R$. We have the superposition of a number of toroidal disturbances, each passing through the north and south poles, with all azimuths of orientation of the tori weighted equally. In line with this picture, A_φ is stronger in the region of overlap near the poles than it is at the equator.

Of the electromagnetic field equation (3), three components unite in saying that A_φ should be independent of φ, and in saying only this: while the remaining, or φ, component, makes the statement

$$\begin{aligned} &(\partial/\partial r)\exp[(\nu-\lambda)/2](\sin\theta)^{-1}\partial A_\varphi/\partial r \\ &\quad+(\partial/\partial\theta)\exp[(\nu+\lambda)/2](r^2\sin\theta)^{-1}\partial A_\varphi/\partial\theta \\ &\qquad-\exp[-(\nu-\lambda)/2](\sin\theta)^{-1}\partial^2 A_\varphi/\partial T^2=0, \end{aligned}$$

whence the radial wave equation

$$d^2R/dr^{*2}+[\Omega^2-l(l+1)e^\nu/r^2]R=0. \quad (26)$$

Here dr^* is an abbreviation for

$$dr^*=[\exp(\lambda-\nu)/2]dr.$$

Let us imagine that we have calculated the time-average value, $\langle T_i{}^k{}_{(\mathrm{I})}\rangle$, of the electromagnetic stress-momentum-energy tensor of the disturbance under discussion: mode number I. This tensor depends only on r and θ. Let there be altogether N modes of different frequencies excited, all of about the same strength and properties as mode I, but differing from it in orientation. Let the distribution of stress, momentum, and energy in a typical one of these disturbances be visualized by mapping out on a rigid sphere the distribution of mode I, and then rotating the north pole of this sphere to an angle α with respect to its original position, and to an azimuth β. Thus $\langle T_i{}^k{}_{(D)}\rangle$, for a typical disturbance, D, is a tensor function of r, θ, and φ, the functional dependence being completely known in principle as soon as we know the angles α and β associated with the mode D. Let us sum the values at a typical point, θ, φ, of the energy density over all N modes, obtaining

$$\langle T_T{}^T\rangle=(N/4\pi)\int\int\langle T_T{}^T{}_{(D)}\rangle\sin\alpha d\alpha d\beta. \quad (27)$$

This result cannot depend upon θ and φ, because of the spherical symmetry. Consequently the point of evaluation can be taken to lie at the original location of the north pole, $\theta=0$. Here the energy density of disturbance D is the same as was the energy density of disturbance I at the polar angle $\theta=\alpha$. Consequently (27) reduces to the form

$$\langle T_T{}^T\rangle=(N/2)\int\langle T_T{}^T{}_{(\mathrm{I})}\rangle\sin\theta d\theta. \quad (28)$$

Similarly we obtain for the radial tension

$$\langle T_r{}^r\rangle=(N/2)\int\langle T_r{}^r{}_{(\mathrm{I})}\rangle\sin\theta d\theta. \quad (29)$$

In evaluating the tangential components of the tension we have to proceed more carefully, for the definition of the relevant directions differs between the original mode and the rotated modes. Consequently we have to employ the standard rules for transformation of the components of a tensor. Fortunately all the transformations are carried out in the local tangent space, and the rules for Cartesian tensors apply:

$$\begin{aligned} \langle T_\theta{}^\theta\rangle &= (N/4\pi)\int[\langle T_\theta{}^\theta\rangle\cos\beta\cos\beta \\ &\quad+\langle T_\theta{}^\varphi{}_{(\mathrm{I})}+T_\varphi{}^\theta{}_{(\mathrm{I})}\rangle\sin\beta\cos\beta \\ &\quad+\langle T_\varphi{}^\varphi{}_{(\mathrm{I})}\rangle\sin\beta\sin\beta]\sin\alpha d\alpha d\beta \\ &= (N/4)\int\langle T_\theta{}^\theta{}_{(\mathrm{I})}+T_\varphi{}^\varphi{}_{(\mathrm{I})}\rangle\sin\theta d\theta. \end{aligned} \quad (30)$$

The expression for $\langle T_\varphi{}^\varphi\rangle$ is identical to this. All other components of the total stress-energy tensor have mixed indices and must vanish on account of one or another symmetry argument. Thus $\langle T_\theta{}^T\rangle$ and $\langle T_\varphi{}^T\rangle$ describe the tangential flow of energy, which has to vanish; the shears $\langle T_r{}^\theta\rangle$, $\langle T_r{}^\varphi\rangle$, $\langle T_\theta{}^\varphi\rangle$ must be null; and the component $\langle T_r{}^T\rangle$ represents the net radial flow of energy, which in a static situation must also be zero. Whether the geon leaks no energy at all, or only a very little energy, depends upon whether we impose upon the electromagnetic field at large distances the requirement that it be a standing wave, or a pure outgoing wave. Which choice we make has negligible effect upon the structure of the geon. We already made our choice when we took the circular frequency, $c\Omega$, to be real and assumed the product representation (24) of the vector potential. From the radial equation (26) it then follows automatically that $R(r)$ must represent a standing wave, and that the radial flux must vanish. In summary, we have for the nonzero components of the average value of the total electromagnetic stress-energy tensor (2) the following values:

$$\begin{aligned}\langle T_r{}^r\rangle &= (N/2)\int \sin\theta d\theta (8\pi)^{-1}(\{r\varphi\}-\{T\varphi\}-\{\theta\varphi\})\\ &=[r\varphi]-[T\varphi]-[\theta\varphi],\\ \langle T_\theta{}^\theta\rangle &= \langle T_\varphi{}^\varphi\rangle = (N/2)\int \sin\theta d\theta (8\pi)^{-1}\{\theta\varphi\}\\ &=[\theta\varphi],\\ \langle T_T{}^T\rangle &= (N/2)\int \sin\theta d\theta (8\pi)^{-1}(\{T\varphi\}-\{r\varphi\}-\{\theta\varphi\})\\ &=[T\varphi]-[r\varphi]-[\theta\varphi]. \end{aligned} \tag{31a}$$

Here we used as abbreviations:

$$\{r\varphi\}\equiv\langle F_{r\varphi}F^{r\varphi}\rangle = e^{-\lambda}(r\sin\theta)^{-2}\langle(\partial A_\varphi/\partial r)^2\rangle \sim \langle H_\theta{}^2\rangle;$$

$$\{\theta\varphi\}\equiv\langle F_{\theta\varphi}F^{\theta\varphi}\rangle = (r^2\sin\theta)^{-2}\langle(\partial A_\varphi/\partial\theta)^2\rangle \sim \langle H_r{}^2\rangle;$$

$$\{T\varphi\}\equiv\langle F_{T\varphi}F^{T\varphi}\rangle = -e^{-\nu}(r\sin\theta)^{-2}\times\langle(\partial A_\varphi/\partial T)^2\rangle \sim -\langle E_\varphi{}^2\rangle,$$

and

$$[\quad] = (N/16\pi)\int\{\quad\}\sin\theta d\theta;\ \text{thus}$$

$$\begin{aligned}[r\varphi] &= (N/16\pi)l(l+1)(2l+1)^{-1}e^{-\lambda}(dR/rdr)^2;\\ [\theta\varphi] &= (N/16\pi)l^2(l+1)^2(2l+1)^{-1}(R/r^2)^2;\\ [T\varphi] &= -(N/16\pi)l(l+1)(2l+1)^{-1}e^{-\nu}(\Omega R/r)^2. \end{aligned} \tag{31b}$$

Having an equation for the influence of gravitation upon the electromagnetic disturbance (26), and expressions (31) for the stress-energy density of this disturbance, we can now complete the circle of the self-consistent system and write down the equations

$$G_{ik} = (8\pi G/c^4)T_{ik} \tag{32}$$

for the determination of the gravitational potentials. In the mixed covariant-contravariant notation (which gives $T_4{}^4$ a sign opposite to T_{44}), we have[11]

$$\begin{aligned} G_r{}^r &= e^{-\lambda}(r^{-1}d\nu/dr+r^{-2})-r^{-2},\\ G_\theta{}^\theta &= G_\varphi{}^\varphi = \tfrac{1}{2}e^{-\lambda}[d^2\nu/dr^2+\tfrac{1}{2}(d\nu/dr)^2+r^{-1}d\nu/dr\\ &\qquad -r^{-1}d\lambda/dr-\tfrac{1}{2}(d\nu/dr)(d\lambda/dr)],\\ G_T{}^T &= e^{-\lambda}(r^{-2}-r^{-1}d\lambda/dr)-r^{-2}. \end{aligned} \tag{33}$$

The field equation $G_\theta{}^\theta=(8\pi G/c^4)T_\theta{}^\theta$ is identical with the equation for $G_\varphi{}^\varphi$ and can be disregarded, for it says nothing in addition to the equations for $G_T{}^T$ and $G_r{}^r$. One has only to differentiate with respect to r the equation for $G_r{}^r$, add to it a proper linear combination of the equations for $G_r{}^r$ and $G_T{}^T$, and employ the wave equation (26) to get the content of the equation for $G_\theta{}^\theta$. This result only says that the tangential tensions in a spherical shell have to be balanced by the radial pressure gradient. We therefore end up with two equations of the first order for the two unknown functions λ (= twice the number of napiers of change of scale in radial distances) and ν (= twice the number of napiers of change of scale in time measurements):

$$e^{-\lambda}(r^{-1}d\nu/dr+r^{-2})-r^{-2}=[l(l+1)GN/2(2l+1)c^4]\times[e^{-\lambda}(dR/rdr)^2+e^{-\nu}(\Omega R/r)^2-l(l+1)(R/r^2)^2]; \tag{34}$$

$$e^{-\lambda}(r^{-2}-r^{-1}d\lambda/dr)-r^{-2}=[l(l+1)GN/2(2l+1)c^4]\times[-e^{-\lambda}(dR/rdr)^2-e^{-\nu}(\Omega R/r)^2-l(l+1)(R/r^2)^2]. \tag{35}$$

The scale invariance of classical geon theory shows itself at once. For geons of the same index l, but different sizes and masses and therefore different circular frequencies $c\Omega$, we introduce the same dimensionless measure of radial coordinate, $\rho=\Omega r$. Also we define the dimensionless measure of potential

$$f(\rho)=[l(l+1)GN/2(2l+1)c^4]^{\frac{1}{2}}\Omega R(r). \tag{36}$$

Furthermore we recall that the Schwarzschild solution for the field of a point mass has the form $e^{-\lambda}=e^{\nu}=1-2r^{-1}L_0$, where $L_0=GM/c^2$ is a measure of the mass of the object. We have to expect a similar result for the metric around the geon at distances where the electromagnetic field has fallen exponentially to negligible values, but where the gravitational field may be still quite strong. Consequently we shall write

$$\begin{aligned} e^{-\lambda} &\equiv 1-2\rho^{-1}L(\rho),\\ e^{\lambda+\nu} &\equiv Q^2(\rho),\\ e^{\nu} &= [1-2\rho^{-1}L(\rho)]Q^2(\rho). \end{aligned} \tag{37}$$

Here the dimensionless measure, $L(\rho)$, of mass inside the radius r is nearly zero from $\rho=0$ to a value of ρ close to the inner surface of the active region of the geon, a value of the order of

$$\rho\sim l^*\equiv[l(l+1)]^{\frac{1}{2}};$$

then $L(\rho)$ rises quickly over a range of ρ of the order

[11] See for example L. Landau and E. Lifschitz, *The Classical Theory of Fields*, translated by M. Hamermesh (Addison-Wesley Press, Cambridge, 1951).

$\Delta\rho\sim l^{*\frac{1}{3}}$; and then $L(\rho)$ stays essentially constant from this point to very large distances. The correction factor, Q, is likewise essentially constant inside and outside the geon, and only makes a sudden rise in the active region. In these notations the geon is described by the self-consistent solution of three equations: the wave equation

$$d^2f/d\rho^{*2}+[1-(l^*Q/\rho)^2(1-2L/\rho)]f=0, \quad (38)$$

where $d\rho^*$ is an abbreviation for

$$d\rho^*=Q^{-1}(1-2L/\rho)^{-1}d\rho; \quad (39)$$

and the two field equations, of which the first gives the change of mass with distance:

$$dL/d\rho^*=(1/2Q)[f^2+(df/d\rho^*)^2+(l^*Qf/\rho)^2(1-2L/\rho)], \quad (40)$$

and the second gives the variation of the factor Q:

$$dQ/d\rho^*=(\rho-2L)^{-1}[f^2+(df/d\rho^*)^2]. \quad (41)$$

It is an interesting feature of the system of equations (38), (40), (41) that they still permit the possibility of a change of scale of distance without a change of form:

$$\begin{aligned}\rho&=b\rho_1,\\ Q&=bQ_1,\\ \rho^*&=\rho_1^*,\\ L&=bL_1,\\ f&=b^{\frac{1}{2}}f_1.\end{aligned} \quad (42)$$

Now $g_{rr}=e^\lambda$ is already so normalized, according to Eq. (37), that it goes over to the Euclidean value of 1 at very large distances. Consequently we wish the corresponding cotime factor, e^ν, also to go to unity. Another value is perfectly possible, but it would correspond to an unhappy choice for the value of the speed of light. Thus, according to (37), we want the factor Q to go to unity at large distances. This normalization means, according to the law (41) of increase of Q, that Q must start off at the origin with a value less than 1. But how much less we do not know until we have integrated the differential equation. Accordingly, we distinguish between the solution Q, L, f that we want and the solution Q_1, L_1, f_1 that we get from our numerical integration of the equations of the self-consistent field. In both solutions the mass factor, L or L_1, and the field factor, f or f_1, start at zero and are well behaved at large distances. In the former Q starts at a value less than one and rises to unity; in the latter we start, for convenience, at $Q_1=1$ and arrive at large distances at a value greater than one. This value then defines the scale factor required in (42) to go from the preliminary solution to the desired solution:

$$b=1/Q_1(\infty); \quad Q_1(0)=1. \quad (43)$$

In preparing for the numerical integration of the geon equations we note that the behavior of the solutions near the origin is obvious and not very interesting

$$\begin{aligned}f_1&=F\rho_1^{l+1}+\cdots,\\ L_1&=2^{-1}(l+1)F^2\rho_1^{2l+1}+\cdots,\\ Q_1&=1+(2l)^{-1}(l+1)^2F^2\rho_1^{2l}+\cdots.\end{aligned} \quad (44)$$

What is important is the circumstance that there is no adjustable constant except the starting value of the field strength. This field strength, or the constant F, must be so chosen as to give the geon stability. We have to do with an eigenvalue, but an eigenvalue of a nonlinear system of equations.

The key to the eigenvalue problem is the wave equation (38). Where the expression in square brackets is positive, there the solution is oscillatory; where it is negative, there the solution rises or falls approximately exponentially. This expression has the following behavior when the factor F in the field strength has been chosen to give the geon stability. (1) At distances less than those that characterize the active region of the geon, $L_1\doteqdot 0$, $Q_1\doteqdot 1$, and the bracket has the approximate form $[1-(l^*/\rho_1)^2]$. Thus, in moving out from $\rho_1=0$ to ρ_1 near l^*, we have until the very end a strongly negative bracket, and therefore a solution that rises approximately exponentially. Consequently we have yet to arrive at the zone of maximum activity. (2) In the active zone near $\rho_1=l^*$ there occurs a quick rise in Q_1 and L_1, with very little change in the radial coordinate. When the field strength is properly chosen, these changes that follow from (40) and (41) are such that the square bracket is positive over a limited range of ρ. Thus f rises to a maximum and starts to fall off exponentially as ρ_1 passes out of the region of oscillation, provided that the field strength is great enough to give the geon the requisite mass for stability in a single wave zone. (3) For values of ρ_1 just slightly larger than l^*, the mass factor L_1 and the quantity Q_1 have attained essentially constant values. These values are such that $(1-2L_1/l^*)$ is much less than unity, but Q_1^2 is much greater than unity, and the product of these two factors also exceeds unity by a considerable margin. Moreover, as ρ_1 increases, the factor $(1-2L_1/\rho_1)$ evidently rises rapidly percentagewise at first, then levels off to a saturation value. The factor $(Q_1l^*/\rho_1)^2$, on the other hand, falls off in a more nearly uniform way. Consequently the product of these two factors rises as we leave the active zone behind, reaches a maximum when the increase, ρ_1-l^*, of the radial coordinate is some substantial fraction of l^*, then falls off and passes through the value unity for ρ_1-l^* of the order of l^* (Fig. 2). The product of factors under discussion in the present classical problem evidently plays the part of the potential in a typical problem of quantum mechanics. The analogy with the theory of alpha decay is complete even to topological identity of the two forms of barrier to be penetrated. In the barrier in the present problem the field factor f of an eigenfunction falls off exponentially by a number of napiers of the order of l^*.

(4) With further increase of the radial coordinate beyond the point of emergence from the barrier, oscillation sets in. The amplitude is of course exceedingly small compared to the amplitude inside the geon. The wavelength is at first long but eventually settles down to the value appropriate to a disturbance of the given frequency in free space:

$$f/\text{constant} \doteqdot \sin(\rho^*+\delta) = \sin(Q_1(\infty)\rho_1+\delta)$$
$$= \sin(\rho+\delta) = \sin(\Omega r+\delta). \qquad (45)$$

To go further, we seem to have to obtain for each conceivable value of the index number l a self consistent numerical solution of three nonlinear equations. However, there exists in addition to the exact scaling law already exploited another and distinct scaling law, valid asymptotically for large l^*, which reduces all these problems to a single one. Similar scaling laws are familiar from other parts of mathematical physics. In quantum mechanics one reduces all hydrogenic atoms to one problem by appropriate choice of scale of length. However, then the radial wave function for the lowest state for each value of l has a different mathematical expression, $\{[(l+1)/2]^{2l+3}(2l+2)!\}^{-\frac{1}{2}}\rho^l \exp[-\rho/(l+1)]$. Yet simple analysis of the behavior of this function in the neighborhood of its maximum, $\rho = l(l+1)$, shows that it has a form,

$$\simeq [\pi l(l+1)^2]^{-\frac{1}{4}} \exp[-(\Delta\rho)^2/2l(l+1)^2],$$

which is the same up to a scale change for all values of l. A similar scaling principle applies to the geon, with two differences: the reduced field factor, f, does not have the form of a harmonic oscillator wave function; and the relevant length variable is not $x = \Delta\rho/l^{\frac{1}{2}}(l+1)$, as in the hydrogenic problem, but

$$x = (\rho^*-l^*)l^{*-\frac{1}{3}}. \qquad (46)$$

The whole of the active region of the geon will be described by a range of x of the order of unity.

To bring this similarity transformation into evidence, we consider large values of l^*, and expand the relevant quantities in inverse powers of $l^{*\frac{1}{3}}$:

$$d\rho^* \equiv l^{*\frac{1}{3}}dx,$$
$$\rho_1 = l^* + l^{*\frac{1}{3}}r_0(x) + \cdots,$$
$$L_1 = l^*\lambda_0(x) + l^{*\frac{2}{3}}\lambda_1(x) + l^{*\frac{1}{3}}\lambda_2(x) + \cdots,$$
$$Q_1 = [1/k(x)] + l^{*-\frac{1}{3}}q_1(x) + l^{*-\frac{2}{3}}q_2(x) + \cdots,$$
$$f_1 = l^{*\frac{1}{3}}\varphi(x) + \varphi_1(x) + l^{*-\frac{1}{3}}\varphi_2(x) + \cdots,$$
$$[1-(Q_1l^*/\rho_1)^2(1-2L_1/\rho_1)] = l^{*-\frac{2}{3}}j(x)k(x) + \cdots. \qquad (47)$$

Inserting these expressions into the system of equations (38), (40), (41), and identifying coefficients of like powers of l^*, we find from the lowest relevant terms the wave equation

$$d^2\varphi/dx^2 + j(x)k(x)\varphi(x) = 0, \qquad (48)$$

and the two field equations:

$$dk/dx + \varphi^2 = 0, \qquad (49)$$
$$dj/dx = 3 - [1 + (d\varphi/dx)^2]/k^2. \qquad (50)$$

If we can find by integration of these three equations the reduced field factor $\varphi(x)$ and the supplementary time scale correction factor $k(x)$ and the reduced oscillation factor $j(x)k(x)$, then we can determine the other leading terms in expressions (47) by simple calculation:

$$dr_0/dx = k(x),$$
$$\lambda_0 = (1-k^2)/2. \qquad (51)$$

We seek a solution of (48), (49), and (50) with the following properties. The field factor $\varphi(x)$ tends to zero both for large positive x and for large negative x. The contraction factor $k(x)$ is unity for large negative x, and falls in the active region near $x=0$, and for large positive x approaches a value which is less than $(1/3)^{\frac{1}{2}} = 0.577$ but still positive. The quantity $j(x)$ is very large and negative for large negative x. It rises with increasing x with a positive slope of two until x reaches the vicinity of the active region. There j succeeds in becoming positive for a limited range of x in the neighborhood of $x=0$. For larger x, the slope dj/dx approaches the negative value $3-k^{-2}(\infty)$. Thus j falls off again to $-\infty$. The oscillation factor, the product $j(x)k(x)$, is positive in only a limited range of x.

There exist a number of solutions having the desired behavior, distinguished from one another by an integer, s, which represents the sum of the number of maxima and minima in the field factor, $\varphi(x)$, in the active region. This integer has the value one for the simplest type of idealized spherical geon. Higher values of s also represent physically acceptable geons. We make the choice between one solution and another when in the integration of the equations we pick one starting magnitude of $\varphi(x)$ or another. The solution $s=1$ is characterized by the largest field strength, for in one-half wave enough energy and mass has to accumulate to hold the geon together. A comparable amount of mass belongs to a geon of higher s, but the energy is distributed over a larger number of half waves and on this account the concentration of energy and the field strength are weaker.

We can summarize the eigenvalue characteristics of our system of equations as follows. We accept that for large negative x, the factors j and k have the form $k(x) \doteqdot 1$ and $j(x) \doteqdot 2x$, and that $\varphi(x)$ is approximately the exponentially rising solution of

$$d^2\varphi/dx^2 = 2x\varphi; \qquad (52)$$

thus,

$$\varphi \doteqdot A(-2x)^{-\frac{1}{4}} \exp[-(-2x)^{\frac{3}{2}}/3]. \qquad (53)$$

Here the constant A, the "amplitude factor," is the sole quantity at our disposal in selecting the character of the solution of (48), (49), (50). If A is chosen very small in comparison with unity, then (49) says that $k(x)$ stays close to 1 for a long range of x, and (50) says that $j(x)$ continues to behave nearly as $2x$ for a considerable range of positive values of x. There likewise (52) remains a good approximation, and the solution, essentially

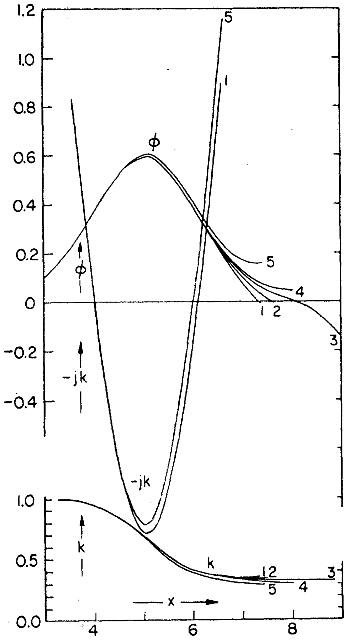

FIG. 3. Results of numerical integration of the differential equations (48), (49), (50) for the simple idealized self-consistent spherical geon, giving a more detailed account of the active region shown in the circle in Fig. 2. Here φ is the dimensionless measure of the radial factor in the electromagnetic potential, jk is the oscillation factor in the differential equation $d^2\varphi/dx^2+jk\varphi=0$, and k is a metric correction factor. The range of integration from start at $x=-4$ up to $x=3$ is not shown owing to the inappreciable difference there between the curves with nearly identical starting values: $\varphi(-4)\equiv\varphi_0$; $\varphi_0\times10^4=1.025$ (case 1); 1.028125 (2); 1.03 (3); 1.03125 (4); 1.0375 (5). The eigenvalue evidently lies between cases (3) and (4). For any given geon mass in the classical range, the curves permit one in principle to find field strengths, metric, and all other relevant physical quantities in the active region of the geon.

an Airy function, goes over from the exponential character (53) to the oscillatory form

$$\varphi \doteqdot 2A(2x)^{-\frac{1}{4}}\sin[(2x)^{\frac{3}{2}}/3+(\pi/4)], \tag{54}$$

passing through the value

$$\varphi \doteqdot 6^{5/6}2^{-1}\pi^{-\frac{1}{2}}\Gamma(4/3)A$$

at $x=0$. Averaging over oscillations in this first part of the active region, we have

$$dk/dx \doteqdot -\langle\varphi^2\rangle \doteqdot -\sqrt{2}A^2/x^{\frac{1}{2}},$$

or

$$k \doteqdot 1-2^{\frac{3}{2}}A^2x^{\frac{1}{2}};$$

and

$$dj/dx \doteqdot 2-2^{\frac{3}{2}}3A^2x^{\frac{1}{2}},$$

or

$$j \doteqdot 2x-2^{5/2}A^2x^{\frac{3}{2}}. \tag{55}$$

These approximations begin to fail when x becomes of the order $1/A^4$. Thereafter we quickly come to the end of the active region. (1) If the amplitude factor has the value, A_s, appropriate to the sth proper solution, then this field factor, $\varphi(x)$, falls off exponentially. (2) If the amplitude factor has a value so little greater than A_s that the difference, $A-A_s$, is small compared to the distance, $A_{s-1}-A_s$, to the next eigenvalue, then there is not quite opportunity to finish the last oscillation in the active region. The field factor starts to fall off after what would have been the last maximum but instead of continuing to fall towards zero, reaches a minimum and then commences to rise exponentially (Fig. 3). The contribution to the mass therefore also rises exponentially. The scale factor k of Eq. (49), instead of decreasing to a reasonable value between 0 and 0.577, continues to fall, and goes to zero at a certain critical value, x^*, as (x^*-x) multiplied by a slowly varying function which has qualitatively the character of a power of $\ln 1/(x^*-x)$. The quantity j therefore becomes singular, and goes to negative infinity as $-(x^*-x)^{-3}$ multiplied again by a slowly varying factor of logarithmic character. Finally, the field factor $\varphi(x)$ goes to infinity qualitatively with much the slowly varying behavior of a power of a logarithm of $[1/(x^*-x)]$. Thus the integration stops at x^*. This limit will be the further beyond the active region, the closer the amplitude factor lies to A_s. (3) As the difference $A-A_s$ becomes larger, the singular point of the integration moves inwards towards what one might still appropriately call the active zone. For a certain A the point x^* has come in to its maximum extent. With further increase in A this critical point moves out again. At or near the value of A for which this turnabout occurs, the sign of φ at the singularity also changes; i.e., the singularity swallows up the last node of the wave function. Thus, for A just less than the critical value, $\varphi(x)$ is described by a quasi-logarithmic function that goes to infinity with one sign; for A just above the critical value, by a singular quasi-logarithmic function of the opposite sign; and for A just right to annihilate a node, by (x^*-x) times a quasi-logarithmic function of (x^*-x). In this particular case the scale factor $k(x)$ goes as $(x^*-x)^3$ times a pseudo-logarithmic function, and the oscillation factor jk goes as $-(x^*-x)^{-2}$ times a pseudo-logarithmic function. (4) As A approaches the proper amplitude factor, A_{s-1}, of the next solution, the singularity moves out towards $x^*=+\infty$ and finally disappears. (5) With further increase in A the sequence (1), (2), (3), (4) repeats itself until finally there are no more nodes in the wave function to be swallowed. At this stage we have passed the last eigenvalue, A_1.

It is appropriate to notice that the system of equations (48), (49), (50) does not contain explicitly the argument x. Hence, if $\varphi(x)$, $k(x)$, $j(x)$ is a solution, so is $\varphi(x+a)$, $k(x+a)$, $j(x+a)$. Consequently there is a certain indeterminacy in the start of the integration process. We remove this ambiguity and completely—though indirectly—define the origin of x when we adopt as convention the asymptotic formula $j(x)=-2x$ and the corresponding asymptotic equations (52) and (53).

With this convention for origin of x, the reduced equations of the spherical geon were integrated numerically to determine approximately the first proper function and first proper value. Thanks are due to Mr. Arthur Komar for checking the algebra and to Mr. Robert Goerss for carrying out the computations on an International Business Machines card-programmed electronic calculator. The logical scheme of the integration is outlined in Fig. 4, where each line symbolizes a successive stage in the calculation, and where Δ represents the size, 0.05, of the interval of x used between one step and the next. The integration was started at $x=-4$ with values derived from the asymptotic formula (53):

$$\begin{aligned} \varphi_0 &\equiv \varphi(-4) = \text{arbitrary}, \\ \delta\varphi_{\frac{1}{2}} &= 8^{\frac{1}{2}}\varphi_0\Delta, \\ k_0 &= 1, \\ j_0 &= -8. \end{aligned} \tag{56}$$

Figure 3 presents the results of the integrations. The first eigenvalue was found to lie between those amplitude factors, A, that correspond to initial values, at $x=-4$, between $\varphi_0=1.03000\times10^{-4}$ and $\varphi_0=1.03125\times10^{-4}$. The active region ($jk>0$) did not begin until $x=4.05$, and reached only to $x=6.12$. The field factor, $\varphi(x)$, reached a peak value of 0.59 for x about equal to 5.1. The scale factor, $k(x)$, approached an asymptotic value, $k(\infty)$, of approximately 0.33.

These curves and numbers give us essentially all the information we need to determine the structure of a simple idealized spherical geon with only one maximum, $s=1$, in the electromagnetic potential and with number of nodes over the surface equal to any arbitrary large number l. Thus, having chosen an l, we calculate $l^*=[l(l+1)]^{\frac{1}{2}}$. This quantity represents the approximate radial coordinate, ρ_1, of the active zone. Then, from our curves and Eqs. (47) and (51) we find the relation between the dimensionless measures of distance, ρ_1; of mass out to a given distance, L_1; of metric correction factor, Q_1; and of potential, f_1. In particular we have for the asymptotic value of Q_1 the value $1/b\doteq 1/0.33$ and for the asymptotic value of L_1 the value $L_1=l^*(1-b^2)/2\doteq 0.45l^*$. Next, we make the scale change of Eqs. (42) and (43) in order to have a metric

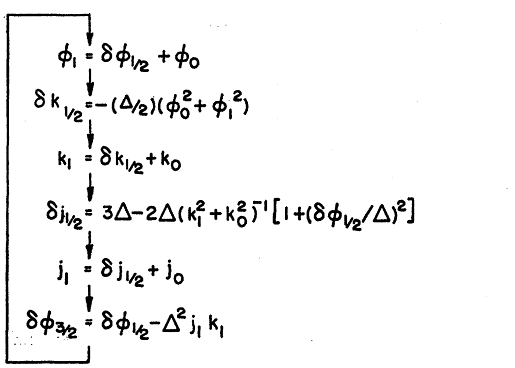

FIG. 4. Flow diagram of numerical integration. In the last term in the last equation there should appear—and did appear in the numerical calculations—a factor ϕ_1.

that approaches the Euclidean values at large distances. The scale factor is $b\doteq 0.33$. In the new dimensionless units the radial coordinate of the active zone is about $0.33l^*$. The metric correction factor Q has the value $b\doteq 0.33$ in the inner inactive part of the geon, rises in the active zone, and has outside a value 1.00. The corresponding new dimensionless measure of mass, L, is essentially zero in the inner inactive region, and outside the active region has the value $L\doteq (b/2)(1-b^2)l^* = 0.15l^*$. Finally, we transform to cgs units of measure via (36) and (37) and the related discussion. A clock ticks at the center of the geon at only about 33 percent of the rate of an identical clock far away from the geon. In terms of the frequency, Ωc, of electromagnetic radiation observed to come from the geon, the radius of the active zone of the system is described by a coordinate

$$r = R \doteq 0.33l^*/\Omega. \tag{57}$$

The mass is

$$M \doteq 0.15c^2l^*/G\Omega. \tag{58}$$

Let an observer far from the geon have a clock that flashes every second, and let an identical clock be placed at the center of the geon. The flashes of light from this second clock will reach the observer, not every second, but about every $b^{-1}\doteq(0.33)^{-1}=3$ seconds.

The mass, circular frequency, and radius of the spherical geon can be expressed in terms of the reduced action, $\mathcal{J}(=\text{action}/2\pi)$, by way of the relation

$$c^2dM = c\Omega d\mathcal{J}, \tag{59}$$

thus:

$$\begin{aligned} M &\doteq 0.54(l^*\mathcal{J}c/G)^{\frac{1}{2}}; \\ c\Omega &\doteq 0.27(l^*c^5/G\mathcal{J})^{\frac{1}{2}}; \\ R &\doteq 1.2(l^*G\mathcal{J}/c^3)^{\frac{1}{2}}. \end{aligned} \tag{60}$$

The root-mean-square value of the electric field, averaged with respect to time and with respect to polar coordinate over a sphere of that radius r which goes with the dimensionless coordinate x is,

$$\begin{aligned} E_{\text{rms}} &= \{-8\pi[T\varphi]\}^{\frac{1}{2}} = \{c^4f^2e^{-\nu}/Gr^2\}^{\frac{1}{2}} \\ &= c^2\Omega G^{-\frac{1}{2}}b^{-\frac{3}{2}}l^{*-\frac{2}{3}}\varphi(x) \\ &= (c^7/\mathcal{J}G^3)^{\frac{1}{2}}l^{*-1/6}[(1-b^2)^{\frac{1}{2}}/2b]\varphi(x) \end{aligned}$$

with a peak value

$$E_{\text{peak rms}} = 0.46l^{*\frac{1}{3}}c^4/G^{\frac{3}{2}}M.$$

4. TRANSFORMATIONS AND INTERACTIONS OF PURE ELECTROMAGNETIC GEONS

Energy leaks out of a simple idealized spherical geon at a rate easily estimated by the methods of the theory of alpha decay. Were the refractive index barrier removed, the energy would disappear from its present region of concentration in a time of the order of one vibration period. The attrition, α, in the equation of definition

$$dM/M = -\alpha\omega dt, \tag{61}$$

would have a value of the order of unity. Owing to the presence of the barrier, the attrition is cut down to a value of the order

$$\alpha \sim \exp(-2P), \tag{62}$$

where P is the barrier penetration integral determined by Eq. (38):

$$P = \int [(d^2f/d^*\rho^2)/f]^{\frac{1}{2}} d\rho^*$$
$$= \int [-1 + (l^*Q/\rho)^2(1-2L/\rho)]^{\frac{1}{2}} \times Q^{-1}(1-2L/\rho)^{-1} d\rho. \tag{63}$$

The integral extends from the outer edge of the active zone to the point of reemergence from the barrier. The relevant range in ρ is of the order of $l^* = [l(l+1)]^{\frac{1}{2}}$, whereas the thickness of the active zone is only of the order $\Delta\rho \sim l^{*\frac{1}{3}}$. For this reason it is legitimate to use for Q and L in the evaluation of the integral (63) the constant values that apply outside the active zone. The error made in the penetration exponent by this approximation will be only of the order of $l^{*\frac{1}{3}}$, whereas the exponent itself will be of the order l^*. Thus in accordance with the results at the end of the last section we write

$$Q = 1,$$
$$\Omega r_{\text{inner}} = \rho_{\text{lower limit}} = bl^* \doteqdot 0.33 l^*,$$
$$L = (b/2)(1-b^2)l^* \doteqdot 0.15 l^*, \tag{64}$$

and find by algebraic examination of the roots of the bracket in (63) that the outer turning point is

$$\Omega r_{\text{outer}} = \rho_{\text{upper limit}} = l^*\{-(b/2) + [1-(3b^2/4)]^{\frac{1}{2}}\} \equiv Bl^* \doteqdot 0.79 l^*. \tag{65}$$

The penetration integral has the value

$$P = l^* \int_b^B \{x^{-2}[1-(1-b^2)(b/x)] - 1\}^{\frac{1}{2}} \times [1-(1-b^2)(b/x)]^{-1} dx$$
$$= 0.76 l^*, \tag{66}$$

where the variable of integration is $x = \rho/l^* = \Omega r/l^*$; and the attrition has the rough value

$$\alpha \sim \exp(-1.52 l^*). \tag{67}$$

As examples, consider two simple idealized spherical geons not very far—on a logarithmic scale—above the limit where quantum effects come in by way of pair production. Let both have masses of 10^{42} gram and only one-half wave of disturbance in the radial direction in the active zone, but let the wavelength, and therefore the thickness of this zone, have quite different values in the two cases, as indicated in Table I. It is evident from the numbers given there that geons, even systems of the same mass and radius, can have fantastically different rates of radiation leakage.

To switch to quantum language, we can speak of the leakage process as single photon emission. In addition to such processes there will occur what can temporarily describe as double photon processes (Fig. 5): two quanta moving tangenially collide and go off in two new directions, not far from parallel or antiparallel to the radius vector, and thus escape from the system simultaneously. To follow the terminology of light rays a little further, we can speak as is well known, of a critical angle required for escape. For rays whose angle of inclination, θ, to the radius vector is greater than this critical amount, there exists a maximum distance to which the ray can go before it falls back on the geon. This maximum distance is obtained by calculating the appropriate root of the equation

$$\frac{l^{*2}\sin^2\theta}{\Omega^2 r^2}\left[1 - \frac{l^* b(1-b^2)}{\Omega r}\right] - 1 = 0. \tag{68}$$

The quantity on the left-hand side of this equation evaluated for the case $\sin\theta = 1$, came into the argument of the barrier penetration integral, another illustration of the close connection between the Hamilton-Jacobi analysis of rays and the differential equation for waves. It provides a simple interpretation of expression (68) to think of $l^* \sin\theta$ as a measure of the angular momentum of the ray: either a ray on the inside trying to get out; or a ray coming from infinity and trying to get in. The turning points for these two cases undergo merger when we have zero not only for the left-hand side of (68), but also for its derivative with respect to r. This happens when

$$\sin\theta = (3^{\frac{3}{2}}/2)b(1-b^2) \doteqdot 0.77 = \sin 50.4°, \tag{69}$$

and the double root then lies at

$$r = (3l^*/2\Omega)b(1-b^2) \doteqdot 0.44 l^*/\Omega, \tag{70}$$

TABLE I. Leakage of radiation from simple idealized spherical geons: illustrative examples. The last column gives the factor of change of scale for a similarity transformation which leaves the geon a classical object.

	Geon I	Geon II	Scale factor
Mass	10^{42} g	10^{42} g	$\times n$
Radial coordinate of active zone (58)	1.67×10^{14} cm	1.67×10^{14} cm	$\times n$
Spherical harmonic index $l^* = [l(l+1)]^{\frac{1}{2}}$	10	8.43×10^{9}	$\times 1$
Circular frequency of emergent radiation (57)	6.00×10^{-4} rad/sec	5.06×10^{5} rad/sec	$\times(1/n)$
Wavelength outside	3.14×10^{14} cm	3.72×10^{5} cm	$\times n$
Approximate attrition (67)	2.5×10^{-7}	$10^{-5570000000}$	$\times 1$
Time to collapse assuming leakage only and classical behavior throughout	212 years	$\sim\infty$	$\times(1/n)$
Rms electric field in most active region [Eq. (60) ff]	4.66×10^{10} esu/cm	4.41×10^{13} esu/cm	$\times(1/n)$
Critical field for pair production	4.41×10^{13} esu/cm	4.41×10^{13} esu/cm	a constant

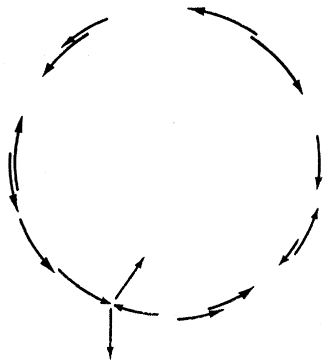

FIG. 5. Photon-photon collisions provide one of the mechanisms for escape of energy from the active region of a geon. The same mechanism is describable from the wave point of view in terms of a coupling between otherwise independent modes of oscillation of the electromagnetic field. The two sources of the nonlinear coupling are virtual pair phenomena and the nonstatic parts of the gravitational field.

that is, at the peak of the barrier in Fig. 2. This distance should be compared with impact parameter, or distance of closest approach in the absence of gravitational forces, of a photon with the same energy and angular momentum:

$$\text{impact parameter} = l^* \sin\theta/\Omega = 0.77 l^*/\Omega. \qquad (71)$$

In contrast, the rays that hold the geon together have in this sense an impact parameter equal to $1.00 l^*/\Omega$, and have the inner and outer turning points,

$$r \doteqdot 0.33 l^*/\Omega \quad \text{and} \quad r \doteqdot 0.79 l^*/\Omega. \qquad (72)$$

Rays with an impact parameter less than $0.77\ l^*/\Omega$ have a component of motion in the radial direction sufficiently great to pass freely from the inner region to the outer region.

Everything just stated in ray language can be rephrased more appropriately in terms of waves. In the quasi-static gravitational field of force of the geon we have already found a solution of the wave equation for the electromagnetic potential which has only one maximum in the active region, varies with angle as a spherical harmonic of order $l \doteqdot l^* - \frac{1}{2}$, has a proper circular frequency $c\Omega$. In addition to waves of such a kind excited to a strength sufficient to hold the geon together, we can have proper solutions of weak amplitude with a great variety of values for the number, s, of radial maxima and minima in the active region and of values of the indices l and m of the spherical harmonic. A typical disturbance of this type will be characterized by a proper value, $\Omega_{s,l}$, and an amplitude factor $A_{s,l,m}$. Let the wavelength of this mode be small compared to the dimensions of the geon. Then it is appropriate to characterize this mode, too, by an equivalent ray-optical impact parameter, $P \equiv \{(l/\Omega_{s,l})$ times a quantity having the significance of $\sin\theta$ and to be calculated via appropriate analysis from the relative values of s and $l\}$. The motion of the trapped ray will have an outer bound equal to the next to the largest solution of the equation

$$(P^2/r^2)[1-(l^*b/\Omega r)(1-b^2)]-1=0. \qquad (73)$$

Outside of this distance the strength of the mode in question falls off exponentially through a refractive index barrier. A standing wave of this kind is the proper transcription of the idea of photon with a radial component of motion too small to allow escape. Evidently the prohibition of escape is not absolute. There will be leakage through the barrier to an extent the greater, the closer P falls to the critical impact parameter for escape, $P = 0.77 l^*/\Omega$. For smaller values of the impact parameter we have no proper solutions with an exponential region of fall off. Instead, we have solutions capable of transporting energy freely from the inside of the geon to outer space, and endowed with a continuum of frequency values. These waves correspond to the notion of photons able to escape from the system.

Instead of speaking of photon-photon collisions, we can talk of excitation of secondary waves by the waves that carry the primary energy supply of the geon. Such excitation is not envisaged in a linear wave equation, derived from a Lagrangian function that contains the fields to the second order. The differentiation of a quadratic Lagrangian gives a time rate of change of the field strength proportional only to the first power of field quantities. However, the Lagrangian actually contains correction terms of the fourth order and higher. They give in the expression for the time rate of change of a field quantity supplementary terms containing the product of three or more field quantities. These terms constitute a nonlinear coupling that takes energy slowly out of the primary modes and redistributes it over secondary radiations. Some of this energy is picked up in characteristic modes. In any such mode the energy is reflected back and forth in a standing wave limited by two values of r. Other parts of the energy go into unbounded waves and are lost to infinity. The existence of the barrier leakage phenomenon means of course that the distinction between the stationary modes and the unbounded waves is only approximate. The stationary modes will themselves leak some of the energy that they get. Moreover these vibrations can transfer energy via nonlinear couplings to still other modes, and also to still other waves that escape.

The relative importance of simple leakage, and of transfer of energy to free-running waves, depends upon the relative strength of excitation of the various modes of the geon. The detailed specification of the state of a general geon is so complicated that there exists the greatest variety of objects, showing amongst them the greatest extremes of behavior.

In a geon where nonlinear terms make the more important contribution to the energy dissipation, we evidently face a problem so complicated that statistical arguments are needed to make any headway. We have to take into account the totality of the stationary modes of the geon in an approximately static gravitational field, and the distribution of energy among these modes. But their number is infinite, as in Rayleigh's paradox of blackbody radiation. No proper account of the distribution can be given without taking into account the quantum of action. We are therefore invited to assign to each mode of characteristic frequency ω an energy $\hbar\omega[\exp(\hbar\omega/\Theta-1]$, where Θ, expressed in energy units, has the significance of a temperature. We can also idealize the energy of each unbounded wave as zero in a first approximation, these waves being most easily distinguished by the criterion (73). The very number of the modes of vibration simplifies the problem. To find in each mode the radial distribution of field strength, and stress and energy, it is no longer necessary to integrate the wave equation numerically. The JWKB method gives the answer in terms of purely algebraic operations. Thus the system of three equations of the theory of spherical geons reduces to two differential equations of the first order for the metric quantities $\lambda(r)$ and $\nu(r)$, or $L(r)$ and $Q(r)$, with the temperature Θ as parameter of integration. No attempt is made here to solve this pair of equations for the "thermal geon."

The thermal geon is obviously an idealization that suggests itself for following out the consequences of the nonlinear interactions between different modes of vibration. In the next approximation one has to make further allowance for the effects of the intermodal couplings, by way of radiation losses through photon-photon collisions. Here the picture will be very different according as the mean free path of the most relevant photons is large or small in comparison with the dimensions of the system. In the second case it becomes necessary to analyze the effective opacity of the system, and to take account of the variation of temperature with radial coordinate in a fashion familiar from stellar theory—a problem again not investigated here.

The nonlinear couplings arise from two sources: the electron pair field, and the gravitational field. The second is classical. The source of the gravitation field—the stress energy tensor—has a reasonably smoothly varying average value, but on top of this are superposed fluctuations due to the fact that the electromagnetic field does after all vary in space and in time. A typical constituent of the fluctuation will have a character qualitatively of the form $A_1f_1(x,y,z)\sin(\omega_1t+g_1)\ A_2f_2(x,y,z)\times\sin(\omega_2t+g_2)$. This term will give rise to a fluctuating component of the gravitational field. Consequently in the differential equation for the electromagnetic field the coefficients of metric origin will not be exactly static functions of position alone, but will have small alternating terms with circular frequencies, $(\omega_1+\omega_2)$ and $(\omega_1-\omega_2)$, and with amplitudes proportional to A_1A_2. Let the uncorrected amplitude of the mode under consideration be $A_3f_3(x,y,z)\sin(\omega_3t+g_3)$. Then the wave equation for this oscillation has to be visualized as containing four supplementary source terms, with frequencies $\omega_1\pm\omega_2\pm\omega_3$, and with amplitudes proportional to the product $A_1A_2A_3$. Thus the primary disturbances in the geon inevitably generate secondary waves.

For an order of magnitude estimate of the strength of the effect, let $\lambda\!\!\!^-$ denote the general scale of the space variations in the relevant primary modes, and let $\mathcal{E}$ denote the order of magnitude of the associated electromagnetic fields. Then a typical inhomogeneity in the distribution of stress and energy will possess a mass of the order of $\mathcal{E}^2\lambda\!\!\!^-{}^3/c^2$. The fluctuations in the gravitational metric will have the general magnitude $\Delta g\sim G\Delta m/c^2\lambda\!\!\!^-\sim(G/c^4)\lambda\!\!\!^-{}^2\mathcal{E}^2$. The square of this dimensionless factor, multiplied by a dimensionless function of the geometry of the geon—depending upon ratio of typical wavelengths to the size of the system, and upon other details—will determine the fraction of the energy of the system lost per cycle via nonlinear coupling processes of gravitational origin.

The correction effects due to virtual production of electron pairs by slowly varying but intense fields bring in a coupling which has the same qualitative consequences as that due to gravitational fields, except that the governing dimensionless factor, Δg, of the previous paragraph is to be replaced by $\sim(e^2/\hbar c)[\mathcal{E}e(\hbar/mc)/(mc^2)]^2$. Thus the pair effects depend upon the electric field itself, whereas the gravitational effects depend upon the electric potential, $A\sim\lambda\!\!\!^-\mathcal{E}$. The two effects become of the same order of magnitude for wavelengths of the order

$$\lambda\!\!\!^-\sim(e^2/Gm^2)(G\hbar/c^3)^{\frac{1}{2}}$$
$$=(4.16\times10^{42})(1.60\times10^{-33}\text{ cm})=0.67\times10^{10}\text{ cm}. \quad (74)$$

To justify the classical analysis of geon structure, we already know that we have to deal with *radii* of the order of the limit (74) or greater. On the wavelengths we have not previously had any limit, except inferiority to the radius. However, in dealing with geons in which many modes are excited, or especially with a thermal geon, it will ordinarily be reasonable to consider wavelengths quite small compared to the limit (74). Then for most purposes gravitational transfer of energy between modes can be neglected in comparison with the coupling due to the charges and currents of virtual pairs.

In analyzing the behavior of a geon classical in the sense of being large compared to the limit (74), we encounter both a large scale static gravitational field and small scale rapid variations in this average. There will exist in addition long scale periodic or secular changes in the configuration of the system, some of which will in certain cases describe significant mechanisms for the disintegration of a geon. It is easy to visualize a slow vibration in the case of a simple toroidal geon. The shape of the torus changes from circular to elliptical first in one sense and then in the other, with the elementary electromagnetic disturbances, or ray tracks if one will, adjusting themselves adiabatically to these slow readjustments in the distribution of refractive index. The gravitational interaction between the various elementary electromagnetic oscillations imposes upon their readjustments of shape a collective character. In these collective motions the coupling between the electromagnetic oscillators by way of gravitational interactions will exceed the coupling via virtual pair effects, so long as we are in the classical domain of geon sizes.

The same mode of vibration of a simple toroidal system, endowed with enough energy, will lead to deformations not far in form from a figure eight, and with still further excitation of this mode scission will occur. Whether the two separate rings then completely break their association depends upon the magnitude of the kinetic energy of recoil from the act of scission. When this energy exceeds the gravitation potential energy of attraction between the two objects, they fly apart to make two distinct systems, and we have a complete act of fission. When the recoil energy is less than the binding, the two rings separate to a limiting distance, reverse their motion, collide, pass through each other, again separate, and so on. At each act of collision some of the energy of relative motion will be redistributed. Some will go into excitation of the electromagnetic modes of the one ring, some into modes of the other ring, some into modes that owe their existence to the combined gravitational field of the pair of rings, some into collective vibrations and rotations of the individual rings, and some will escape into space as free radiation. Ultimately the relative motion of the two tori will be damped down and they will come together to form a single geon with a smaller mass, and a much more complicated distribution of energy among modes, than characterized the original system.

Fission can in principle take place spontaneously in any classical geon. The principal requirement is adequate time for the exchange of energy between the various modes of the system, so that ultimately, by a statistical fluctuation in the distribution of energy, enough becomes concentrated upon a collective mode of distortion to lead to a critical deformation followed by fission. As between the mechanisms of energy dissipation, and compared to radiation leakage and nonlinear coupling to free running waves, the probability of fission will vary in a complicated way depending upon whether the distribution of energy among modes is finegrained or coursegrained, whether the gravitational field is symmetric or has large scale irregularities, and whether the total angular momentum is small or large. These circumstances will also affect the distribution in size of the fragments from fission in those classes of cases where this process occurs with appreciable probability.

A geon is characterized not only by its internal structure and by the genetics of its radioactive decay processes, but also by its interaction with other geons. When the two systems pass by each other at a separation large compared with the size of either of them, they will act on each other like the bodies of classical physics. The interaction is the relativistic generalization of the simple Newtonian law of force, $F=-GM_1M_2/r_{12}{}^2$. The electromagnetic interaction between the two systems will give rise to an additional force which is enormously weaker than the gravitational force. Thus, the flux of outgoing leakage radiation from one of the systems undergoes scattering by the other geon and gives rise to a radiation force of the order

$$\begin{aligned} F &\sim (-c^2dM_1/cdt)[\pi R_2{}^2/4\pi r_{12}{}^2] \\ &\sim (cM_1\alpha_1\omega_1)[R_2(GM_2/c^2)/r_{12}{}^2] \\ &\sim (GM_1M_2/r_{12}{}^2)(\alpha_1 l_1 R_2/R_1). \end{aligned} \quad (75)$$

An additional force will arise from the pressure of the radiation emitted by the other body. In expression (75) the attrition, α_1, is an exponentially small function of the order, l_1; thus the dimensionless product $\alpha_1 l_1$ will be extremely small for all stable geons. Of course, if the two geons are exceedingly unlike in size, or one of them is decaying so fast as to constitute an explosion, then the radiation force between the objects can become comparable to the gravitational force.

When two geons have large intrinsic angular momenta, the forces depend upon orientation as well as upon distance. Let two simple toroidal geons be oriented with their principal planes perpendicular to the line that connects their centers. When the rays of light in one geon all go around one way, and those in the second go around the other way, then the attraction is stronger than it is in the case of parallel orientation of the two angular momentum vectors. This circumstance is a reminder that a gravitational field is described by quantities more complicated than a static potential. The supplementary orientation-dependent forces constitute in this case a fraction $\sim R_1R_2/r_{12}{}^2$ of the total gravitational force.

As one geon flies by another at the minimum separation r_{12} and with a relative velocity v perpendicular to this line, it not only sets up in the other geon a bulk gravitational field of the order $G_1M_1/r_{12}{}^2$, but also creates tide producing forces of the order $GM_1R_2{}^2/r_{12}{}^4$. Both forces have frequency components ranging from $\omega=0$ to $\omega\sim v/r_{12}$. The first acts on the center of mass, a degree of freedom with zero natural frequency, and produces a bulk motion in accord with the classical concept of "body." The second acts on a mode of deformation with a natural frequency of the order c/R_2, very much higher than that of the driving force. Consequently the response will be adiabatic and a negligible part of the energy of motion will be left in internal degrees of freedom after the two geons have gone far apart.

When two geons pass by each other at a distance which brings the outer boundaries of their active regions to a separation, ΔR, small in comparison with geon radii, then the interaction between the two objects becomes more complicated. First, on account of the tidal deformations the gravitational forces will experience fractional increases of the order $R_2{}^2/r_{12}{}^2$ and $R_1{}^2/r_{12}{}^2$.

Second, there will come into being fluctuating forces of electromagnetic origin, of the order of magnitude of the product of typical field strengths, $\mathcal{E}_1$ and $\mathcal{E}_2$, in the active regions of the two geons, multiplied by an exponential decrement factor of the character $\exp(-\Delta R/\lambdabar)$, multiplied by an area of contact factor of the order $R\lambdabar$. Here we have assumed that the important wavelengths in the two geons are of the same general magnitude. Also we have used R to represent the radius of the smaller of the two geons; or better, R can be identified with the reduced radius, $R_1R_2/(R_1+R_2)$. Let f_1 denote the fraction of the volume of the first geon occupied by its active region, so that

$$\mathcal{E}_1^2 f_1 R_1^3/c^2 \sim M_1 \sim c^2 R_1/G. \quad (76)$$

Then the electric field in the active region of this geon will be of the order

$$\mathcal{E}_1 \sim f_1^{-\frac{1}{2}} G^{\frac{1}{2}} M_1/R_1^2, \quad (77)$$

and the expression for the electromagnetic force between the two systems will have the character

$$F \sim (f_1^{-\frac{1}{2}} f_2^{-\frac{1}{2}} \lambdabar/R)(GM_1M_2/r_{12}^2)\exp(-\Delta R/\lambdabar). \quad (78)$$

As the first dimensionless combination of factors will be of the order of unity for many classes of geons, it follows that the interaction (78) is of the order of the gravitational force, multiplied by a characteristic exponential function of distance.

Two geons which almost touch as they pass will display not only an anomalous interaction but also energy exchanges and energy losses. Where the refractive index barrier between the two systems is thinnest, radiation will leak across from one body to the other. Some modes previously excited weakly or not at all will gain energy, and other important modes will lose energy. Thus the two bodies will part company with a relation of masses different from what they had when they met. Their total mass will be diminished at the same time, both via momentary stimulation of the leakage of radiation to the outside, and by way of local perturbations of the gravitational field that deflect some light rays to the outside from orbits that previously were relatively stable.

Let two geons collide still more directly, so that they pass through each other. It will be rare for the two systems to emerge from the encounter without much loss of mass or redistribution of mass. The frequency spectrum of the perturbations experienced by one of the geons during the encounter will range from tidal forces of frequency $\sim c/R$ at one end of the spectrum, through strong components of frequency $\sim c/$(thickness of active region), to fluctuations with the scale of wavelengths and frequencies associated with the individual electromagnetic modes themselves. As a result almost everything that can happen ordinarily will happen. There will be a substantial loss of energy in the form of free radiation. Collective modes of motion will be excited. Individual electromagnetic modes will have quite new amplitudes. Moreover, forces will be at work to induce fission of the original geons into smaller fragments. In view of all these circumstances, it is not appropriate to describe the close interaction of two classical geons in terms of an effective law of force, or potential curve. Instead, one has to use the much more complicated language of transmutation physics, and to try to describe the distribution in mass of the products of a geon reaction. Evidently the geon decay processes and interaction mechanisms form an extensive subject for a more detailed investigation, not undertaken here.

5. INFLUENCE OF VIRTUAL PAIRS ON GEON STRUCTURE

As we go down in the mass and size spectrum of geons, we eventually pass out of the purely classical realm (Region I) into conditions (Region II) where the field strengths are no longer negligible compared to the critical field, $\mathcal{E}_{\text{crit}} = m^2c^3/\hbar e$, of the theory of electron pairs; and from here it is but a short step to Region III, where the fields are strong enough to turn a problem of virtual pairs into one where real pairs are present in large numbers. About III we say nothing; and about II our considerations are very primitive. The induced charges and currents of the virtual pairs alter the electromagnetic properties of the medium, so that it acquires a refractive index, n. It will be reasonable to consider all the electromagnetic modes of the geons under consideration to have wavelengths very long in comparison with the Compton length, $\hbar/mc$. Under these conditions the refractive index will be independent of frequency but will be dependent upon field strength, $\mathcal{E}$:

$$n-1 \sim (e^2/\hbar c)(\mathcal{E}/\mathcal{E}_{\text{crit}})^2. \quad (79)$$

To be more specific, let the geon be a simple toroid of azimuthal index number a. Then the relevant field strength in (79) is some appropriate average over the active region. The speed of light around the torus is reduced to c/n. The connection between radius of the geon and frequency of its leakage radiation will be altered to the form $\omega \sim nac/R$, where the integer, a, represents as before the azimuthal index number. The general relation between changes in energy and changes in reduced action, $\mathcal{J} = \text{action}/2\pi$, becomes

$$dM = c^{-2}\omega d\mathcal{J} \sim nad\mathcal{J}/cR = [1 + a(\mathcal{J}_{\text{crit}}/\mathcal{J})]ad\mathcal{J}/cR. \quad (80)$$

In the last line for convenience we have translated expression (79) for the refractive index correction from a field strength dependence to an action dependence, according to relation (5) between the two variables, the uncorrected formula (5) being legitimate in the evaluation of a first order correction. The deflection of the radiation in the gravitational field is more difficult to discuss. In lieu of an accurate analysis, for which only the bare formalistic bones are presented below, the following order of magnitude analysis is presented, which may well be in error through oversight of some significant factor. A photon which outside has a frequency ω has inside a mass of the order $\hbar\omega/c^2$, and experiences in the gravitational field of the geon a force of attraction of the order $(GM/R^2)(\hbar\omega/c^2)$. This expression should equal the time rate of change of the momentum of the photon: the product of its momentum, $n\hbar\omega/c$, by the angular rate of revolution in its orbit, c/nR; that is, $\hbar\omega/R$. Equating the gravitational and dynamic terms gives the same relation that we had in the absence of a refractive index correction:

$$M \sim (c^2R/G)(?).$$

Multiplying this formula by (80), we have

$$MdM \sim (acd\mathcal{J}/G)[1+a(\mathcal{J}_{\text{crit}}/\mathcal{J})](?). \qquad (81)$$

Here the question mark indicates that the relation in question is subject to possible correction. If this relation is correct, it states that the square of the geon mass, though nearly a linear function of action for values of the action large in comparison with a certain characteristic value, nevertheless for lower values of the action curves towards a steeper dependence on $\mathcal{J}$. The region where (81) should apply is too small on the scale of Fig. 6 to allow showing this curvature there.

The dependence of effective refractive index increment upon the square of the field strength lets itself be seen in two different ways.[12] From the photon point of view the torus is a channel filled by two streams of quanta moving in opposite directions. The space density of photons going one way is of the order

$$n_1 \sim \mathcal{E}^2/\hbar\omega_1. \qquad (82)$$

The photons travelling the other way are not able to make pairs by the process[13] $\hbar\omega_2+\hbar\omega_1 \rightarrow e^+ + e^-$ through want of energy: $\hbar\omega_1 = \hbar\omega_2 \ll mc^2$; but could do so were their quantum energies greater than $\hbar\omega^* = (mc^2)^2/\hbar\omega_1$. For this process the absorption cross section starts at zero at the threshold $\hbar\omega_2 = \hbar\omega^*$, rises to a peak value $\sigma \sim (e^2/mc^2)^2$ at a small multiple of this frequency, and then falls off. The absorption presented by the medium at high frequencies implies a contribution to the refractive index, n, at low frequencies, according to the formula

$$\begin{aligned} n(\omega_2)-1 &= (c/\pi)\int_{\omega^*}^{\infty} n_1\sigma(\omega_2')d\omega_2'/(\omega_2'^2-\omega_2^2) \\ &\sim (c\mathcal{E}^2/\hbar\omega_1)(e^2/mc^2)^2/\omega^* \\ &\sim (e^2/\hbar c)[\mathcal{E}e(\hbar/mc)/(mc^2)]^2, \end{aligned} \qquad (83)$$

in agreement with (79). Alternatively stated, there exists a process of scattering of light by light,[6] for which the differential cross section at low frequencies in the forward direction,

$$(d\sigma/d\Omega)_{\text{forward}} \sim (e^2/\hbar c)^2(e^2/mc^2)^2(\hbar\omega_2/mc^2)^4(\hbar\omega_1/mc^2)^2, \qquad (84)$$

makes a contribution to the refractive index given by Rayleigh's formula,

$$n-1 = n_1 2\pi(c/\omega_2)^2(d\sigma/d\Omega)^{\frac{1}{2}}_{\text{forward}}; \qquad (85)$$

this contribution is in agreement with (79).

This analysis of the nonlinear behavior of a strong electromagnetic field makes it clear that the corrections to be applied in Region II will be very different for a toroidal geon according as it has an angular momentum of zero, so that half the photons go each way, or has the maximum possible angular momentum, so that the photons meet no counterstream. In the second case the onset of strong pair corrections will evidently be postponed to higher field strengths and smaller bodies.

The picture of photon-photon collisions and a refractive index is not suitable for a precise evaluation of the

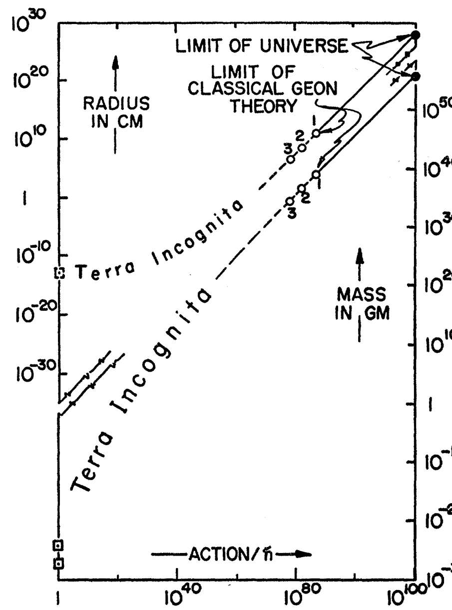

FIG. 6: Mass and radius as functions of action for classical geons (slanting full lines in upper right hand portion of diagram). With diminishing mass and radius, the electric fields active in an electromagnetic geon increase to a point, 1, where electron pairs are produced, below which point a classical analysis of geons completely fails (dot-dash line in diagram). For pure neutrino geons it may be possible to carry a simple analysis down as far as point 2 in the diagram before pair effects appear (Sec. 6 of text). Point 3 indicates where densities of the nuclear order of magnitude would be attained in either electromagnetic or neutrino geons were extrapolation of simple geon theory justified, which it certainly is not. Neither is any further extrapolation of the curves for mass and radius at all allowed (*Terra Incognita*). Masses and radii of electron and proton are indicated on the diagram for orientation as to magnitudes. The representation of geon mass or radius by a single line, or the action variables of the system by a single number, gives a rather oversimplified picture, as is clear from Eqs. (5) for even the simplest variety of toroidal geons. The numbers in the present diagram may be thought of as applying to simple toroidal geons with azimuthal index number, a, equal to 10 or less.

[12] John S. Toll, Ph.D. thesis, Princeton University, 1952 (unpublished); J. S. Toll (to be published); J. S. Toll and J. A. Wheeler, Phys. Rev. **81**, 654 (1951) and more detailed account, to be published.

[13] G. Breit and J. A. Wheeler, Phys. Rev. **46**, 1087 (1934).

pair corrections. For example, in a simple toroidal geon of zero angular momentum the standing electromagnetic wave with which we deal will, if strong, produce changes in the properties of the medium with twice the periodicity of the wave itself. This phase relation means that no simple averaging of the properties of the medium is appropriate. For a detailed treatment of the corrections it is therefore appropriate to go back to first principles, as typified by the variational principle (4) of classical relativity theory. Heisenberg and Euler[14] have analyzed the case of slowly varying but strong fields and have shown that the charges and currents of the virtual pairs have the same effect as if there were no pairs, but instead the Lagrangian of the variation principle of the electrodynamics of special relativity theory were corrected from $\mathfrak{L}$, where

$$-\mathfrak{L}=(\mathbf{E}^2-\mathbf{B}^2)/8\pi=(1/8\pi)\mathcal{E}_{\mathrm{crit}}{}^2S, \qquad (86)$$

to $\mathfrak{L}^*$, where

$$(-\mathfrak{L}^*)-(-\mathfrak{L})=(1/8\pi^2)(e^2/\hbar c)\mathcal{E}_{\mathrm{crit}}{}^2\int_0^\infty dx e^{-x}x^{-3}$$
$$\times\left\{ix^2P\frac{\cos[x(S+2iP)^{\frac{1}{2}}]+\mathrm{conj.}}{\cos[x(S-2iP)^{\frac{1}{2}}]-\mathrm{conj.}}-(x^2S/3)+1\right\}. \qquad (87)$$

Here S represents the scalar, $(\mathbf{E}^2-\mathbf{B}^2)/\mathcal{E}_{\mathrm{crit}}{}^2$ and P represents the pseudoscalar, $(\mathbf{E}\cdot\mathbf{B})/\mathcal{E}_{\mathrm{crit}}{}^2$. For small field strengths,

$$-8\pi\mathfrak{L}^*/\mathcal{E}_{\mathrm{crit}}{}^2=S+\pi^{-1}(e^2/\hbar c)$$
$$\times\{(S^2+7P^2)/45+(26P^2S+4S^3)/315+\cdots\}. \qquad (88)$$

In general relativity the appropriate scalar and pseudoscalar quantities are

$$\mathcal{E}_{\mathrm{crit}}{}^2S=-\tfrac{1}{2}F_{\alpha\beta}g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta},$$
$$\mathcal{E}_{\mathrm{crit}}{}^2P=\tfrac{1}{8}(-g)^{\frac{1}{2}}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}. \qquad (89)$$

Here g represents the determinant of the g_{ik}. The quantities $\epsilon^{\alpha\beta\gamma\delta}$ do not form a tensor. They are defined instead by the statements that $\epsilon^{1234}=1$ and that ϵ changes sign on the interchange of any two indices. To correct for the effect of virtual pairs in geon theory we have only to replace $\mathfrak{L}/c=(1/16\pi c)\ F_{\alpha\beta}F^{\alpha\beta}$ in the variational principle (4) of general relativity by $\mathfrak{L}^*/c$, calculated as just defined. In taking the variation of the integral, which includes the volume factor $(-g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$, we regard as the quantities to be varied the 16 gravitational potentials g^{ik} and the four electromagnetic potentials A_i in $F_{ik}=(\partial A_k/\partial x^i)-(\partial A_i/\partial x^k)$. The coefficient of δg^{ik} contains two parts, of which one is the Einstein analog, $G_{ik}\equiv R_{ik}-\frac{1}{2}g_{ik}R$, of the d'Alembertian of the gravitational potential, and the other is the stress-energy tensor, T_{ik}:

$$T_{ik}=2(-g)^{-\frac{1}{2}}\delta[(-g)^{\frac{1}{2}}\mathfrak{L}^*]/\delta g^{ik}$$
$$=2(\partial\mathfrak{L}^*/\partial S)(\delta S/\delta g^{ik})$$
$$+2(\partial\mathfrak{L}^*/\partial P)(\delta P/\delta g^{ik})-g_{ik}\mathfrak{L}^*$$
$$=-2F_{i\beta}g^{\beta\delta}F_{k\delta}\mathcal{E}_{\mathrm{crit}}{}^{-2}(\partial\mathfrak{L}^*/\partial S)$$
$$-g_{ik}(P\partial\mathfrak{L}^*/\partial P+\mathfrak{L}^*). \qquad (90)$$

Thus the gravitational field equations take their usual form, $G_{ik}=(8\pi G/c^4)T_{ik}$, with only the change (90) in the form of the stress energy tensor. Similarly the coefficient of the variation, δA_i, of a typical electromagnetic potential, equated to zero, gives the ith electromagnetic field equation,

$$0=4\pi(-g)^{-\frac{1}{2}}(\partial/\partial x^\alpha)[(-g)^{\frac{1}{2}}\partial\mathfrak{L}^*/\partial(\partial A_i/\partial x^\alpha)]$$
$$=4\pi(-g)^{-\frac{1}{2}}(\partial/\partial x^\alpha)\{(-g)^{\frac{1}{2}}(\partial\mathfrak{L}^*/\partial S)$$
$$\times(2F^{i\alpha}/\mathcal{E}_{\mathrm{crit}}{}^2)+(-g/2)(\mathcal{E}_{\mathrm{crit}}{}^{-2}\partial\mathfrak{L}^*/\partial P)\epsilon^{\alpha i\gamma\delta}F_{\gamma\delta}\}, \qquad (91)$$

where the F's are considered to be expressed in terms of the A's. Equations (90) and (91) define the theory of geons in Region II.

NEUTRINO-CONTAINING GEONS

There is little difference between the theory of a geon built out of neutrinos, and one built out of electromagnetic fields, apart from the fact that each neutrino state will accommodate only one quantum of energy. The general relativity version of Maxwell's equations is replaced by the general relativity form of Dirac's equation[15] for an entity of zero rest mass. The stress-energy tensor for Dirac particles of zero rest mass forms the source term in the gravitational field equations. The equation $GM/c^2R\sim 1$ still connects mass and radius of the object. The modes of dissipation of energy are still leakage through the refractive index barrier, coupling through nonlinear effects to waves that run freely to infinity, and various forms of fission. Consequently there is little point to reformulating in terms of neutrinos our discussion of the properties of electromagnetic geons.

One difference appears in the process of interpenetration of two geons. When both are made of neutrinos, the overlap of the active regions of the two objects will force a promotion of their constituent neutrinos to states of higher momentum and energy, in accordance with the Pauli principle. The uptake of energy implies a strong effective repulsion compared with the forces that would otherwise have been at work, as for example in the collision of an electromagnetic geon and a neutrino geon, or the impact of a neutrino geon on an anti-

[14] W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936); see also V. Weisskopf, Kgl. Danske Videnskab. Selskab., Mat.-fys. Medd. **14**, No. 6 (1936–7).

[15] See for example Marcel Riesz, "L'équation de Dirac en relativité générale" (to be published). I am indebted to Professor Riesz for the opportunity to see this paper in advance of publication.

neutrino geon. Thus there is a certain interesting specificity about the interaction of geons.

A second and fundamental difference shows between pure neutrino geons and pure electromagnetic geons when one asks how small the object can be before quantum effects come in. In the neutrino case quantum effects are of course in principle present right from the start, in the action of the Pauli exclusion principle. But no other quantum effects are evident, and one does not at first sight see any reason why one should not be able to follow the properties of neutrino geons down to very small sizes, enormously less than the limit $\sim 10^{10}$ cm placed on classical electromagnetic geons by the onset of virtual pair phenomena. However, one at least knows where the limit is in the electromagnetic case, while in the neutrino case the critical limit could be anywhere, as indicated by the following analysis. Just as the nonlinear phenomena of electromagnetic geons commence when photon-photon collisions become important, so neutrino-neutrino collisions represent a potential critical mechanism to set a lower bound on the size, and an upper limit on the density, of those pure neutrino geons that are susceptible to simple analysis. The penetrating power of a neutrino through ordinary matter is so enormous[16] that it seems at first sight ridiculous to consider the collision of neutrino with neutrinos. However, if the continuous spectrum of electrons from μ-meson decay is correctly interpreted in terms of the simultaneous emission of two neutrinos (or a neutrino and an antineutrino) then it necessarily follows that two neutrinos of the appropriate character running towards each other with sufficiently high energy, $(\hbar\omega_1)\cdot(\hbar\omega_2)\gtrsim(207mc^2/2)^2$, will necessarily be capable of provoking the reaction $\nu+\nu\rightarrow\mu^-+e^+$; and from the Fermi type of beta-decay theory it follows that the reaction cross section will vary at sufficiently high energies as

$$\sigma\sim(g^2/\hbar^2c^4)\omega_1\omega_2, \tag{92}$$

where the coupling constant is of the order

$$g\sim10^{-49}\text{ erg cm}^3. \tag{93}$$

The existence of this absorption mechanism for high energies implies that even low-energy neutrinos can produce virtual μ,e pairs, which reannihilate and send the neutrinos off into new directions. The cross section for this neutrino-neutrino scattering process, evaluated in the forward direction by use of the dispersion relation[12]

$$(d\sigma/d\Omega_2)_{\text{forward}} = \left|(\omega_2^2/2\pi^2c)\int_0^\infty \sigma_{\text{absn}}(\omega_2')(\omega_2'^2-\omega_2^2)^{-1}d\omega_2'\right|^2, \tag{94}$$

is easily seen to diverge logarithmically,

$$(d\sigma/d\Omega_2)_{\text{forward}} \sim(\omega_1\omega_2^2g^2/\hbar^2c^5)^2\ln^2(\hbar\omega_{\text{upper}}/100\,mc^2), \tag{95}$$

unless one inserts for the indicated upper limit of infinity a finite upper limit, $\hbar\omega_{\text{upper}}$, for which one so far has no evidence. Similarly, the refractive index presented by a medium containing per unit volume n_1 neutrinos of circular frequency ω_1 differs from unity, according to (83), by the amount

$$n(\omega_1)-1\sim(g^2/\hbar^3c^3)(n_1\hbar\omega_1)\ln(\hbar\omega_{\text{upper}}/100\,mc^2), \tag{96}$$

again a divergent expression.

In view of the divergence of the neutrino-neutrino interaction as evaluated from Fermi beta theory and the dispersion relation, we really have no basis to discuss neutrino geons at all, much less to set a lower limit of masses at which the theory takes a new form, as in the case of electromagnetic geons. Nevertheless, it is still of interest to see what we can say if we assume that the logarithm, instead of diverging, has a value of the order of 10^2 or less. In this case we can employ (96) to draw two conclusions. First, it will be beyond the scope of simple neutrino theory to analyze geons in which the refractive index correction is of the order of unity, or the energy density exceeds

$$\begin{aligned} n_1\hbar\omega_1 &\sim(\hbar^3c^3/g^2)[1/\ln(\hbar\omega_{\text{upper}}/100\,mc^2)] \\ &\sim10^{47}\text{ erg/cm}^3 \\ &\sim(10^{26}\text{ g/cm}^3)c^2. \end{aligned} \tag{97}$$

To yield such a density, neutrino states must be occupied up to momenta of the order p, or energies of the order cp, where

$$10^{47}\text{ erg/cm}^3\sim\int_0^p(cp)2(4\pi p^2dp/h^3), \tag{98}$$

or

$$\begin{aligned} cp &\sim(h^3c^3 10^{47}\text{ erg/cm}^3)^{\frac{1}{4}} \\ &\sim1\text{ erg}\sim10^6\,mc^2. \end{aligned} \tag{99}$$

This energy is so great that it carries us beyond the domain of virtual μ,e processes to real production of pairs of this kind. This circumstance leads to the second conclusion, that the energy density of any geon which can be analyzed in terms of simple neutrino theory can at most be equal to the right-hand side of (98), evaluated for a maximum neutrino energy, cp, of the threshold value, $100\,mc^2$:

$$\begin{aligned} \text{energy density} &\sim2\pi(cp)^4/h^3c^3 \\ &=3.62\times10^{31}\text{ erg/cm}^3 \\ &=(4.02\times10^{10}\text{ g/cm}^3)c^2, \end{aligned} \tag{100}$$

corresponding according to (96) to a refractive index correction of only one part in 10^{15}. This estimate implies a mass density considerably less than a typical estimate for nuclear matter, $\sim1.4\times10^{14}$ g/cm^3. It is conceivable that there is some limitation of which physics is not

[16] See, for example, F. Reines and C. Cowan, Phys. Rev. **92**, 830 (1953).

aware that puts a bound to the density of pure neutrino geons much smaller than (100). However, if real production of μ,e pairs sets the only limit, then it is possible in principle to treat in terms of existing theory objects built up solely out of neutrinos and gravitational forces down to a radius

$$R=[(M/R)(3/4\pi)(4\pi R^3/3M)]^{\frac{1}{2}} \sim[(c^2/G)1.69\times10^{11}\text{ g/cm}^3)]^{\frac{1}{2}} =2.83\times10^8\text{ cm}$$

and mass

$$M\sim(c^2/G)R=3.82\times10^{36}\text{ g.} \qquad (101)$$

This lower limit for pure neutrino geons is less than the corresponding limit (10) for pure electromagnetic geons (10^{11} cm, 10^{39} g) by not more than a few powers of ten. In one case the bound comes from the limiting electromagnetic field strength, or field energy density, at which pairs of positive and negative electrons begin to appear; in the other case, electron-mu-meson pairs.

Mixed geons, energized by a combination of neutrinos and electromagnetic fields, will have properties similar to either kind of pure system, and hardly need separate consideration here. However, the presence of neutrinos and photons on an equal footing does raise again the question of the neutrino theory of light, developed by Jordan, Kronig, and others, and brought to a halt by the discovery of Pryce that the theory in its then existing form did not possess proper relativistic invariance.[17] The theory assumes that there exists between neutrinos a physical interaction, never introduced explicitly and never discussed, such that a photon is described by a pair consisting of one neutrino and one antineutrino; or rather, by a quantum-mechanical linear superposition of very many such pairs of states. The description implies an unavoidable complementarity, such that a statement of the occupation numbers of the photon states, and a prescription of the occupation numbers of the neutrino states, stand to each other in a mutually exclusive relation. When only neutrino states are occupied, the theory speaks of a pure neutrino field. When in addition a few antineutrino states are filled, a mixture of light and neutrinos is said to be present. With equal numbers of neutrinos and antineutrinos a pure photon field is considered to exist; and so on, up to the case of a pure antineutrino field. Such a description would evidently subsume geons of purely electromagnetic character, of purely neutrino type, and of mixed constitution, all into a unified class of systems.

Regarding the status of the neutrino theory of light, I am kindly informed by Professor Eugene Wigner that the conceivable mechanisms for the combination of the spinors and momentum vectors of the neutrino and antineutrino states to form in an invariant way the vector magnitudes of the photon states are far from having been explored in a comprehensive way in the literature, so that is not necessarily clear that the objections of Pryce will forever retain their force. Second, recent studies of the decay of the μ meson[18] show that the lifetime and the form of the electron spectrum[19] are together consistent with a universal Fermi interaction of the kind met in beta decay only if the two neutrinos are of opposite character. This result suggests, though it does not prove, that μ decay produces one neutrino and one antineutrino. The consequences are the existence of the $\nu+\nu^*\rightarrow\mu+e$ reaction at high energies, and ν,ν^* scattering at low energies. In other words, a physical background for the interaction of the neutrino theory of light does not have to be postulated; it exists. Third, no such argument exists for an interaction between two neutrinos of the same character. Consequently it is conceivable that the theory of pure neutrino geons, or pure antineutrino geons, can be carried without meeting new physical effects to energy densities larger, and sizes smaller, than the limits of (100) and (101).

In summary, if we assume the existence of a cutoff in the Fermi interaction at high energies in Eqs. (95) and (96), then the door is open to analyzing the properties of neutrino geons over an enormous range of sizes without going outside the scope of existing theory; but below an uncertain critical limit of sizes interesting and fundamental physical questions raise themselves.

7. ELECTRICITY, GAUSS' THEOREM, AND GRAVITATIONAL FIELD FLUCTUATIONS

What of free electricity? Consistently to complete the scheme of classical physics we find no alternative but to regard the geon as exemplar of the concept of body; but neither in this object nor in the divergence free theory behind it is there any place for charge. All lines of force continue without end. Let a sphere be drawn around the immediate neighborhood of a point. Then as many lines of force go in as out.

In applying the theorem of Gauss we have tacitly assumed space to be simply connected. One knows, however, that the notion of Riemannian manifold by itself places no such requirement upon the space-time continuum. One can consider a metric which on the whole is nearly flat except in two widely separated regions, where a double-connectedness comes into evidence as symbolized in Fig. 7. The general divergence-free electromagnetic disturbance holding sway in the space around one of these "tunnel mouths" will send forth lines of force into the surrounding space, and appear to have a charge. However, an equal number of lines of force must enter the region of disturbance from

[17] P. Jordan, Ergeb. exakt. Naturwiss. **7**, 158 (1928); Z. Physik, numerous papers ending with **105**, 229 (1937); R. de L. Kronig, several papers ending with Z. Physik **100**, 569 (1936) and Physica **3**, 1120 (1936); M. H. L. Pryce, Nature **141**, 976 (1938). See also de Broglie, Heisenberg, and Kramers in *L. de Broglie, Physicien et Penseur* (Editions Albin Michel, Paris, 1953). I am indebted to Professor Arthur Wightman for several discussions of the present status of Jordan's idea.

[18] L. Michel and A. S. Wightman, Phys. Rev. **93**, 354 (1954).

[19] J. Vilain and R. W. Williams, Phys. Rev. **92**, 1586 (1953).

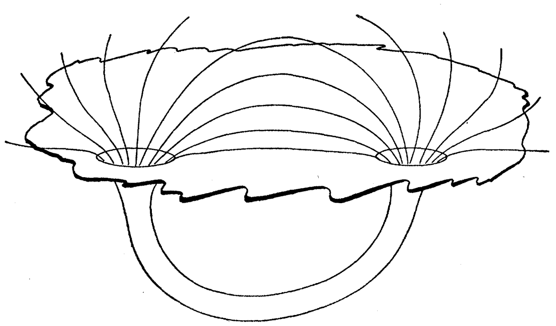

FIG. 7. Schematic representation of lines of force in a doubly-connected space. In the upper continuum the lines of force behave much as if the tunnel mouths were the seats of equal and opposite charges.

the tunnel. Consequently the other mouth of the tunnel must manifest an equal and opposite charge. In such a doubly-connected space it is evidently a matter of definition whether we say that divergence-free field equations do or do not permit the existence of electric charge. It will be convenient to say yes if the width of the tunnel is small compared to the separation of its mouths. So far we have inquired only after the behavior of the electromagnetic field in a metric assumed to be pre-existing. However, in classical relativity theory the metric cannot be taken arbitrarily, but must be found by solution of the gravitational field equations. No investigation is known to have been made of the possibility of a self-consistent solution that is double-connected. Yet one would not be surprised to find that no reasonable choice of boundary conditions would permit such a classical solution.

Let one pass from the classical theory defined by the action principle (4) to the corresponding quantum theory, either by the prescription of Feynman,[20]

$$\psi \sim \sum_{\substack{\text{all conceiv-}\\ \text{able relevant}\\ \text{field histories}}} \exp[(i/\hbar)\text{ (classical action for each field history)}], \tag{102}$$

or by any other standard method. Then one is forced to recognize the existence of fluctuations in all fields. Their magnitudes depend upon the size, L, of the space-cotime regions under consideration, and are given under suitable conditions by formulas of the form $\Delta F \sim (\hbar c)^{\frac{1}{2}}/L^2$; $\Delta A \sim (\hbar c)^{\frac{1}{2}}/L$; and $\Delta g \sim (\hbar G/c^3)^{\frac{1}{2}}/L$. So long as one deals with distances large in comparison with $(\hbar G/c^3)^{\frac{1}{2}} \sim 10^{-33}$ cm, one can disregard for most purposes the fluctuations in the metric, and consider space to be simply connected. But in deriving Gauss' theorem one is driven to consider an integral over the whole of the region in question, including regions of the very smallest spatial extension. Here the inevitable fluctuations force on space time a most complicated structure. Because it is the essence of quantum mechanics that *all* field histories contribute to the probability amplitude, the sum (102) not only may contain doubly and multiply connected metrics; it must do so. General relativity, quantized, leaves no escape from topological complexities of which Fig. 7 is only an oversimplified symbol. In this sense the door is open for the existence of charges in the quantum version of a theory that contains no charges.

[20] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

Little progress has so far been made in studying the quantization of general relativity.[21] (1) It is not yet certain whether the method of summing over configurations gives in the case of nonlinear theories results that are identical to those derived from other methods of quantization. (2) It is not certain that the action function of general relativity can be given a well-defined meaning for those field configurations, classically unrealizable, that make the factor $(-g)^{\frac{1}{2}}$ in the action function a pure imaginary. This square root recalls the similar factor in relativistic electron theory, $(-dx_\mu dx^\mu)^{\frac{1}{2}}$, where likewise similar difficulties of interpretation arise for nonclassical paths, and where the root has been replaced by Dirac's matrix expression, $\Gamma_\mu dx^\mu$, associated with half-integral spins and Fermi-Dirac statistics. Thus, as of today we cannot exclude that charges will show themselves naturally and inevitably in the spinor quantization of the only comprehensive and divergence-free classical theory of fields that we possess.

8. CONCLUSION

Taking seriously and following out the consequences of the forty year old theory of general relativity, we have been led to recognize the relative stability of certain types of electromagnetic field disturbances held together by gravitational forces. These geons furnish for the first time a completely classical, divergence-free, self-consistent picture of the Newtonian concept of body over the range of masses from $\sim 10^{39}$ g to $\sim 10^{57}$ g. Two such geons interacting at a distance large compared to their characteristic dimensions behave as elementary objects. However, when one geon is followed for a long time or is allowed to interact closely with another it undergoes interesting and characteristic transformation processes.

Classical geons are not objects for study in the laboratory, nor is there any evident reason to believe that geons of the classical range of sizes now exist in nature, or ever did exist. Even were such large geons once present, a sufficient lapse of time would guarantee the decay of all but extraordinarily stable ones to a mass below the limit where these systems are capable of classical analysis.

On entry into the quantum domain of sizes, electromagnetic geons build up field strengths strong enough to produce pairs of electrons. At not very different sizes

[21] I am indebted to Professor James Anderson and Professor Peter Bergmann for instructive discussions of the literature of this problem.

neutrino geons commence to give birth to μ meson-electron pairs. Consequently the projection of geon theory to objects of dimensions less than $\sim 10^{10}$ cm to $\sim 10^{8}$ cm requires the analysis of new phenomena.

In conclusion, the geon makes only this visible contribution to science: it completes the scheme of classical physics by providing for the first time an acceptable classical theory of the concept of body.

One's interest in following geon theory down into the quantum domain will depend upon one's considered view of the relation between very small geons and elementary particles.

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Angular Momentum of a Real Field

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It is paradoxical that a field oscillating in a mode which on grounds of symmetry contains no angular momentum should carry angular momentum when quantized. This angular momentum is shown to result from zero-point oscillations in other modes. The operator for the square of the angular momentum of the field is discussed.

The angular momentum of a second quantized real field has been discussed a number of times in the past in connection with liquid drop oscillation[1] and multipole expansions of the electromagnetic field.[2] These articles discuss the z component of the angular momentum in a satisfactory fashion but the discussion of the square of the angular momentum seems to have been first carried out in a proper way only recently.[3] It would hardly seem necessary to raise again the question of the angular momentum of a second quantized field, but there seems to be considerable confusion in the literature as regards to the square of the angular momentum. Also in at least three of these references[2] there are remarks which are not quite correct. The fact that there is a good deal of evidence that this old but important question is still not properly understood has convinced the author that someone should write still another note on this subject even if it does not contain very much that is new.

In order to illustrate the type of question that we wish to consider, we discuss first a liquid drop oscillating in a surface mode given by the spherical harmonic $l=2$, $m=0$, namely an oscillation back and forth between prolate and oblate spheroidal shapes. It is obvious from symmetry arguments that such a mode of oscillation should carry no angular momentum. How then in second quantization for a one-quantum state of this mode, does one expect to get a squared angular momentum of $6\hbar^2$ rather than zero? The usual answer which one encounters is that this result of $l=2$ for the quantum number of squared angular momentum of the excited mode follows in an elementary way from the fact that the excited mode is described by a spherical wave with that index. This, however, is not in accord with the facts. In classical field theory the square of the angular momentum is given by the square of the m index of the spherical mode which is excited. It might be expected that after the inclusion of quantum fluctuation effects the angular momentum of the field for a one particle state would be $m(m+1)\hbar^2$. However, in second quantization for a one-particle state it becomes a function of the l index rather than the m index for reasons which are quite subtle. The easiest way to get the correct result is to remember that an alternative description of the quantized field is a first quantized many-particle system where for one particle the square of the angular momentum is, of course, given by the l index of the ψ field. However, in field theory the problem is much less transparent.

To come back again to the example given above, the symmetry arguments (reflections in x, y, and z planes) which show that classically there should be zero angular momentum associated with this mode, somehow become invalid when the system is described in quantum-mechanical language. Since each of the normal modes of oscillation constitutes a separate dynamical system quite independent of the others, it is clear that their symmetry properties exist independent of their level of oscillation or of possible zero-point fluctuations in other modes, and that the angular momentum of this mode should, in fact, be zero. If this seems parodoxical, it is, in part, because of a confusion in terminology. For a classical field it is possible to speak of the angular momentum of a normal mode of oscillation in an

[1] M. Fierz, Helv. Phys. Acta **16**, 365 (1943).

[2] W. Heitler, Proc. Cambridge Phil. Soc. **32**, 112 (1936); W. Heitler, *Quantum Theory of Radiation* (University Press, Oxford, 1949); J. B. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952); B. Stech, Z. Naturforsch. **7a**, 401 (1952); R. G. Sachs and J. G. Brennan, Phys. Rev. **88**, 825 (1952).

[3] C. M. DeWitt and J. H. D. Jensen, Naturforsch. **8a**, 267 (1953). It is proposed to consider here questions not covered in this reference.