Double Scattering of Electrons with Magnetic Interaction*

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The Mott theory of double scattering of electrons by nuclei is extended to the case where a constant homogeneous magnetic field intervenes between the two scattering centers. The problem is treated in the Foldy-Wouthuysen representation. It is found that the presence of the magnetic field causes the asymmetry in the double-scattering cross section to be altered, and that this change in the asymmetry can be utilized to determine the gyromagnetic ratio of the free electron.

I. INTRODUCTION

THE Mott theory[1] of double scattering of electrons by nuclei is extended to the case where a constant homogeneous magnetic field intervenes between the two targets. If there is no anomalous moment interaction, it is found that the double-scattering cross section into a given direction is the same cross section, as in the field-free case, into another direction. The new direction (magnetic field present) can be obtained from the first direction (field-free) by rotating about the direction of the field by an amount equal to the space rotation of the particles between the scatterers. However, with an anomalous-moment interaction, there is an additional change in the cross section which depends upon the relative orientation of the constant homogeneous magnetic field with the directions of motion of the initial beam and first scattered beam. The way in which the asymmetry is altered is discussed, and it is demonstrated how it can be employed to determine the gyromagnetic ratio of the free electron.

II. MOTT THEORY

We will briefly outline the Mott theory of double scattering in a vacuum, and then show how the intervention of a magnetic field between the two scatterers alters the cross section.

An unpolarized beam along the direction IA (Fig. 1) is incident upon the first target at A. The part of the beam which is scattered into the direction of AB is allowed to fall upon the target at B. The beam is then scattered into direction BC. It is found that for a given angle θ_2 (Fig. 1), the double scattering cross section exhibits an asymmetry about the azimuthal direction in the $x'y'z'$ coordinate system.

Although the Mott theory starts with the Dirac equation, it restricts itself to transitions between positive energy states and neglects all contributions to the cross section arising from transitions from negative energy states. This approximation is quite good because it is found that the asymmetry in the double scattering cross section washes out for energies such that $\beta=v/c$ is greater than about 0.7.[2] This can be explained by noting that transitions between the positive and negative energy states can cause a depolarization of the first scattered beam. Thus, only where the contribution to the cross section from the negative-energy states is negligible can we expect that the asymmetry will not be negligible.

Because we limit the discussion to positive-energy states only, we can describe the spin polarization of the beam in terms of a two-by-two spin density matrix.[3] The original beam which is incident on the first target is unpolarized. The spin density matrix is given by

$$\rho_{mm'}{}^{(s)}=\tfrac{1}{2}\delta_{mm'} \tag{1}$$

where m and m' refer to two states of spin polarization. The beam of particles are considered to have the same momentum, and so the momentum density matrix describes a pure momentum state. In order that our notation be close to that of Mott,[1] we take the z component Pauli spin matrix diagonal. The scattering is described by a scalar transition operator (scalar under simultaneous rotations of spin and space) which transforms an initial momentum and spin state to a final state. It is given by

$$V=f(\theta)+ig(\theta)(\sigma_x\sin\varphi+\sigma_y\cos\varphi). \tag{2}$$

The $f(\theta)$ and $g(\theta)$ are the Mott f and g functions which depend on the polar angle θ between the beam incident on the target and the scattered beam (Fig. 1), and φ is the azimuthal angle. The angles θ and φ describe the difference in direction between the initial momentum and momentum after the scattering. The σ's are the usual Pauli spin matrices.

After the initial scattering, the density matrix is

$$\bar{\rho}=V(\theta_1,\varphi_1)\rho V^\dagger(\theta_1,\varphi_1). \tag{3}$$

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[1] N. F. Mott, Proc. Roy. Soc. (London) **A124**, 425 (1929); **A135**, 429 (1932).

[2] N. F. Mott, Proc. Roy. Soc. (London) **A135**, 429 (1932); H. A. Tolhoek and S. R. deGroot, Physica **17**, 1 (1951). This does not mean that the single scattering Mott formula is not exact, because there it is legitimate to consider only the positive energy states.

[3] Density matrices are discussed by: J. von Neuman, Göttingen Nachr. 245 (1927); R. C. Tolman, *Principles of Statistical Mechanics* (Oxford University Press, London, 1938), p. 327; H. A. Tolhoek and S. R. deGroot, Physica **17**, 1 (1951); U. Fano, National Bureau of Standard Report 1214 (unpublished) and others.

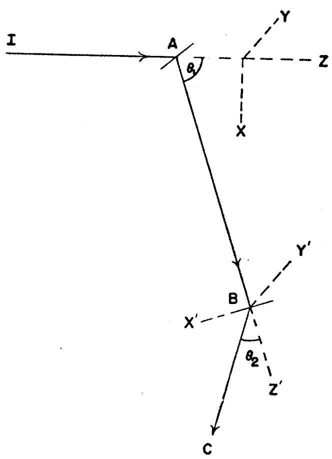

FIG. 1. Schematic diagram of double-scattering experiment.

Since V is a function of the angle between the initial and scattered directions, the density matrix after the first scattering $\bar{\rho}$ is a function of θ_1, φ_1 (Fig. 1). The direction of the incident beam being taken as the polar axis. The cross section can be obtained by taking the trace over the spin states, and is

$$d\sigma(\theta_1,\varphi_1)\sim[|f(\theta_1)|^2+|g(\theta_1)|^2], \tag{4}$$

which is independent of the azimuthal angle φ_1. If instead of measuring the cross section after the first scattering, the beam is then scattered by the target at B (Fig. 1), the cross section after the second scattering will exhibit an asymmetry in the azimuthal direction.

Following Mott, we consider the first scattered beam to be in the direction $\theta=\theta_1$ and $\varphi_1=0$. It is incident on the second target. The spin density matrix of this beam is, from Eq. (3),

$$\bar{\rho}^{(s)}=\tfrac{1}{2}N\{|f(\theta_1)|^2+|g(\theta_1)|^2 -i\sigma_y[f(\theta_1)g^*(\theta_1)-f^*(\theta_1)g(\theta_1)]\}, \tag{5}$$

where N is a normalization constant. It will be dropped in future calculations but its presence is implied. Note that the form of $\bar{\rho}^{(s)}$ differs from the spin density matrix of the incident beam. This means that the portion of the scattered beam in a specified direction in space is polarized. The degree of polarization of the beam incident on the second target is given by[4]

$$P=\frac{2|f(\theta_1)|^2|g(\theta_1)|^2-[f(\theta_1)g^*(\theta_1)+f^*(\theta_1)g(\theta_1)]}{[|f(\theta_1)|^2+|g(\theta_1)|^2]}. \tag{6}$$

If the f and g functions are real, the polarization vanishes as has been pointed out by Mott.

The beam incident on the second target is a partially polarized beam and the momentum density matrix describes a pure momentum state along the z' direction (Fig. 1). The second scattering is described by the scalar operator $V(\theta_2,\varphi_2)$ which is given by Eq. (2) in that representation where $\sigma_{z'}$ is diagonal. The spin density matrix $\bar{\rho}^{(s)}$ is transformed to this representation by rotating about the y axis through an angle θ_1 (Fig. 1).

Since the y direction remains invariant, the spin density matrix $\rho'^{(s)}$ in the representation where $\sigma_{z'}$ is diagonal is given by Eq. (5) with σ_y replaced by $\sigma_{y'}$. The density matrix after the second scattering is

$$\bar{\rho}'=V(\theta_2,\varphi_2)\rho' V^\dagger(\theta_2,\varphi_2), \tag{7}$$

and the double-scattering cross section is obtained by taking the trace of $\bar{\rho}'$ over the spin states. Thus,

$$d\sigma\sim\{[|f_1|^2+|g_1|^2][|f_2|^2+|g_2|^2] -[f_1g_1^*-f_1^*g_1][f_2g_2^*-f_2^*g_2]\cos\varphi_2\}, \tag{8}$$

where f_1 means $f(\theta_1)$ and f_2 means $f(\theta_2)$, etc. This can be written as

$$d\sigma\sim 1-\delta\cos\varphi_2, \tag{9}$$

where

$$\delta=\frac{[f_1g_1^*-f_1^*g_1][f_2g_2^*-f_2^*g_2]}{[|f_1|^2+|g_1|^2][|f_2|^2+|g_2|^2]}. \tag{10}$$

Equation (9) gives the Mott double-scattering formula in free space.

If the transition operator V were independent of the spin, the first scattered beam would have remained unpolarized. The cross section of the double scattering would have been given by the first term of (8), and would not have exhibited any asymmetry in the azimuthal direction. Thus, the asymmetry is seen to arise from the spin dependence of the scattering potential. Therefore, in principle, one could determine the spin magnetic moment by measuring the asymmetry. However, it is difficult to measure an absolute cross section in the double-scattering experiment, and so it would not be feasible to employ this method to measure the gyromagnetic ratio of the free electron. The fact that there is an asymmetry in the cross section demonstrates the existence of the spin magnetic moment of the free electron. In order to make a quantitative measurement of the gyromagnetic ratio of an electron not bound to an atom, one can allow the spin magnetic moment to interact with a magnetic field between the two scattering events. The magnetic interaction causes the asymmetry to be altered and this can be utilized to determine the gyromagnetic ratio.

III. EXTENSION OF THE MOTT THEORY

The Mott theory is adequate to describe the double scattering experiment in free space. However, it has to

[4] The degree of polarization is defined as $P=|a|^2$, where the spin density matrix is defined in terms of the Pauli spin vector as

$$\rho^{(s)}=\tfrac{1}{2}[1+\mathbf{a}\cdot\boldsymbol{\sigma}].$$

This is the same as Fano's definition in reference 3. Fano's definition differs from that of Tolhoek and deGroot, but both are related to the same invariant properties of the density matrix.

be modified where there is a magnetic field present between the two scattering centers.

It is assumed that within a region of about 10^5 wavelengths from the scatterer, about 10^{-5} cm for an electron with energy as low as one hundred kilovolts, the effect of the magnetic field upon the particle is negligible compared to the scattering potential. This is justified, because for magnetic field of about one hundred gauss, the spin precession frequency is given by

$$\omega_s=(e\mathcal{H}/m_0c)(1-\beta^2)^{\frac{1}{2}}g/2, \qquad (11)$$

where g=gyromagnetic ratio (about 2), e=electronic charge, $\mathcal{H}$=magnetic field strength, m_0=rest mass of electron, c=velocity of light, $\beta=v/c$ (the ratio of electron's velocity to the velocity of light), and the spin precession will be about the order of 10^{-6} radian. The change in orbit direction will be of the same order of magnitude. For electrons of higher energy this approximation will be even better. Within this region the particle can be considered as traveling in free space. The scattering is then completely determined by the scattering potential. The same assumption is made for the second scattering. Thus the only effect of the magnetic field will be to change the wave function between the two scattering events.

The equations of motion of the wave function is discussed in the Foldy-Wouthuysen representation.[5] Although the Mott theory is based on the Dirac representation,[1] Mott actually employs a Pauli type approximation[6] as has been noted previously. It has been demonstrated that in the low-energy limit the positive energy eigenfunctions of this new representation readily go over to the Pauli limit.

If we consider only positive-energy states, the Hamiltonian is shown in the appendix to be, up to terms of first order in $\mu_0\mathcal{H}$ (μ_0 is the Bohr magneton),

$$H=\left\{\boldsymbol{\mathcal{E}}_\pi-\mu_0\boldsymbol{\sigma}\cdot\boldsymbol{\mathcal{H}}[(1-\beta^2)^{\frac{1}{2}}+a]+a\mu_0\frac{(\boldsymbol{\sigma}\cdot\mathbf{v})(\mathbf{v}\cdot\boldsymbol{\mathcal{H}})}{v^2}[1-(1-\beta^2)^{\frac{1}{2}}]\right\}. \qquad (12)$$

Furthermore $\boldsymbol{\mathcal{E}}_\pi$ is the energy operator in the absence of spin and is

$$\boldsymbol{\mathcal{E}}_\pi=[m_0^2c^4+c^2\pi^2]^{\frac{1}{2}}, \qquad (13)$$

where $\boldsymbol{\pi}=\mathbf{p}-e\mathbf{A}/c=m\mathbf{v}$, $\mathbf{p}$=momentum, $\mathbf{A}$=vector potential, and $a=\alpha/2\pi$ is the second order radiative correction to the spin magnetic moment.[7] This can be shown to be independent of the momentum[8] for a particle in a constant magnetic field.

The equations of motion of the coordinate operator $\mathbf{r}$ is given by

$$\dot{\mathbf{r}}=\frac{1}{i\hbar}[\mathbf{r},H]=c^2(\mathbf{p}-e\mathbf{A}/c)/\boldsymbol{\mathcal{E}}_\pi, \qquad (14)$$

which is just the equation of motion of a particle in a magnetic field rotating with the relativistic cyclotron frequency $\omega_L=e\mathcal{H}(1-\beta^2)^{\frac{1}{2}}/m_0c$. If we consider the terms in the Hamiltonian containing the spin operator (defined as $\boldsymbol{\sigma}$ acting in this representation), we find the equations of motion for the spin operator for the following interesting cases are given as:

$$(1)\ \boldsymbol{\mathcal{H}}\parallel\mathbf{v}:\quad d\boldsymbol{\sigma}/dt=[1+a]\omega_L\boldsymbol{\sigma}\times\mathbf{h}, \qquad (15a)$$

$$(2)\ \boldsymbol{\mathcal{H}}\perp\mathbf{v}:\quad d\boldsymbol{\sigma}/dt=[1+a/(1-\beta^2)^{\frac{1}{2}}]\omega_L\boldsymbol{\sigma}\times\mathbf{h}; \qquad (15b)$$

$\mathbf{h}$ is a unit vector in the direction of the field. Thus, we find that the spin can be considered as precessing about the direction of the field with a frequency different from the cyclotron frequency by an amount that depends upon the relative orientation of the field and the particle's velocity.

The effect of the Hamiltonian can be considered as changing the space state and the spin state separately. The Hamiltonian acting on the initial space part of the wave function gives a wave function which describes a particle which has been displaced along the direction of the field and rotated about the direction of the field by an amount ω_Lt, t being the time spent between the two scattering events. Similarly one can see that the action of the field on the spin state is to rotate the spin state about the direction of the field by an amount ω_st, where ω_s is given by Eq. (15).[9]

Thus, if the wave function after the first scattering is Ψ, then just prior to the second scattering it is

$$\Psi_{inc_2}=e^{i\gamma t}P(t)Q(t)\Psi. \qquad (16)$$

[5] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950); K. M. Case, Phys. Rev. **95**, 1323 (1954); H. Mendlowitz, thesis, University of Michigan, 1954 (unpublished), gives a detailed treatment in the Foldy-Wouthuysen representation of an electron with an anomalous moment in a constant magnetic field.

[6] This is because he allows only positive energy eigenfunctions of the Dirac equation to be incident upon the second target.

[7] J. Schwinger, Phys. Rev. **76**, 790 (1949); **82**, 664 (1951).

[8] Schwinger has shown in Phys. Rev. **76**, 790 (1949), Eq. (1.114) that the extra term appears as

$$(\alpha/2\pi)\mu_0c\frac{\partial}{\partial x_\nu}\int F_0(x-x')m_{\mu\nu}(x')d\omega'A_\mu(x).$$

Also

$$F_0(x)=\delta(x)+(1/6\kappa^2)\square^2\delta(x)+\cdots O(\square^4),$$

where

$$\kappa=\hbar/m_0c.$$

Following K. M. Case, Phys. Rev. **76**, 1 (1949), and S. Borowitz and W. Kohn, Phys. Rev. **76**, 818 (1949), we integrate by parts and apply the d'Alembertian operator on the vector potential. For a constant magnetic field, the terms in $\square^2$ and all higher order terms in this operator fall out.

[9] Note that ω_s depends on the relative directions of the motion of the particle and the magnetic field.

The operators $P(t)$ and $Q(t)$ are operators which rotate the space and spin states respectively, and are given by

$$P(t)=\exp\left[\frac{i}{\hbar}\omega_L \mathbf{L}\cdot\mathbf{h}t\right], \tag{17a}$$

$$Q(t)=\exp\left[\frac{i}{\hbar}\frac{\omega_s}{2}\boldsymbol{\sigma}\cdot\mathbf{h}t\right], \tag{17b}$$

where $\mathbf{L}$ is the angular momentum operator. The term $e^{i\gamma t}$ in (16) can be considered as a constant phase factor arising from the displacement of the particle along the direction of the field, and it will be dropped in future calculations.

After the first scattering, the spin density matrix in the representation where $\sigma_{z'}$ is diagonal is given by [from Eq. (15)]

$$\rho'^{(s)}=\tfrac{1}{2}\{[|f_1|^2+|g_1|^2]-i\sigma_{y'}[f_1g_1^*-f_1^*g_1]\}. \tag{18}$$

The momentum density matrix describes a beam of particles traveling along the z' direction, a pure momentum state. If there were no magnetic field, the density matrix ρ' would describe the beam incident on the second target. However, in the presence of the magnetic field, both the spin and space states of the beam are altered. The density matrix (in the primed coordinate system) describing the beam incident on the second target, after spending a time t in the field, is

$$\rho''=P(t)Q(t)\rho' Q^{-1}(t)P^{-1}(t). \tag{19}$$

The density matrix after the second scattering is then

$$\bar{\rho}''=VP(t)Q(t)\rho' Q^{-1}(t)P^{-1}(t)V^\dagger, \tag{20}$$

where V is defined by Eq. (2) in the representation where $\sigma_{z'}$ is diagonal.

Let

$$Q(t)=Q_P(t)Q_\epsilon(t), \tag{21}$$

where $Q_P(t)$ corresponds to the same angle of rotation as $P(t)$, and $Q_\epsilon(t)$ rotates the spin state through the angle corresponding to the difference between $Q(t)$ and $Q_P(t)$. Then

$$\bar{\rho}''=VP(t)Q_P(t)Q_\epsilon(t)\rho' Q_\epsilon^{-1}(t)Q_P^{-1}(t)P^{-1}(t)V^\dagger. \tag{22}$$

Since V is a scalar under simultaneous spin and space rotations, $\bar{\rho}''$ can be expressed as

$$\bar{\rho}''=P(t)Q_P(t)VQ_\epsilon(t)\rho' Q_\epsilon^{-1}(t)V^\dagger Q_P^{-1}(t)P^{-1}(t). \tag{23}$$

The scattering cross section for the double scattering is obtained by taking the trace of Eq. (23) over the spin states. This gives

$$d\sigma\sim P(t)\left\{\operatorname*{trace}_{\text{spin}}[VQ_\epsilon(t)\rho' Q_\epsilon^{-1}(t)V^\dagger]\right\}P^{-1}(t). \tag{24}$$

Consider $Q_\epsilon(t)$ to be given in the primed coordinate system by

$$Q_\epsilon(t)=\exp(i\boldsymbol{\sigma}\cdot\mathbf{n}\xi/2), \tag{25}$$

where $\mathbf{n}$ is a unit vector in the direction specified by

$$n_x=\sin\theta\cos\beta,\quad n_y=\sin\theta\sin\beta,\quad n_z=\cos\theta,$$

where θ is the polar angle β is the azimuthal angle in the primed coordinate system. The cases of interest are when $\mathbf{n}$ is along the direction of the field. Now

$$\begin{aligned}\operatorname*{trace}_{\text{spin}}\{VQ_\epsilon(t)\rho' Q_\epsilon^{-1}(t)V^\dagger\}\\ =\{[|f_1|^2+|g_1|^2][|f_2|^2+|g_2|^2]\}\\ -\{[f_1g_1^*-f_1^*g_1][f_2g_2^*-f_2^*g_2]\}\\ \times\{\cos\varphi_2[\cos^2(\xi/2)\cos\theta\sin^2(\xi/2)]\\ -\sin^2\theta\sin^2(\xi/2)\cos(2\beta-\varphi_2)\\ -\sin\varphi_2\cos\theta\sin\xi\}.\end{aligned} \tag{26}$$

Therefore, the cross section given by (24) can be obtained from Eq. (26) by rotating the primed coordinate system about the direction of the field in the direction of rotation of the particle and through the same angle.

Consider the case $Q_\epsilon(t)=1$. This is true when $\xi=0$, which means the anomalous moment vanishes. Then (26) reduces to

$$\begin{aligned}\{[|f_1|^2+|g_1|^2][|f_2|^2+|g_2|^2]\\ -[f_1g_1^*-f_1^*g_1][f_2g_2^*-f_2^*g_2]\cos\varphi_2\}.\end{aligned} \tag{27}$$

The double-scattering cross section of (24) into a given direction is the same as the field-free cross section into another direction. The new direction is obtained by rotating the direction specified by Eq. (27) in the primed coordinate system about the field by an angle corresponding to $P(t)$. If the space state was rotated through angle $2n\pi$ (n being an integer), the cross section would correspond to the field free cross section.

If the magnetic field is oriented along the y' direction, the direction perpendicular to the initial and first scattered beams, then

$$Q_\epsilon(t)=\exp(i\sigma_{y'}\epsilon/2),$$

and

$$\theta=\pi/2,\quad \beta=\pi/2,\quad \xi=\epsilon.$$

This is substituted into Eq. (26) and gives Eq. (27). Thus, this case does not differ from the case where there is no anomalous moment, and the effects of the anomalous moment cannot be detected.

If the magnetic field is along the z' direction, the direction of the first scattered beam,

$$Q_\epsilon(t)=\exp(i\sigma_{z'}\epsilon/2),$$

and

$$\theta=0.$$

Then Eq. (26) gives

$$\{[|f_1|^2+|g_1|^2][|f_2|^2+|g_2|^2] \\ -[f_1g_1^*-f_1^*g_1][f_2g_2^*-f_2^*g_2]\cos(\varphi_2+\epsilon)\}. \quad (28)$$

The scattering cross section for the double-scattering experiment is obtained from (28) by rotating the direction about the z' axis. If, before the rotation, a direction is specified by θ and φ in the primed system, after it has been rotated through an angle $\omega_L t$, the direction is specified by θ and $\varphi+\omega_L t$ in the primed system. Therefore, the cross section (24) is, in the primed system,

$$d\sigma\sim\{[|f_1|^2+|g_1|^2][|f_2|^2+|g_2|^2] \\ -[f_1g_1^*-f_1^*g_1][f_2g_2^*-f_2^*g_2] \\ \times\cos(\varphi_2+\omega_L t+\epsilon)\}, \quad (29)$$

or

$$d\sigma\sim1-\delta\cos(\varphi_2+\omega_L t+\epsilon). \quad (30)$$

For these experimental conditions, if ϵ is not too small, one can determine both the magnitude and sign of the anomalous moment. If $\omega_L t=2n\pi$ (n being an integer), the effect of the anomalous moment on the asymmetry could be detected. In the experiment of Louisell, Crane, and Pidd,[10] the time of flight between the two scatterers was too small for ϵ to be significantly different from zero, and so they effectively measured the rotation of the asymmetry about the direction of the field which was caused by the normal moment. This corresponds to the case where $Q_\epsilon=1$ discussed above.

If the magnetic field is along the x' direction, then

$$Q_\epsilon(t)=\exp(i\sigma_{x'}\epsilon/2),$$

and

$$\theta=\pi/2,\quad \beta=0,\quad \xi=\epsilon.$$

Equation (26) gives

$$[|f_1|^2+|g_1|^2][|f_2|^2+|g_2|^2] \\ -[f_1g_1^*-f_1^*g_1][f_2g_2^*-f_2^*g_2]\cos\varphi_2\cos\epsilon. \quad (31)$$

If $P(t)$ corresponds to a rotation of $2n\pi$ radians (n being an integer), then the double scattering cross section, for the case where the magnetic field is in the same plane as the initial and first scattered beams but perpendicular to the scattered beam, is given by

$$d\sigma\sim1-\delta\cos\varphi_2\cos\epsilon. \quad (32)$$

It is seen that the effects of the anomalous moment on the cross section are detectable because of the alteration of the asymmetry. However, in this case the sign of the anomalous moment cannot be determined because the change is independent of the sign of ϵ. Also, the direction of the field along the x' axis is not important. In the limit of small ϵ,

$$d\sigma\sim1-\delta\cos\varphi_2. \quad (32a)$$

It is therefore necessary that the time between the two scatterings be long enough that ϵ be at least a significant fraction of π.

[10] Louisell, Pidd, and Crane, Phys. Rev. 94, 7 (1954).

In the experiment to measure the magnitude of the gyromagnetic ratio to such accuracy that the second order radiative corrections are significant, it will be necessary for the number of spatial revolutions about the field to be of the order of 10^3 to 10^4.

IV. CONCLUSION

It is shown, that up to first order of $\mu_0\mathcal{H}$ in the energy, the precession frequency of the spin about the direction of the magnetic field is given by:

(1) Field parallel to velocity of the particle,

$$\omega_s=\omega_L[1+a]. \quad (33)$$

(2) Field perpendicular to the direction of the velocity of the particle,

$$\omega_s=\omega_L[1+a/(1-\beta^2)^{\frac{1}{2}}]. \quad (34)$$

The ratio of the spin precession frequency ω_s to the cyclotron frequency ω_L is for the two cases:

$$(1)\quad \omega_s/\omega_L=g/2=[1+a], \quad (35)$$

$$(2)\quad \omega_s/\omega_L=g/2=[1+a/(1-\beta^2)^{\frac{1}{2}}], \quad (36)$$

where g is the gyromagnetic ratio for the spin moment. Although the magnitude of the second order correction to the moment is constant in a time-independent homogeneous magnetic field, the gyromagnetic ratio depends upon the relative orientation of the field to the velocity of the particle and also on the magnitude of the velocity.

The Mott double-scattering cross section is modified by the intervention of a magnetic field between the two scattering centers. For the case where the magnetic field is along the direction of the first scattered beam (z' direction), the cross section is given by

$$d\sigma_1\sim1-\delta\cos[\varphi_2+\omega_L t(1+a)]. \quad (37)$$

When the magnetic field is perpendicular to the direction of the first scattered beam but parallel to the plane of the initial and first scattered beams, the expression for the cross section is more complicated. If the particle executes an integral number of revolutions about the direction of the field, the cross section is given by

$$d\sigma_2\sim1-\delta\cos\varphi_2\cos[a\omega_L t/(1-\beta^2)^{\frac{1}{2}}]. \quad (38)$$

If the particles do not make an integral number of revolutions, the double scattering cross section can be obtained from Eq. (38) by rotating the primed coordinate system about the field through the same angle that the electrons have been rotated. The cross section is given by Eq. (38) in the new coordinate system.

The effects of depolarization of the electron beam on the double-scattering cross section will be discussed in a later paper.

APPENDIX

Consider the Dirac equation for a particle in a constant homogeneous magnetic field,

$$H_D=c\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\pi}+\rho_3m_0c^2 \\ =\rho_1O_1+\rho_3O_3, \quad (A\text{-}1)$$

where O_1 and O_3 are the operators which are the coefficients of the ρ_1 and ρ_3, respectively. This Hamiltonian is then transformed to another representation by the following transformation

$$H=e^{-i\rho_2\varphi/2}H_D e^{i\rho_2\varphi/2}, \quad \text{(A-2)}$$

where φ is independent of the ρ operators and assumed to be independent of time. Then

$$H=\rho_3[O_3\cos\varphi-O_1\sin\varphi] +\rho_1[O_1\cos\varphi+O_3\sin\varphi]. \quad \text{(A-3)}$$

If φ is chosen such that

$$\tan\varphi=-O_1/O_3, \quad \text{(A-4)}$$

then

$$H=\rho_3[O_1^2+O_3^2]^{\frac{1}{2}}. \quad \text{(A-5)}$$

This new Hamiltonian is the Hamiltonian of the particle in the Foldy-Wouthuysen representation, and for the above case it is

$$H=\rho_3[m_0^2c^4+c^2\pi^2-e\hbar c\boldsymbol{\sigma}\cdot\mathcal{H}]^{\frac{1}{2}}=\rho_3\mathbf{E}_\pi. \quad \text{(A-6)}$$

If ρ_3 is taken diagonal with the eigenvalues of ±1, the energy eigenvalues are just $\pm$ the square bracket above.

In this representation the positive eigenvalue of ρ_3 refers to the positive-energy states and the negative eigenvalues to the negative-energy states.

In the usual Dirac representation, a particle with an anomalous moment of magnitude $a\mu_0$ is described by a Hamiltonian given by

$$H_D'=c\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\pi}+\rho_3(m_0c^2-a\mu_0\boldsymbol{\sigma}\cdot\mathcal{H}). \quad \text{(A-7)}$$

This is transformed to the Foldy-Wouthuysen representation and gives

$$H'=e^{-i\rho_2\varphi/2}H_D'e^{i\rho_2\varphi/2}=A-a\mu_0B, \quad \text{(A-8)}$$

where A is given by Eq. (A-6) and B is

$$B=e^{-i\rho_2\varphi/2}\boldsymbol{\sigma}\cdot\mathcal{H}e^{i\rho_2\varphi/2}. \quad \text{(A-9)}$$

Therefore

$$B=\rho_3\{\cos(\varphi/2)\boldsymbol{\sigma}\cdot\mathcal{H}\cos(\varphi/2)-\sin(\varphi/2)\boldsymbol{\sigma}\cdot\mathcal{H}\} +\rho_1\{\sin(\varphi/2)\boldsymbol{\sigma}\cdot\mathcal{H}\cos(\varphi/2) +\cos(\varphi/2)\boldsymbol{\sigma}\cdot\mathcal{H}\sin(\varphi/2)\}. \quad \text{(A-10)}$$

Now φ is given by Eq. (A-4), so that $\cos(\varphi/2)$ is a function of

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})]^n\sim[\pi^2-(e\hbar/c)\boldsymbol{\sigma}\cdot\mathcal{H}]^n.$$

Since $\mathcal{H}$ is a constant homogeneous field, $\cos(\varphi/2)$ commutes with $\boldsymbol{\sigma}\cdot\mathcal{H}$. However, $\sin(\varphi/2)$ is a function of

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})]^n\sim(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})[\pi^2-(e\hbar/c)\boldsymbol{\sigma}\cdot\mathcal{H}]^n$$

so that

$$\sin(\varphi/2)(\boldsymbol{\sigma}\cdot\mathcal{H})=[(1-\cos\varphi)/2]^{\frac{1}{2}}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathcal{H})/|\boldsymbol{\pi}|. \quad \text{(A-11)}$$

Therefore

$$B=\rho_3\left[\boldsymbol{\sigma}\cdot\mathcal{H}\cos(\varphi/2)-\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathcal{H})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})}{\pi^2}\sin^2(\varphi/2)\right] +\rho_1\left[\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathcal{H})+(\boldsymbol{\sigma}\cdot\mathcal{H})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})}{|\boldsymbol{\pi}|}\sin(\varphi/2)\cos(\varphi/2)\right],$$

where the trigonometric functions in Eq. (A-11) are now independent of $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}$. Now

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathcal{H})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})=2(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathcal{H}) -(e\hbar/c)\mathcal{H}^2-\pi^2(\boldsymbol{\sigma}\cdot\mathcal{H}),$$

and

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\mathcal{H})+(\boldsymbol{\sigma}\cdot\mathcal{H})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})=2\boldsymbol{\pi}\cdot\mathcal{H}.$$

Therefore,

$$B=\rho_3\{\boldsymbol{\sigma}\cdot\mathcal{H}-(1-m_0c^2/\mathbf{E}_\pi) \times[(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathcal{H})/\pi^2-(e\hbar c/2)\mathcal{H}^2/c^2\pi^2\} -\rho_1(\boldsymbol{\pi}\cdot\mathcal{H}/|\boldsymbol{\pi}|)(1-m_0^2c^4/E_\pi^2). \quad \text{(A-12)}$$

Since we are restricting ourselves only to positive-energy solutions of the wave equation, the terms in ρ_1 are dropped, and for ρ_3 we substitute its positive eigenvalue. The Hamiltonian is then given by

$$H=\mathcal{E}_\pi[1-e\hbar c\boldsymbol{\sigma}\cdot\mathcal{H}/\mathcal{E}_\pi^2]^{\frac{1}{2}}-a\mu_0\{\boldsymbol{\sigma}\cdot\mathcal{H}-(1-m_0c^2/\mathbf{E}_\pi) \times[\boldsymbol{\sigma}\cdot\boldsymbol{\pi}(\boldsymbol{\pi}\cdot\mathcal{H})/\pi^2-(e\hbar c/2)(\mathcal{H}^2/c^2\pi^2)]\}. \quad \text{(A-13)}$$

The first term in (A-13) is expanded to give

$$\mathbf{E}_\pi=\mathcal{E}_\pi-\frac{e\hbar}{2\mathcal{E}_\pi/c}\boldsymbol{\sigma}\cdot\mathcal{H}-\frac{1}{2\mathcal{E}_\pi}\left(\frac{e\hbar\mathcal{H}}{2\mathcal{E}_\pi/c}\right)^2-\cdots.$$

Since the minimum eigenvalue of $\mathcal{E}_\pi$ is m_0c^2, then up to terms which are linear in $\mu_0|\mathcal{H}|$, we can write

$$\mathbf{E}_\pi=\mathcal{E}_\pi-\left(\frac{e\hbar}{2\mathcal{E}_\pi/c}\right)\boldsymbol{\sigma}\cdot\mathcal{H}.$$

The correction terms are smaller by a factor of 10^{-11} for a field of about one hundred gauss. Therefore up to terms which are linear in $\mu_0|\mathcal{H}|$, the Hamiltonian in Eq. (A-13) becomes

$$H=\mathcal{E}_\pi-\frac{e\hbar}{2\mathcal{E}_\pi/c}\boldsymbol{\sigma}\cdot\mathcal{H}-a\mu_0\boldsymbol{\sigma}\cdot\mathcal{H} +a\mu_0(1-m_0c^2/\mathcal{E}_\pi)(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\mathcal{H})/\pi^2. \quad \text{(A-14)}$$

If we consider wave functions which are product functions of eigenfunctions of $\mathcal{E}_\pi$ and a function of σ, then we can replace $\mathcal{E}_\pi$ in the denominator by its eigenvalue $m_0c^2/(1-\beta^2)^{\frac{1}{2}}$, which gives

$$H=\mathcal{E}_\pi-u_0\boldsymbol{\sigma}\cdot\mathcal{H}[(1-\beta^2)^{\frac{1}{2}}+a] +a\mu_0\frac{(\boldsymbol{\sigma}\cdot\mathbf{v})(\mathbf{v}\cdot\mathcal{H})}{v^2}[1-(1-\beta^2)^{\frac{1}{2}}], \quad \text{(A-15)}$$

because

$$m\mathbf{v}=\boldsymbol{\pi}.$$

Thus this now corresponds to Eq. (12) in the text.