Use of Causality Conditions in Quantum Theory

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It is shown that the derivation of the Kramers-Kronig dispersion relations given by Gell-Mann, Goldberger, and Thirring may be carried out without the use of perturbation theory. It is further shown that the results are essentially independent of the form of the coupling between the electromagnetic field and the matter fields. The demonstration is facilitated by the construction of simple rigorous expressions for the commutator of two vector potential operators and for the S-matrix describing photon scattering.

I. INTRODUCTION

In a recent paper[1] with the above title, it was shown that the well-known Kramers-Kronig dispersion relations followed from conventional quantum electrodynamics by the imposition of the requirement that signals cannot propagate faster than the velocity of light. The situation discussed was that of a system interacting with the electromagnetic field according to the interaction Hamiltonian, H, given by

$$H=-\int d^3x j_\mu(x)A_\mu(x). \tag{1.1}$$

The proof of the dispersion relations was carried out by treating the above interaction as a perturbation and only terms of second order in the electric charge were retained. It was conjectured that this perturbation limitation was unnecessary and that the results were rigorously correct. It is one of the purposes of this paper to prove that conjecture. We shall also show that the electromagnetic interaction with the matter system need not be restricted to the above simple form which is not even sufficiently general to include the practical case of a system of nucleons and pions. We need, in fact, assume only that the current density operator, defined symbolically by

$$j_\mu(x)=\partial L/\partial A_\mu(x), \tag{1.2}$$

where L is the Lagrangian density of the system and $A_\mu(x)$ is the vector potential operator, does not contain any time derivatives of $A_\mu(x)$. The explicit form of the interaction Hamiltonian will not be needed.

In Sec. II an exact evaluation of the commutator of two A's will be given and in Sec. III we present a rigorous expression for the photon scattering amplitude. The results are briefly discussed in Sec. IV.

II. EVALUATION OF $[A_\mu(x),\ A_\nu(y)]$

The causality condition used in GGT was the requirement that the commutator of two Heisenberg field operators of the electromagnetic field, $A_\mu(x)$ and $A_\nu(y)$, shall be zero if the points x and y have space-like separation. It was in the evaluation of this commutator that perturbation theory was employed in GGT and only terms of order e^2 were retained in the case when the interaction was given by Eq. (1.1). We shall show that the expression derived there, GGT, Eqs. (3.1) and (3.6), are rigorously correct provided the interaction representation current density operators in those equations are replaced by the corresponding Heisenberg representation operators. There are, of course, other modifications, to be given below, when electromagnetic interactions more general than (1.1) are permitted.

The unrenormalized vector potential operator is taken to satisfy the Heisenberg equation of motion,

$$\square^2 A_\mu(x)=-j_\mu(x), \tag{2.1}$$

where we assume that the matter system current density operator, $j_\mu(x)$, may involve $A_\mu(x)$ (as is the case for spin-zero fields) but does not contain any time derivations of $A_\mu(x)$. For our purposes there is no need to use the explicitly renormalized operators.[2] We write, following Yang and Feldman[3] and Källén,[4] in place of (2.1) the formal solution

$$A_\mu(x)=A_\mu{}^{\text{in}}(x)+\int d^4x' D_r(x-x')j_\mu(x'), \tag{2.2}$$

where $D_r(x)$ is the usual retarded Green's function. The "in" fields, $A_\mu{}^{\text{in}}(x)$, satisfy the commutation relation

$$[A_\mu{}^{\text{in}}(x),A_\nu{}^{\text{in}}(y)]=i\delta_{\mu\nu}D(x-y). \tag{2.3}$$

If $\eta(x)$ is the step function which is zero for $x_0<0$ and unity for $x_0>0$, we have

$$D_r(x)=-\eta(x)D(x). \tag{2.4}$$

We now form the commutator of two A's, using

[1] Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954). This paper will be referred to hereafter as GGT.

[2] It is of course possible to use the renormalized operators, but the subsequent equations become more lengthy, since $j_\mu(x)$ then involves time derivatives of $A_\mu(x)$. See, for example, S. N. Gupta, Proc. Phys. Soc. (London) **A64**, 426 (1951). We shall note the modifications of our formulae which would arise in this case.

[3] C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).

[4] G. Källén, Arkiv Fysik **2**, 371 (1950).

Eq. (2.2):

$$[A_\mu(x),A_\mu(y)]$$
$$=i\delta_{\mu\nu}D(x-y)+\int d^4x'\int d^4y' D_r(x-x')$$
$$\times[j_\mu(x'),j_\nu(y')]D_r(y-y')$$
$$+\int d^4y' D_r(y-y')[A_\mu{}^{\mathrm{in}}(x),j_\nu(y')]$$
$$+\int d^4x' D_r(x-x')[j_\mu(x'),A_\nu{}^{\mathrm{in}}(y)]. \quad (2.5)$$

In order to evaluate the commutator of the j's with the "in" fields, we use the following convenient expression given by Källén[5] for $A_\mu{}^{\mathrm{in}}(x)$:

$$A_\mu{}^{\mathrm{in}}(x)=\int d^4x'\left\{D(x-x')\eta(z-x')j_\mu(x')-\delta(x_0'-z_0)\right.$$
$$\left.\times\left[A_\mu(x')\frac{\partial D(x-x')}{\partial x_0}+\frac{\partial A_\mu(x')}{\partial x_0'}D(x-x')\right]\right\}. \quad (2.6)$$

In this expression z is an arbitrary point. In order to evaluate $[A_\mu{}^{\mathrm{in}}(x),j_\nu(y')]$ we choose z to coincide with y' and similarly in the evaluation of $[j_\mu(x'),A_\nu{}^{\mathrm{in}}(y)]$, z is taken to be x'. We find then for Eq. (2.5) the expression

$$[A_\mu(x),A_\nu(y)]$$
$$=i\delta_{\mu\nu}D(x-y)+\int d^4x'\int d^4y'\left\{D_r(x-x')D_r(y-y')\right.$$
$$\times[j_\mu(x'),j_\nu(y')]+D_r(y-y')D(x-x')$$
$$\times\left[\eta(y'-x')[j_\mu(x'),j_\nu(y')]-\delta(x_0'-y_0')\right.$$
$$\left.\times\left[\frac{\partial A_\mu(x')}{\partial x_0'},j_\nu(y')\right]\right]+D_r(x-x')D(y-y')$$
$$\times\left[\eta(x'-y')[j_\mu(x'),j_\nu(y')]\right.$$
$$\left.\left.-\delta(x_0'-y_0')\left[j_\mu(x'),\frac{\partial A_\nu(y')}{\partial y_0'}\right]\right]\right\}. \quad (2.7)$$

With the exception of the terms involving the commutator of the current density and the time derivative of the vector potential, this is exactly the expression found in GGT, Eq. (3.6), provided that one replaces their interaction representation operators by our Heisenberg operators, $j_\mu(x)$. The additional commutators in (2.7) are of course zero if j_μ is independent of A_μ as it was in the case treated in GGT. Incidentally, if we had been using the renormalized photon operators, the term in Eq. (2.6) involving $\partial D/\partial x_0$ would have made a contribution to Eq. (2.7).

[5] G. Källén, Helv. Phys. Acta **25**, 417 (1952); and Copenhagen Lectures, 1952 and 1953 (unpublished).

If now, following GGT, we restrict our attention to the case $x_0\to+\infty$, $y_0\to-\infty$, the second and third terms on the right hand side of Eq. (2.7) vanish and using Eq. (2.4) and the fact that $D(x)=-D(-x)$, we have

$$[A_\mu(x),A_\nu(y)]\to i\delta_{\mu\nu}D(x-y)$$
$$+\int d^4x'\int d^4y' D(x-x')D(y'-y)$$
$$\times\left\{\eta(x'-y')[j_\mu(x'),j_\nu(y')]\right.$$
$$\left.-\delta(x_0'-y_0')\left[j_\mu(x'),\frac{\partial A_\nu(y')}{\partial y_0'}\right]\right\}. \quad (2.8)$$

Needless to say, in practice the vanishing of the other terms in (2.7) requires some precise specification of how the limit $x_0\to+\infty$, $y_0\to-\infty$ is to be taken. We shall not discuss this point here.

It is useful to introduce into Eq. (2.8) the Fourier representation of the D function:

$$D(x)=-\frac{i}{(2\pi)^3}\int d^4k\,\epsilon(k)\delta(k^2)e^{ik\cdot x}. \quad (2.9)$$

We obtain then, finally,

$$[A_\mu(x),A_\nu(y)]\to i\delta_{\mu\nu}D(x-y)-i\int\frac{d^4k'}{(2\pi)^3}\int\frac{d^4k}{(2\pi)^3}$$
$$\times e^{ik'\cdot x-ik\cdot y}\epsilon(k')\epsilon(k)\delta(k^2)\delta(k'^2)\mathfrak{M}_{\mu\nu}(k',k), \quad (2.10)$$

where

$$\mathfrak{M}_{\mu\nu}(k',k)=-i\int d^4x'\int d^4y' e^{-ik\cdot x'+ik\cdot y'}$$
$$\times\left\{\eta(x'-y')[j_\mu(x'),j_\nu(y')]\right.$$
$$\left.-\delta(x_0'-y_0')\left[j_\mu(x'),\frac{\partial A_\nu(y')}{\partial y_0'}\right]\right\}. \quad (2.11)$$

The matrix element of this operator between an initial and final state of the matter system yields the generalization of the analogous quantity introduced in GGT, Eq. (3.10). In the next section we shall study the relation between such a matrix element and an exact expression for the photon scattering amplitude. The requirement that (2.10) shall vanish for space-like separations of x and y together with the relation between $\mathfrak{M}_{\mu\nu}(k',k)$ to the photon scattering amplitude to be discussed below leads, as is shown by GGT, to the Kramers-Kronig dispersion relations.

III. THE EXACT PHOTON SCATTERING AMPLITUDE

We wish to derive an expression for the scattering amplitude which describes the situation of a photon of four-momentum k and polarization ν being scattered by a matter system in state i, say, which makes a transition to state f producing a photon of four-momentum k', polarization μ. For this purpose it is useful to consider the propagation function

$$G_+(x,y)=\langle f|P(A_\mu(x),A_\nu(y))|i\rangle. \tag{3.1}$$

It is easy to see that as $x_0\rightarrow+\infty$, $y_0\rightarrow-\infty$, (3.1) is proportional to a matrix element of the S-matrix, since $A_\mu(x)\rightarrow A_\mu{}^{\text{out}}(x)$ and $A_\nu(y)\rightarrow A_\nu{}^{\text{in}}(y)$ and $A_\mu{}^{\text{out}}(x)=S^{-1}A_\mu{}^{\text{in}}(x)S$.[3] We assume that the states i and f are steady. The matrix element of the R matrix ($S=1-iR$) which gives the scattering amplitude for the process described above may be found from the propagation function (3.1) in a familiar fashion.[6] The procedure given in reference 6 is equivalent to the following instruction:

$$\langle f,k',\mu|R|i,k,\nu\rangle=-i\int d^4x\int d^4y e^{-ik'\cdot x} \times\{(-\Box_x{}^2)(-\Box_y{}^2)G_+(x,y)\}e^{ik\cdot y}. \tag{3.2}$$

We now carry out the differentiations indicated in (3.2). It follows from the canonical commutation rules and from the equation of motion for $A_\mu(x)$, Eq. (2.1), together with the definition of the P bracket that

$$-\Box_x{}^2G_+(x,y)=\langle f|P(j_\mu(x),A_\nu(y))|i\rangle -i\delta(x-y)\delta_{fi}\delta_{\mu\nu}. \tag{3.3}$$

Let us assume that the states i and f are different so that we may ignore the second term in (3.3). Next we apply $-\Box_y{}^2$ and obtain

$$(-\Box_x{}^2)(-\Box_y{}^2)G_+(x,y)=\langle f|P(j_\mu(x),j_\nu(y)) -\delta(x_0-y_0)[j_\mu(x),\partial A_\nu(y)/\partial y_0]|i\rangle. \tag{3.4}$$

In deriving Eq. (3.4) we have dropped a term involving $\delta(x_0-y_0)[A_\nu(y),j_\mu(x)]$ in accordance with our assumptions about $j_\mu(x)$. If we had been using renormalized photon operators, this term would make a contribution. It is, however, a C-number contribution if the only way time derivatives entered $j_\mu(x)$ were through the explicit charge-renormalization terms, and such contributions may again be dropped if the states i and f are different. The final expression for the scattering amplitude is then found to be

$$\langle f,k',\mu|R|i,k,\nu\rangle =-i\int d^4x\int d^4y e^{-ik'\cdot x}\langle f|P(j_\mu(x),j_\nu(y)) -\delta(x_0-y_0)[j_\mu(x),\partial A_\nu(y)/\partial y_0]|i\rangle e^{ik\cdot y}. \tag{3.5}$$

This looks exactly the same as the corresponding formula in GGT, Eq. (3.13) (remember that the commutator term should not appear in the case considered there), except for the fact that now the exact Heisenberg current operators appear in place of the interaction representation operators. Note that we have made no use of the fact that k and k' refer to real photons, i.e., $k^2=k'^2=0$, so that Eq. (3.5) is equally valid both on and off the energy shell.

A special case of Eq. (3.5) has previously been given by Low[7] in his treatment of the scattering of photons by nucleons interacting with charged mesons. Low's derivation is based on Dyson's expression for the S matrix in the interaction representation and will be published shortly by him. Similar results for special cases have also been given by Nambu.[8]

Our argument is now essentially completed. The matrix element appearing in Eq. (3.5) is the same as that which is found by forming $\langle f|\mathfrak{M}_{\mu\nu}(k',k)|i\rangle$ from Eq. (2.11) except that the P-bracket of the current operators appears instead of the commutator times a step function. This is just the situation found in the perturbation treatment of GGT, and the same relations among the dispersive and absorptive parts of the on-energy-shell matrix elements for forward scattering ($k'=k$) given there [GGT, Eqs. (3.15) and (3.16)] now hold as rigorous relations. The discussion given by them about the analytic properties of the scattering amplitudes may be taken over directly.

IV. CONCLUSIONS

It is, of course, not surprising that the Kramers-Kronig relations which follow from the above results are correct, independent of perturbation theory, and under very general assumptions on the form of the coupling between the electromagnetic field and the matter field. It is gratifying, however, that such a simple derivation may be given.

The rigorous expression for the scattering amplitude derived in Sec. III appears to be quite useful. An exactly analogous expression may be derived for the problem of meson-nucleon scattering. From such expressions many general properties of the scattering amplitudes may be deduced in a very simple way. These together with other applications will be discussed in a subsequent publication.

Note added in proof.—After this manuscript had been submitted for publication, the author received a letter from Dr. S. S. Schweber outlining a very similar treatment of the problem discussed here which he had independently given at about the same time.

Dr. Schweber's derivation differs from ours in the following respects: (1) He uses both "in" and "out" fields, expressing [see Eqs. (2.2) and (2.5)] $A_\mu(x)$ in terms of $A_\mu{}^{\text{in}}(x)$ and the retarded Green's function and $A_\nu(y)$ in terms of $A_\nu{}^{\text{out}}(y)$ and the advanced Green's function. (2) By limiting himself to a consideration of $\langle f|[A_\mu(x), A_\nu(y)]|i\rangle$ in the limit as $x_0\rightarrow+\infty$ and $y_0\rightarrow-\infty$, he shows that the use of Eq. (3.6) may be avoided, at least in the case where j_μ is independent of A_μ, which is the only case treated by him.

[6] Karplus, Kivelson, and Martin, Phys. Rev. **90**, 1073 (1953).

[7] F. E. Low, Phys. Rev. **96**, 1428 (1954). The author wishes to thank Prof. Low for a preprint of his paper and for a discussion of his derivation of Eq. (3.2) for his problem.

[8] Y. Nambu (private communication).