Collision Matrices for the Compound Nucleus*

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The collision matrix found by Newton to satisfy at all excitation energies the requirement that it describe an excited nucleus whose decay is independent of the mode of formation is shown to imply the vanishing of the absorption cross sections at high energies where the levels overlap and therefore does not describe a compound nucleus in the usual sense. The essential characteristic of this matrix is the high degree of the correlations of the signs of the square roots $\gamma_{\lambda c}$ of the reduced level widths for the various levels λ and channels c. On the other hand, a collision matrix, which is similar to one first considered by Bethe and involves $\gamma_{\lambda c}$ whose signs are uncorrelated, implies energy average decays that are independent of the formation mode and absorption cross sections that are of the order of magnitude of the nuclear area at high energies. These matrices are derived and discussed by using the R-matrix theory of Wigner, Eisenbud, and Teichmann. It is shown that the Bethe form of the collision matrix, which is valid only if all of the partial level widths are much less than the spacings, may be modified by means of the Teichmann-Wigner channel elimination procedure so that it is also valid in situations where some of the partial widths exceed the spacings. The form of the compound-nucleus collision matrix thus obtained is similar to one deduced by Weisskopf by considerations involving the compound-nucleus hypothesis and the reciprocity theorem. The pole strength functions s_c, which are the averages of the ratios of reduced widths $\gamma_{\lambda c}^2$ to level spacings, and their Stieltjes energy transforms are decisive in the determination of the behavior of these collision matrices and their associated cross sections. The s functions and their transforms are presented and discussed in the cases of the strong-coupling and complex square well potential representations of the particle-nuclei interactions. The latter representation with an additional surface absorption is also considered. The present theory indicates that the imaginary part of the complex square well potential should increase with the absorption width, and it suggests a "giant resonance" interpretation of the average cross-section behaviors. The effect of the compound-nucleus on non-compound-nucleus processes such as stripping and pickup is also mentioned.

I. INTRODUCTION[1]

THE concept of the compound nucleus has been useful for the understanding of many nuclear reaction phenomena ever since it was first proposed by Bohr.[2] Subsequent to this proposal, derivations were given of the form of the collision matrix U describing the compound nucleus under various conditions. In particular, Wigner and Eisenbud[3] derived by means of their R-matrix theory a generalized one-level collision matrix which revealed that if the average total level width $\langle\Gamma_\lambda\rangle_{Av}$[4] is small enough compared with the mean level spacing D, then at all excitation energies in the vicinity of the levels the nuclear system described by U will decay in accordance with the well known one-level Breit-Wigner resonance formula as a compound nucleus independently of its mode of formation.

Recently there has appeared an interesting deduction by Newton[5] of a form of U which satisfies the requirement of independent decay at *all* excitation energies and for any value of the ratio $\langle\Gamma_\lambda\rangle_{Av}/D$. The essentials of his deduction involved consideration of only the unitary and symmetry properties of U. It was shown that this U implied that the square-roots of the reduced level widths of the associated R matrix are of the highly correlated form $\gamma_{\lambda c}=a_\lambda b_c$, the factor a_λ referring to the levels λ and the factor b_c to the channels c. However, according to the present considerations, the absorption cross sections implied by Newton's U tend to vanish at sufficiently high nuclear excitations where there are many decay channels and the levels overlap, contrary to experience and to the compound-nucleus concept.

One aim of the investigation reported here was therefore to obtain by means of the R-matrix theory a form of the collision matrix which would satisfy the compound-nucleus requirement of independent decay at least *on the average*, rather than at *all* excitation energies, and for any value of $\langle\Gamma_\lambda\rangle_{Av}/D$, thus extending the Wigner-Eisenbud result. This was accomplished by assuming that the signs of the $\gamma_{\lambda c}$ appearing in Wigner's[6] many-level expansion for U are uncorrelated, quite contrary to the conditions involved in Newton's U. The collision matrix and its associated average cross sections thereby obtained are similar to those deduced by Bethe[7] many years ago. In Bethe's derivation it was assumed that the signs of the matrix elements for the formation and decay of the various intermediate levels are uncorrelated. When the levels overlap, both results lead to average absorption cross sections of the order of the nuclear area, in conformity with the compound-nucleus concept, but they are restricted to situations where *all* of the average partial level widths $\langle\Gamma_{\lambda c}\rangle_{Av}$ for the channels c are less than D.

The recent extensive fast neutron total cross-section measurements by Barschall, Nereson and others,[8] and

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[1] A preliminary account of some aspects of this paper was given at the September 1953 meeting of the American Physical Society [R. G. Thomas, Phys. Rev. **92**, 1094(A) (1953)].

[2] N. Bohr, Nature **137**, 344 (1936); Science **86**, 161 (1937).

[3] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).

[4] The symbol $\langle\ \rangle_{Av}$ is often used to denote averages with respect to the levels λ, the number of contributing λ being roughly $\langle\Gamma_\lambda\rangle_{Av}/D$ when this ratio exceeds unity, where D is the mean level spacing.

[5] T. D. Newton, Can. J. Phys. **30**, 53 (1952).

[6] E. P. Wigner, Phys. Rev. **70**, 606 (1946).

[7] H. A. Bethe, Revs. Modern Phys. **9**, 69 (1937), Sec. 56D.

[8] H. H. Barschall, Phys. Rev. **86**, 431 (1952); Miller, Adair, Bockelman, and Darden, Phys. Rev. **88**, 83 (1952); Norris Nereson and Sperry Darden, Phys. Rev. **89**, 775 (1953); **94**, 1678 (1954).

their interpretation by Feshbach, Porter, and Weisskopf[9] have revealed that a complex square well potential provides a satisfactory representation of the neutron-nucleus interaction. One implication of this model is that at certain excitation energies the mean partial width $\langle\Gamma_{\lambda n}\rangle_{Av}$ for the incident neutron channel n considerably exceeds D, thus violating the criterion for applicability of the Bethe collision matrix. The second aim was therefore to find a U which would also be valid in such circumstances. This was effected by means of the channel elimination procedure and reduced R-matrix theory developed by Teichmann and Wigner.[10] In the final form the compound-nucleus collision matrix leads to expressions for the nuclear reaction cross sections which are of the form predicted by Weisskopf[11] from considerations of the compound-nucleus assumption and the reciprocity theorem; these expressions are restricted to circumstances where the mean of the $\langle\Gamma_{\lambda c}\rangle_{Av}$ for the participating channels c is less than D. One object of the present work may thus be stated as the determination of the conditions for the existence of the compound nucleus which has been assumed in the previous work.

There are observed to be certain nuclear reaction phenomena which are not interpretable as proceeding through the intermediary of a compound nucleus. Noteworthy are the high-energy direct and exchange collisions, the stripping and pickup reactions which are prominent at intermediate energies as well. The signs of the $\gamma_{\lambda c}$ associated with these processes no doubt have significant correlations, although this matter has not yet been quantitatively investigated. It is hoped that the material presented herein will provide not only a better understanding of the compound-nucleus processes but also permit the inclusion of their contribution to and effect on the non-compound-nucleus processes in the theoretical interpretation of the latter.

II. NOTATION

Although the present treatment utilizes the various R-matrix relations given by Teichmann and Wigner,[10] it is desirable to introduce the modifications of their notations that are indicated in Table I. The use of a reduced width having the dimensions of energy, rather than energy-times-distance, is natural and leads to dimensionless quantities in standard notation for the quantities characterizing the external wave functions at the nuclear surface. The present notation will also facilitate comparisons with the results of Feshbach and Weisskopf.[12,13] The R matrix will be dimensionless, and its diagonal component referring to the entrance channel will be related to the reciprocal of the important quantity f which appears in their work.

[9] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).
[10] T. Teichmann and E. P. Wigner, Phys. Rev. **87**, 123 (1952), referred to as TW.
[11] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Chap. VIII.
[12] H. Feshbach and V. F. Weisskopf, Phys. Rev. **76**, 1550 (1949).
[13] Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 1451 (1947).

TABLE I. Relations between the present notation and that of TW.

Present notation	Teichmann and Wigner
$\gamma_{\lambda c}$	$\gamma_{\lambda c}a_c^{-\frac{1}{2}}$
R_{st}	$a_s^{-\frac{1}{2}}R_{st}a_t^{-\frac{1}{2}}$
$\mathcal{U}_c$	$\alpha_c a_c^{-\frac{1}{2}}$
$\mathcal{D}_c=\rho_c\mathcal{U}_c'$ $=\mathfrak{D}_c+B_c\mathcal{U}_c$	$\alpha_c a_c^{-\frac{1}{2}}+\beta_c a_c^{\frac{1}{2}}$
$\mathcal{D}_c a_c^{-\frac{1}{2}}-\mathcal{U}_c a_c^{-\frac{1}{2}}$	β_c
B_c	$-a_c\bar{b}_c=1-a_c b_c$
$F_c,\quad G_c$	$k_c^{\frac{1}{2}}F_c,\quad k_c^{\frac{1}{2}}G_c$
$'\equiv d/d\rho_c$	$'\equiv d/dr_c$

The regular and irregular external radial functions for channels c are designated $F_c(\rho)$ and $G_c(\rho)$, respectively, as is customary when discussing charged particle reactions.[14] These quantities and their derivatives with respect to $\rho_c=k_c r_c$ appear in the theoretical development evaluated at the channel radii where $r_c=a_c$. However, these four quantities are related through the Wronskian,

$$F_c'G_c-G_c'F_c=1,$$

so that it is actually necessary to introduce only three independent quantities. For these we use the real and imaginary parts of a complex quantity L which is the radius a times the logarithmic derivative at $r=a$ of an outgoing wave $O=G+iF$ (omitting the subscript c):

$$L=\rho O'/O=S+iP,$$
$$S=\rho(FF'+GG')/(F^2+G^2),$$
$$P=\rho/(F^2+G^2),$$

and

$$\phi=\tan^{-1}(F/G),$$

which is the negative of the *hard-sphere potential scattering phase shift*. The real part S of L appears as a factor in the level shift expressions and is therefore referred to as the *shift factor* while the imaginary part P appears as a factor in the level width expressions and is referred to as the *penetration factor*. If the energy of relative motion in a channel is negative, the analytical continuation of the outgoing wave O is the real, exponentially-decaying Whittaker function $W_{-\eta,\,l+\frac{1}{2}}(2\rho)$, where $\eta=Z_1Z_2e^2/\hbar v$ and hl is the channel orbital angular momentum, so that[15]

$$S=\rho W'/W,\quad P=0.$$

The development actually involves the barred quantities

$$\bar{L}=L-B=\bar{S}+i\bar{P},$$
$$\bar{S}=S-B,\quad \bar{P}=P,$$

where the B are the real, constant boundary conditions satisfied by the proper solutions X_λ of energy E_λ at the radii a_c of the channels c. The bar will always be

[14] Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. **23**, 147 (1951).
[15] R. G. Thomas, Phys. Rev. **88**, 1109 (1952).

omitted. The various quantities L_c, S_c, P_c, B_c, etc. appear as the components of the diagonal matrices L, S, P, B, etc., these matrices being referred to respectively as the logarithmic derivative, shift, penetration, boundary-condition matrices.

III. THE GENERAL EXPRESSIONS FOR THE COLLISION MATRIX

The general expressions for the collision matrix in terms of the R matrix and its parameters $\gamma_{\lambda c}$, E_λ as derived by Wigner, Eisenbud, Teichmann, and Newton[3,5,10] are summarized here in the present notation. The collision matrix is conveniently written

$$U=\Omega V\Omega, \tag{1}$$

the factors being the unitary, diagonal, potential-scattering matrix,[16]

$$\Omega=\exp(-i\phi), \tag{1a}$$

and the unitary, symmetric matrix

$$V=1+2iP^{\frac{1}{2}}(1-RL)^{-1}RP^{\frac{1}{2}}, \tag{1b}$$

which will also be referred to as a collision matrix. L and P are the diagonal, logarithmic derivative and barrier penetration matrices, the components of which are given in II. It is often useful to separate the real, symmetrical R matrix into two terms

$$R=R^0+R', \tag{2}$$

with

$$R'=\sum_\lambda(\gamma_\lambda\times\gamma_\lambda)/(E_\lambda-E),$$

and the R^0 matrix as the sum of the contributions from the remaining levels not included in the λ sum, or if R' is a uniform R matrix, similar to Wigner's uniform R function,[17] as the difference $(R-R')$. With this separation,

$$(1-RL)^{-1}R=(1-R^0L)^{-1}R^0 +(1-R^0L)^{-1}(1-R'L')^{-1}R'(1-LR^0)^{-1}, \tag{3}$$

where

$$L'=L(1-R^0L)^{-1}.$$

By means of the procedure indicated in Wigner's paper on resonance reactions[6] or in Appendix A of this paper, it may be deduced that

$$(1-R'L')^{-1}R'=\sum_{\lambda\mu}A_{\lambda\mu}(\gamma_\lambda\times\gamma_\mu), \tag{4}$$

where the components of the symmetrical level matrix A are obtained from the level matrix relation,

$$A=(\bar{E}-E-\xi)^{-1}, \tag{5}$$

and the subscripts λ, μ refer to the levels λ of R'. Here the components of the complex, symmetrical level matrix,

$$\xi=-\Delta+\tfrac{1}{2}i\Gamma, \tag{6}$$

are given by the channel scalar products,

$$\xi_{\lambda\mu}=(\gamma_\lambda,L'\gamma_\mu); \tag{6a}$$

the real part Δ of ξ is referred to as the shift matrix and twice the imaginary part $\frac{1}{2}\Gamma$ to the width matrix, their respective components being given by the scalar products

$$\tfrac{1}{2}\Gamma_{\lambda\mu}=(\alpha_\lambda{}^*,P\alpha_\mu),\quad \Delta_{\lambda\mu}=(\alpha_\lambda{}^*,\,(L^*R^0L-S)\alpha_\mu), \tag{6b}$$

where

$$\alpha_\lambda=(1-R^0L)^{-1}\gamma_\lambda.$$

The components of the real, diagonal matrix $\bar{E}$ are the proper values E_λ, and the real, diagonal matrix E is the energy E times the unit matrix. By substituting (4) into (3) one obtains

$$(1-RL)^{-1}R=(1-R^0L)^{-1}R^0+\sum_{\lambda\mu}A_{\lambda\mu}(\alpha_\lambda\times\alpha_\mu). \tag{7}$$

If only one level is considered in the λ sum, the substitution of (7) into (1) leads to the Wigner-Eisenbud generalized one-level collision matrix.[3]

If the matrix R^0 is diagonal, (1) may be rewritten

$$U=\Omega'V'\Omega', \tag{8}$$

where

$$\begin{aligned}
&\Omega'=\exp(-i\phi'),\\
&\phi'=\phi-\tan^{-1}[R^0P/(1-R^0S)],\\
&V'=1+2iP'^{\frac{1}{2}}(1-R'L')^{-1}R'P'^{\frac{1}{2}},\\
&L'=S'+iP',\\
&S'=[S(1-R^0S)-R^0P^2]/|1-R^0L|^2,\\
&P'=P/|1-R^0L|^2.
\end{aligned}$$

$-\phi'$ is a potential scattering phase which includes the contribution to the hard-sphere phase from R^0, and P' is a penetration factor similarly modified. The components of the width and shift matrices given by (6b) simplify to

$$\begin{aligned}
&\tfrac{1}{2}\Gamma_{\lambda\mu}=\tfrac{1}{2}\textstyle\sum_c\Gamma_{\lambda\mu c},\\
&\Gamma_{\lambda\mu c}=2P_c'\gamma_{\lambda c}\gamma_{\mu c};
\end{aligned} \tag{8a}$$

$$\begin{aligned}
&\Delta_{\lambda\mu}=\textstyle\sum_c\Delta_{\lambda\mu c},\\
&\Delta_{\lambda\mu c}=-S_c'\gamma_{\lambda c}\gamma_{\mu c}.
\end{aligned} \tag{8b}$$

The primes will henceforth be omitted.

If the symmetric matrix $(\bar{E}-\xi)$ has no double characteristic values, it may be put in the diagonal form $H=F-\frac{1}{2}i\Gamma$ by means of a complex, orthogonal matrix T,[6]

$$H=T(\bar{E}-\xi)T^{-1},$$

so that

$$A=\tilde{T}(H-E)^{-1}T,$$

and

$$(1-RL)^{-1}R=\sum_\nu(\delta_\nu\times\delta_\nu)/(F_\nu-E-\tfrac{1}{2}i\Gamma_\nu), \tag{9}$$

where

$$\delta_\nu=\textstyle\sum_\lambda T_{\nu\lambda}\alpha_\lambda.$$

The $\delta_{\nu c}$ and H_ν are in general complex and energy dependent in contrast with the parameters $\gamma_{\lambda c}$ and E_λ of

[16] The Coulomb phase for charged particle channels has not been included.

[17] E. P. Wigner, Proc. Cambridge Phil. Soc. 47, 790 (1951).

the R matrix which are real and constant. From the nature of the T matrix it may be deduced that[18]

$$\Gamma_\nu \leqslant 2\sum_c P_c|\delta_{\nu c}|^2. \tag{9a}$$

A procedure for the determination of the $\delta_{\nu c}$ and H_ν has been indicated by TW. The H_ν are the roots of the determinantal equation

$$|1-R(H_\nu)L|=0, \tag{10a}$$

and the $\delta_{\nu c}$ are the solutions of the associated homogeneous equations,

$$[1-R(H_\nu)L]\delta_\nu=0, \tag{10b}$$

with the normalization

$$(\delta_\nu, (\dot{L}+L\dot{R}L)\delta_\nu)=1, \tag{10c}$$

the dot denoting differentiation with respect to energy. This procedure leads to essentially the same results as obtained by Kapur and Peierls[19] using complex, energy-dependent boundary conditions $B_c=L_c$ (unbarred) rather than the real, energy-independent B_c of the R-matrix theory.

IV. NEWTON'S COLLISION MATRIX

Newton's collision matrix[5] satisfies for any value of $\langle\Gamma_\lambda\rangle_{Av}/D$ the requirement that the probabilities for the various modes of decay be independent of the formation mode *at all excitation energies*. A much simpler derivation of this matrix may be obtained by proceeding directly from the R-matrix expressions (4) and (1b). From these expressions it is evident that if the ratios (V_{st}/V_{ut}), s, $u\neq t$, are to be independent of t, the $\gamma_{\lambda c}$ must be factorizable as

$$\gamma_{\lambda c}=a_\lambda b_c, \tag{11}$$

so that the collision matrix becomes

$$V=1+2i(G\times G)(a,Aa), \tag{12}$$

where the vector

$$G=P^{\frac{1}{2}}b,$$

and the scalar product with respect to levels,

$$(a,Aa)=\sum_{\lambda\mu} a_\lambda A_{\lambda\mu}a_\mu,$$

being a function of energy but not of the formation mode. With this form the components of the ξ matrix simplify to

$$\xi_{\lambda\mu}=\zeta a_\lambda a_\mu,$$

where the channel scalar product

$$\zeta=-\Delta+\tfrac{1}{2}i\Gamma=(b,Lb),$$
$$\Delta=-\sum_c S_c b_c^2, \quad \Gamma=\sum_c \Gamma_c, \quad \Gamma_c=2G_c^2.$$

It may be deduced that

$$(a,Aa)=(t-\zeta)^{-1},$$

where

$$t^{-1}=\sum_\lambda a_\lambda^2/(E_\lambda-E).$$

Newton's collision matrix is therefore

$$V=1+2i(G\times G)/(t+\Delta-\tfrac{1}{2}i\Gamma), \tag{13}$$

which closely resembles the collision matrix for the one-level R matrix. As noted by Teichmann,[20] this resemblance is to be expected because with the $\gamma_{\lambda c}$ given by (11) the R matrix reduces to the expression $(b\times b)/t$ which is of rank one as is the one-level R matrix, $(\gamma_\lambda\times\gamma_\lambda)/(E_\lambda-E)$.

According to (13) the various reaction cross sections resulting from incident waves of angular momentum $\hbar l$ are

$$\sigma_{st}^{(l)}=(\pi/k_s^2)(2l+1)(4\Gamma_s\Gamma_t/\Gamma^2)\sin^2\tau, \tag{14}$$

where

$$\tau=\tan^{-1}[\tfrac{1}{2}\Gamma/(t+\Delta)].$$

It was also shown by Teichmann[20] that these cross sections may readily be averaged if $a_\lambda^2=a_\mu^2=\cdots 1$ and if the levels are uniformly spaced by an amount D, in which case

$$t=-(D/\pi)\tan(\pi E/D).$$

With this expression for the quantity t the averages are found to be

$$\langle\sigma_{st}^{(l)}\rangle_{Av}=(\pi/k_s^2)(2l+1)(4\Gamma_s\Gamma_t/\Gamma^2)\times\frac{(\pi\Gamma/2D)[1+(\pi\Gamma/2D)]}{(\pi\Delta/D)^2+[1+(\pi\Gamma/2D)]^2}. \tag{15}$$

By choosing the $B_c=S_c$ at a particular energy of interest, the level shift Δ will vanish at that energy and will be negligible in the vicinity of it. Therefore, in the region of overlapping levels where $\pi\Gamma/2D\gg 1$, (15) simplifies to

$$\langle\sigma_{st}^{(l)}\rangle_{Av}=(\pi/k_s^2)(2l+1)(4\Gamma_s\Gamma_t/\Gamma^2),$$

and the total absorption of the lth partial wave is

$$\langle\sigma_s^{(l)}\rangle_{Av}=\sum_t\langle\sigma_{st}^{(l)}\rangle_{Av}=(\pi/k_s^2)(2l+1)(4\Gamma_s/\Gamma).$$

If $\rho=ka\gg 1$, many partial waves contribute so that the usual approximate procedure[20] for summing these may be applied to obtain

$$\langle\sigma_s\rangle_{Av}\approx\sum_{l=0}^{\rho}\langle\sigma_s^{(l)}\rangle_{Av}\approx\pi a^2(1+\rho^{-1})^2\langle 4\Gamma_s/\Gamma\rangle_{Av}. \tag{16}$$

It is apparent that if there are many channels for absorption so that $\Gamma\gg 4\Gamma_s$, $\langle\sigma_s\rangle_{Av}$ becomes much less than the magnitude πa^2 suggested by the compound-nucleus concept, and it does not seem likely that this is a consequence of the special assumptions made in connection

[18] G. Breit, Phys. Rev. **69**, 472 (1946). In this paper Breit has also succeeded in deriving an expansion similar to (9) for a certain nuclear model having an arbitrary number of channels.

[19] P. L. Kapur and R. Peierls, Proc. Roy. Soc. (London) **A166**, 277 (1937).

[20] T. Teichmann, Phys. Rev. **77**, 506 (1950).

with (15). This tendency for the absorptions to vanish is evidently due to destructive interference of the various contributing levels caused by the extreme correlations of the signs of the $\gamma_{\lambda c}$, the arrangement of these signs being the same at each level to within a common factor ± 1. Such destructive interference is also evident in the results obtained by Kalckar, Oppenheimer, and Serber[21] as well as Bohr, Peierls, and Placzek.[22]

In view of this defect, we shall seek another form for the compound-nucleus collision matrix. However, Newton's result does show that the requirements of independent decay and of absorption cross sections of the order πa^2 cannot be satisfied at all excitations, and therefore that the appropriate form can be expected to satisfy these only on the average.

V. "BETHE'S COLLISION MATRIX"

Bethe has derived an expression for the averages of the reaction cross sections which is valid even if the levels overlap provided that all of the average partial level widths $\langle\Gamma_{\lambda c}\rangle_{Av}$ are smaller than the mean spacing D. His derivation is based on the assumption that the signs of the matrix elements for the formation and decay of the various intermediate nuclear states are uncorrelated, and the result thereby obtained shows that the excited nucleus will decay on the average with probabilities which are independent of the formation mode and that the average total absorption cross sections will be of the order of magnitude of the nuclear area, in conformity with the compound-nucleus hypothesis. Although the concept of a collision matrix had not been introduced at the time of his work, the form of this matrix can of course be inferred from his equations. The purpose of this section is to rederive Bethe's result using the more rigorous R-matrix theory and to show that in this theory the signs of the $\gamma_{\lambda c}$ for the states of the compound nucleus must be considered as uncorrelated.[23]

The set of equations (10) for the determination of the parameters of the collision matrix can be solved approximately in the case of uncorrelated signs and overlapping levels by the standard perturbation-theory procedures such as those reviewed by Morse and Feshbach.[24] However, for simplicity we shall use a less rigorous procedure which leads to essentially the same result.

With the recognition that in the case of interest the nondiagonal components of the ξ matrix given by (6) will be smaller in absolute magnitude than the diagonal components, one can attempt an expansion of the matrix expression for A given by (5) in a power series about the diagonal part $\epsilon=\bar{E}-E-\xi+\xi'$ whose components are $\epsilon_\lambda=E_\lambda+\Delta_\lambda-E-\frac{1}{2}i\Gamma_\lambda$, where ξ' is the nondiagonal part of ξ. The result is that

$$A=\epsilon^{-1}+\epsilon^{-1}\xi'\epsilon^{-1}+\cdots, \tag{17}$$

and therefore

$$(1-RL)^{-1}R=\sum_\lambda(\gamma_\lambda\times\gamma_\lambda)/\epsilon_\lambda +\sum_{\lambda\mu}\epsilon_\lambda^{-1}(\gamma_\lambda\times\gamma_\mu)\xi_{\lambda\mu}'\epsilon_\mu^{-1}+\cdots \tag{18}$$

the components of which are

$$[(1-RL)^{-1}R]_{st}=\sum_\lambda(\gamma_{\lambda s}\gamma_{\lambda t}/\epsilon_\lambda) +i\sum_c P_c\sum_\lambda(\gamma_{\lambda s}\gamma_{\lambda c}/\epsilon_\lambda)\times\sum_{\mu(\neq\lambda)}(\gamma_{\mu c}\gamma_{\mu t}/\epsilon_\mu)+\cdots. \tag{18a}$$

For simplicity we have set $B_c=L_c$(unbarred) so that the level shift matrix may temporarily be disregarded, and the expressions given by (8a) for the resulting components of ξ' have been used to arrive at (18a) from (18). We proceed now to determine the conditions for the validity of the approximation (19), below, for the collision matrix which is obtained by neglecting all but the first term of (18) or (18a). For this determination the sum over the channels c of (18a) is considered in two parts: (1) the contributions from the channels s and t; (2) the contributions from the remaining channels. If the $\mu=\lambda$ terms are added to the μ sum of (18a), which it is permissible to do if the total level widths are much larger than the spacings, the contribution from the channel t is observed to be $iP_t\sum_\mu\gamma_{\mu t}^2/\epsilon_\mu$ times the contribution from the first sum of (18a). By replacing this μ sum by an integration, it may readily be estimated as $\pi\langle\gamma_{\mu t}^2\rangle_{Av}/D$. The contribution from the channel s may be determined in a similar manner. The magnitude of the contribution from the s and t channels together is thus $\frac{1}{2}\pi\langle\Gamma_{\lambda s}\rangle_{Av}D^{-1}+\frac{1}{2}\pi\langle\Gamma_{\lambda t}\rangle_{Av}D^{-1}$ times that from the first sum of (18) or (18a). In the consideration of the contribution (2) from the remaining channels, it is evident that no two of the same $\gamma_{\lambda c}$ appear multiplied together. Since the signs of these are presumed to be uncorrelated, the most-probable value of this contribution is zero; its root-mean-square magnitude may be estimated as follows: The rms magnitude of the contribution to the components of ξ' from the remaining channels may be estimated as equal to $\frac{1}{2}\langle\Gamma_\lambda\rangle_{Av}$, the average magnitude of the diagonal components of ξ, divided by the square root of the number of open channels $c\neq s,t$, this number being approximately equal to $\langle\Gamma_\lambda/\Gamma_{\lambda c}\rangle_{Av}$, the average being with respect to both λ and c. The magnitude of the contribution from the μ sum of ϵ_μ, the individual terms having random signs, may be estimated as $(2\pi/\langle\Gamma_\lambda\rangle_{Av}D)^{\frac{1}{2}}$. The rms magnitude of the contribution to the second sum of (18a) from these channels is thus estimated as the channel average of $(\pi\langle\Gamma_{\lambda c}\rangle_{Av}/2D)^{\frac{1}{2}}$ times the first sum. A necessary condition for the validity of

[21] Kalckar, Oppenheimer, and Serber, Phys. Rev. **52**, 273 (1937).
[22] Bohr, Peierls, and Placzek, Nature **144**, 200 (1939).
[23] These signs should be random for those λ in an energy interval of width comparable to $\langle\Gamma_\lambda\rangle_{Av}$ centered at the energy E of interest. According to the TW sum rule, $\Sigma_\lambda\gamma_{\lambda s}\gamma_{\lambda t}=0$ if $s\neq t$ so that they cannot be completely uncorrelated with respect to arbitrarily large energy intervals.
[24] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, New York, 1953), Vol. II, Chap. 9.

the approximation

$$V=1+2iP^{\frac{1}{2}}\sum_{\lambda}\frac{(\gamma_\lambda\times\gamma_\lambda)}{E_\lambda-E-\frac{1}{2}i\Gamma_\lambda}P^{\frac{1}{2}} \tag{19}$$

is therefore that all of the ratios $\pi\langle\Gamma_{\lambda c}\rangle_{Av}/2D$ must be much less than unity, as imposed by the contribution (1). The third and higher terms of the expansion (18a) are proportional to the square and higher powers of these ratios. If the level shifts Δ_λ are included in the denominators of (19), it is apparent that the ratios $\pi\langle\Delta_{\lambda c}\rangle_{Av}/D$ must also be less than unity. Equation (19) is recognized as the many-level approximation to the collision matrix which is valid in general when the total level widths are much less than their spacings.[6] It is also noted that this result corresponds to the equality condition of the relation (9a).

The reaction cross sections obtained from (19) are

$$\sigma_{st}{}^{(l)}=\frac{4\pi}{k_s^2}(2l+1)\sum_{\lambda\mu}\frac{P_sP_t\gamma_{\lambda s}\gamma_{\mu s}\gamma_{\lambda t}\gamma_{\mu t}}{(E_\lambda-E-\frac{1}{2}i\Gamma_\lambda)(E_\mu-E+\frac{1}{2}i\Gamma_\mu)}. \tag{20}$$

Following the procedure indicated by Bethe, the averages of these with respect to an energy interval δ, which is large compared with the total widths, are found to be

$$\langle\sigma_{st}{}^{(l)}\rangle_{Av}=\frac{4\pi}{k_s^2}(2l+1)\frac{\pi}{\delta}\sum_{\lambda\mu(\text{in}\delta)}\frac{P_sP_t(\Gamma_\lambda+\Gamma_\mu)\gamma_{\lambda s}\gamma_{\mu s}\gamma_{\lambda t}\gamma_{\mu t}}{(E_\lambda-E_\mu)^2+\frac{1}{4}(\Gamma_\lambda+\Gamma_\mu)^2}, \tag{21}$$

the sums being now restricted to levels within the interval δ. For each level in the λ sum there will be about $\langle\Gamma_\lambda\rangle_{Av}/D$ levels contributing from the μ sum and thus about $\delta\langle\Gamma_\lambda\rangle_{Av}/D^2$ pairs altogether. In view of the random nature of the $\gamma_{\lambda c}$ signs, the contribution from the terms with $\lambda\neq\mu$ will be proportional to the square-root of this number of pairs in contrast with the number δ/D of contributions from the positive-definite terms with $\lambda=\mu$. The cross-product terms may therefore be neglected so that

$$\langle\sigma_{st}{}^{(l)}\rangle_{Av}=(\pi/k_s^2)(2l+1)(T_sT_t/\textstyle\sum_c T_c), \tag{22}$$

where the

$$T_c=2\pi\langle\Gamma_{\lambda c}\rangle_{Av}/D$$

are quantities similar to Weisskopf's transmission factors.[11] Evidently the decay of the compound nucleus is on the average independent of the formation mode. By summing over decay channels t, the total absorption cross section for the lth partial wave is obtained,[25]

$$\langle\sigma_s{}^{(l)}\rangle_{Av}=(\pi/k_s^2)(2l+1)T_s. \tag{23}$$

When the bombarding energy is greater than the barrier height, the penetration factors $P_l\approx ka$, and the strong-coupling theory (see Sec. IX) suggests that the ratios $\langle\gamma_{\lambda c}^2\rangle_{Av}/D\approx(\pi Ka)^{-1}$ where $K\approx1\times10^{13}$ cm^{-1} is a wave number characteristic of nucleon motion within the nucleus. Under these conditions the transmission factor is $T=(4k/K)$, which is of the order of magnitude unity at moderate and high energies, and the summing procedure leading to (16) indicates that the total absorption is of the order of magnitude of the nuclear area. However, it is noted that when T_c is of the order unity, the ratio $\pi\langle\Gamma_{\lambda c}\rangle_{Av}/2D$ is also of this order, and the expansion (19) is not valid. It is also noted that transmission factors calculated from (22) can exceed unity for high energies (that is, if $4k$ exceeds K) in violation of the requirement that the collision matrix be unitary. The next two sections are devoted to the development of an alternative procedure which avoids these difficulties by dealing with an expansion similar to (18a) except for the nonappearance of the objectionable contributions from the s and t channels in the ξ' matrix. This avoidance is accomplished by means of the channel elimination procedure and the reduced R-matrix theory of TW. The result thereby obtained is similar to (22) though valid under the less restrictive condition, imposed by the channel contribution (2) above, that the mean with respect to channels c of the $\langle\Gamma_{\lambda c}\rangle_{Av}$ be less than D; the transmission factors that are obtained cannot exceed unity and are of the form proposed by Weisskopf.

[25] This sum includes compound elastic scattering (see Sec. VIII).

VI. THE REDUCED R AND U MATRICES

In the present notation the basic R-matrix relation, Eq. (6′) of TW, is

$$\mathfrak{U}=R(\mathfrak{D}-B\mathfrak{U})=R\bar{\mathfrak{D}}. \tag{24}$$

According to this expression the value $\mathfrak{U}$ of the radial part of the wave function on any particular channel surface is linearly related by the components of the R matrix to the derivative quantities $\mathfrak{D}$ and the boundary conditions B of all channels. Following TW the channels are separated into groups e (eliminated) and r (retained), whereupon the matrix relation (24) reduces to two submatrix relations,

$$\mathfrak{U}_e=R_{ee}\bar{\mathfrak{D}}_e+R_{er}\bar{\mathfrak{D}}_r,\quad \mathfrak{U}_r=R_{rr}\bar{\mathfrak{D}}_r+R_{re}\bar{\mathfrak{D}}_e. \tag{25}$$

It is to be noted that the subscripts e and r refer respectively to the groups of e and r channels, rather than to particular channels. If the group of e channels comprises only reaction and negative-energy channels, which have no incident waves, then its logarithmic derivatives at the nuclear surface are known to be those of an outgoing wave O or a negative-energy function W, so that

$$\bar{L}_e=\bar{\mathfrak{D}}_e\mathfrak{U}_e^{-1}=\mathfrak{D}_e\mathfrak{U}_e^{-1}-B_e=L_e-B_e. \tag{26}$$

As usual the bar will be omitted. By substituting (26) into (25) and solving, one obtains

$$\mathfrak{U}_e=\mathfrak{R}_{er}\mathfrak{D}_r,\quad \mathfrak{U}_r=\mathfrak{R}_{rr}\mathfrak{D}_r, \tag{27}$$

where

$$\mathfrak{R}_{rr}=R_{rr}+R_{re}L_e(1-R_{ee}L_e)^{-1}R_{er},$$
$$\mathfrak{R}_{er}=(1-R_{ee}L_e)^{-1}R_{er}.$$

$\mathfrak{R}_{rr}$ is the symmetrical reduced R matrix; it is not necessarily real as is the ordinary R matrix.

By means of the procedures indicated in the various papers on the R-matrix theory, it is possible to obtain directly from (27) the general expressions for the collision matrices; they are

$$V_{rr}=1+2iP_r^{\frac{1}{2}}(1-\mathfrak{R}_{rr}L_r)^{-1}\mathfrak{R}_{rr}P_r^{\frac{1}{2}}, \quad (28a)$$

$$V_{er}=2iP_e^{\frac{1}{2}}\mathfrak{R}_{er}(1-L_r\mathfrak{R}_{rr})^{-1}P_r^{\frac{1}{2}}. \quad (28b)$$

V_{rr} is referred to as the reduced collision matrix; it is not necessarily unitary as is the ordinary collision matrix. As in the deduction of (4), the inverse of the channel matrix appearing in (27) in the definitions of $\mathfrak{R}_{rr}$ and $\mathfrak{R}_{er}$ can be expressed in terms of the inverse of a level matrix A:

$$\mathfrak{R}_{rr}=R^0{}_{rr}+R^0{}_{re}L'_{ee}R^0{}_{er}+\sum_{\lambda\mu}(\omega_{\lambda r}\times\omega_{\mu r})A_{\lambda\mu}, \quad (29)$$

where

$$\omega_{\lambda r}=\gamma_{\lambda r}+R^0{}_{re}L'_{ee}\gamma_{\lambda e};$$

and

$$\mathfrak{R}_{er}=\sum_{\lambda\mu}(\alpha_{\lambda e}\times\gamma_{\mu r})A_{\lambda\mu},$$

where

$$\alpha_{\lambda e}=(1-R^0{}_{ee}L_e)^{-1}\gamma_{\lambda e}, \quad L'_{ee}=L_e(1-R^0{}_{ee}L_e)^{-1}.$$

As in (5) and (6) the components of the symmetrical level matrix A are obtained from the matrix relation

$$A=(\bar{E}-E-\xi)^{-1}, \quad (30a)$$

in which the components of the complex, symmetrica level matrix $\xi=-\Delta+\frac{1}{2}i\Gamma$ are given by the scalar products with respect to *eliminated* channels only,

$$\xi_{\lambda\mu}=(\gamma_{\lambda e},L'_{ee}\gamma_{\mu e}), \quad (30b)$$

in contrast with the corresponding expression (6a) in which the scalar product was with respect to *all* channels; as before the components of the real, diagonal matrix $\bar{E}$ are the proper values E_λ and the real, diagonal matrix E is the energy E times the unit matrix. Expressions similar to (6b) are obtained for the components of the total width and shift matrices Γ and Δ of the eliminated channels. The derivation of (29), which is lengthy though straightforward, is indicated in Appendix A. Henceforth we shall consider that $R^0=0$, so that Eqs. (29) simplify to

$$\mathfrak{R}_{rr}=\sum_{\lambda\mu}(\gamma_{\lambda r}\times\gamma_{\mu r})A_{\lambda\mu}, \quad (29a)$$

$$\mathfrak{R}_{er}=\sum_{\lambda\mu}(\gamma_{\lambda e}\times\gamma_{\mu r})A_{\lambda\mu}. \quad (29b)$$

The expressions for $\mathfrak{R}_{er}$ and V_{er} have been given here for completeness and will not be needed in the further developments.

If there is only one r channel, the entrance channel, and if $R^0=0$, (29a) reduces to the quadratic form

$$\mathfrak{R}_r=\sum_{\lambda\mu}\gamma_{\lambda r}A_{\lambda\mu}\gamma_{\mu r}. \quad (31)$$

$\mathfrak{R}_r$ is referred to as the reduced R function and is related to the quantity f_r, which appears in the work of Feshbach, Peaslee, and Weisskopf:[13]

$$\mathfrak{R}_r=(f_r-B_r)^{-1}. \quad (31a)$$

VII. "WEISSKOPF'S COLLISION MATRIX"

If the signs of the $\gamma_{\lambda c}$ are uncorrelated, the arguments used to justify the approximation (19) to (4) and (1) may also be used to justify the approximation

$$\mathfrak{R}_{rr}=\sum_\lambda(\gamma_{\lambda r}\times\gamma_{\lambda r})/(E_\lambda+\Delta_\lambda-E-\tfrac{1}{2}i\Gamma_\lambda) \quad (32)$$

to (29a), the widths Γ_λ and shifts Δ_λ of this expression being the diagonal components of the total width and shift matrices for the eliminated channels. However, (32) is valid under the less restrictive condition that the means with respect to λ and c of the partial level widths and shifts for the eliminated channels be less than the spacings, because the contributions to the second sum from the channels s, t have been eliminated in the expansion corresponding to (18a) for the components of $\mathfrak{R}_{rr}$.

When the Γ_λ are much larger than the spacings D, it is permissible to replace the individual widths and shifts in (32) by suitable averages with respect to an energy interval of size comparable to the $\langle\Gamma_\lambda\rangle_{Av}$:

$$\Gamma_\lambda\approx\langle\Gamma_\lambda\rangle_{Av}\equiv\Gamma(E), \quad \Delta_\lambda\approx\langle\Delta_\lambda\rangle_{Av}\equiv\Delta(E). \quad (33)$$

Both Γ and Δ are expected to be mildly energy-dependent because of the presence of the energy dependent factors P and S in the individual contributing terms and of possible systematic long-range variations of the reduced widths of the eliminated channels. This approximation is reasonable because it is not expected that the individual level widths and shifts will deviate significantly from one another, and the results obtained are usually not sensitive to the actual values of Γ and Δ. We will also apply (33) in those circumstances where the Γ_λ are narrow or comparable to D by allowing the Γ and Δ to vary in an appropriate manner from level to level.

Approximations (32) and (33) together lead to the interesting result that the various components of the reduced R matrix can be obtained approximately by simply evaluating the components of the ordinary R matrix at the complex energy $\mathcal{E}=E-\Delta+\frac{1}{2}i\Gamma$:

$$\mathfrak{R}_{rr}(E)\approx R_{rr}(\mathcal{E}). \quad (34)$$

This result is particularly useful because the latter components are analytical functions of the complex energy $\mathcal{E}$.[26] Thus the diagonal components of the R matrix are meromorphic functions, the imaginary parts of which are non-negative on the upper-half plane and nonpositive on the lower; their poles E_λ are confined to the real axis and have negative residues $-\gamma_{\lambda c}{}^2$. In this connection frequent reference will be made to the *pole strength function*[16,26] s_c for channel c which is defined as the sum of the $\gamma_{\lambda c}{}^2$ per unit energy interval of the E_λ,

[26] E. P. Wigner, Ann. Math. **53**, 36 (1951); **55**, 7 (1952).

averaged with respect to an interval of appropriate length.

The following procedure will now be used for the determination of the various components of the collision matrix. For the determination of a particular diagonal component, one eliminates explicit reference to all but the channel referred to by that component, while for the determination of a particular nondiagonal component, one eliminates explicit reference to all but the two channels referred to by that component. First we consider the diagonal components.

Diagonal Components

If there is only one r channel, the reduced collision matrix given by (28) together with (1) becomes the reduced collision function for the r channel; it is conveniently written (the r subscripts being omitted)

$$U=\exp(-2i\phi)(1-\mathfrak{R}L^*)/(1-\mathfrak{R}L). \tag{35}$$

It may be noted that if the reduced R function is approximated by one level of the diagonal component of the R matrix of (34), Eq. (35) reduces to the familiar one-level approximation to the collision function as given by Feshbach, Peaslee, and Weisskopf.[13]

By the introduction of Wigner's statistical R function R',[17,26]

$$R'=\sum_{\lambda'}\gamma_{\lambda'}{}^2/(E_{\lambda'}-E), \tag{36}$$

it is possible to develop a useful representation for the approximation (34) which is expected to be rather accurate when the absorption width is small (as well as large) compared with the mean level spacing D. The R' function has the property that in an energy interval I containing the energy E of interest and of a length such that $D\ll I\ll s(ds/dE)$, where $s(E)$ is the strength of R in the vicinity of E, its poles $E_{\lambda'}$ and residues $-\gamma_{\lambda'}{}^2$ are equal to those of R whereas outside of I they maintain the same statistical distributions as within and thus will in general differ from those of R which may evidence long-range fluctuations. The strength s of R' is thus everywhere equal to the strength $s(E)$ of R at the energy E. R' has therefore the useful property of approaching $i\pi s(E)$ when evaluated at a complex energy whose imaginary part is large compared with D, as is evident by replacing the sum over levels in (36) by an integration for the evaluation. The R' function is then added to and subtracted from the right side of (34) and the sums in the combination $(R-R')$ approximated by integrations, whereupon one obtains

$$\mathfrak{R}(E)=R'(\mathcal{E})+\bar{R}(\mathcal{E})-i\pi s(E), \tag{37}$$

in which

$$\bar{R}(\mathcal{E})=R^{\mathrm{Re}}+i\bar{R}^{\mathrm{Im}}=\int s(E_\lambda)dE_\lambda/(E_\lambda-\mathcal{E}) \tag{37a}$$

is the Stieltjes transform[27] of $s(E)$, the inverse of which is

$$s(E-\Delta)=\lim_{\Gamma\to 0_+}(1/\pi)\bar{R}^{\mathrm{Im}}(\mathcal{E}). \tag{37b}$$

The representation (37) has the expected behavior in the two extremes: $\Gamma\gg D$, $\mathfrak{R}=\bar{R}$; $\Gamma\ll D$, $\mathfrak{R}=\bar{R}^{\mathrm{Re}}+R'$. The former result may be obtained simply by replacing the sum over the levels of R by an integration; according to (35) the corresponding collision function is

$$\bar{U}=\exp(-2i\phi)(1-\bar{R}L^*)/(1-\bar{R}L), \tag{38a}$$

the bar having been placed over U to indicate that $\mathfrak{R}$ has been replaced by $\bar{R}$. This function manifests no resonances and is essentially constant in the interval I. In the latter extreme, the term $\bar{R}^{\mathrm{Re}}$ represents the net contribution to R from the levels outside of I; in the limit $\Gamma=0$, it is the principal part of the integral of (37a). The collision function (35) for this case is conveniently rewritten

$$U=\exp(-2i\phi')(1-R'L'^*)/(1-R'L'), \tag{38b}$$

where

$$L'=L/(1-\bar{R}^{\mathrm{Re}}L),$$

and

$$\phi'=\phi-\tan^{-1}[\bar{R}^{\mathrm{Re}}P/(1-\bar{R}^{\mathrm{Re}}S)]$$

is the negative of the actual potential scattering phase shift. For example, in the collisions of slow neutrons with nuclei, the potential scattering term appearing in elastic scattering is found to be modified by a factor $(1-\bar{R}^{\mathrm{Re}})$ by which the radius is multiplied; according to (37b) the average of the reduced widths $\gamma_\lambda{}^2$ of the resonances is $D\bar{R}^{\mathrm{Im}}/\pi$.

Nondiagonal Components

For the evaluation of the st nondiagonal component of U, one can eliminate explicit reference to all but the s and t channels by introducing the reduced R matrix with components $\mathfrak{R}_{ss}$, $\mathfrak{R}_{tt}$, $\mathfrak{R}_{st}$ into the expression (28a) for the reduced collision matrix. By inverting the two-channel matrix $(1-\mathfrak{R}_{rr}L_r)$, one finds the nondiagonal component of the V_{rr} matrix to be

$$V_{st}=2iP_s{}^{\frac{1}{2}}\mathfrak{R}_{st}P_t{}^{\frac{1}{2}}\mathfrak{D}^{-1}, \tag{39}$$

the determinant being

$$\mathfrak{D}=(1-L_s\mathfrak{R}_{ss})(1-L_t\mathfrak{R}_{tt})-L_s\mathfrak{R}_{st}{}^2L_t.$$

The approximations (34) may be used for the various $\mathfrak{R}$ components, the Γ_λ and Δ_λ now including contributions from the partial widths and shifts of all but the s and t channels. When $\langle\Gamma_\lambda\rangle_{\mathrm{Av}}\gg D$, it can be shown by means of Radamacher's theorem[20] that $|\mathfrak{R}_{st}|^2$ averaged with respect to all possible choices of the random signs of the $\gamma_{\lambda c}$ is equal to $2\pi s_s s_t D/\langle\Gamma_\lambda\rangle_{\mathrm{Av}}$, which is negligible compared with the product $\mathfrak{R}_{ss}\mathfrak{R}_{tt}=\bar{R}_s\bar{R}_t\approx\pi^2 s_s s_t$. Equation (39) may therefore be replaced

$$V_{st}=2iP_s{}^{\frac{1}{2}}\sum_\lambda\frac{\theta_{\lambda s}\theta_{\lambda t}}{E_\lambda+\Delta_\lambda-E-\frac{1}{2}i\Gamma_\lambda}P_t{}^{\frac{1}{2}}, \tag{40}$$

[27] D. M. Widder, *The Laplace Transform* (Princeton University Press, Princeton, 1941).

where

$$\theta_{\lambda s}=(1-L_s\bar{R}_s)^{-1}\gamma_{\lambda s}.$$

Since $L_c\bar{R}_c\approx\frac{1}{2}i\pi\langle\Gamma_{\lambda c}\rangle_{\text{Av}}D^{-1}-\pi\langle\Delta_{\lambda c}\rangle_{\text{Av}}D^{-1}$, it is apparent that as expected (40) differs from (19) when the absolute magnitudes of these quantities approach or exceed unity. There is the additional difference that the widths and shifts in the denominators of (19) include contributions from all channels. However, when $\langle\Gamma_\lambda\rangle_{\text{Av}}\gg D$, this difference is negligible and for convenience the widths and shifts of (40) may be regarded as the respective totals, as in (19). The collision matrix thereby obtained has the advantage of being a valid approximation when $\langle\Gamma_\lambda\rangle_{\text{Av}}\ll D$ as well as when $\langle\Gamma_\lambda\rangle_{\text{Av}}\gg D$. One would also presume it to be reasonably accurate when $\langle\Gamma_\lambda\rangle_{\text{Av}}\sim D$, although in this case one can refer directly to (39). However, it is noted that this form of the collision matrix may violate the relation (9a), although to order of the channel mean of the ratios $\langle\Gamma_{\lambda c}\rangle_{\text{Av}}/D$, which necessarily must be small. This slight defect is readily amended by considering the partial widths and shifts which contribute to the respective totals to be

$$\Gamma_{\lambda c}=2P_c|\theta_{\lambda c}|^2,\quad \Delta_{\lambda c}=-S_c|\theta_{\lambda c}|^2, \tag{41}$$

rather than $2P_c\gamma_{\lambda c}^2$ and $-S_c\gamma_{\lambda c}^2$. The equality alternative of (9a) is then satisfied. As shown in the next section, the expressions for the average reaction cross sections which may be deduced from (40) are of the same form as those derived by Weisskopf.[11] We will therefore refer to (40) with the modifications indicated above as Weisskopf's collision matrix, although it was not explicitly considered by him in this manner. Finally it should be noted that although the nondiagonal components of V have been expressed in the expansion form (9), the diagonal components have not. No particular significance is attributed to this departure other than that it does appear to make V unitary.

VIII. AVERAGE CROSS SECTIONS

Scattering and Absorption

Convenient expressions for the energy averages of the cross sections for scattering and absorption of an incident beam of particles have been given by Feshbach, Porter, and Weisskopf.[9] For the lth partial wave and in units of $(\pi/k^2)(2l+1)$ they are:

total,	$\langle\sigma_0\rangle_{\text{Av}}=2-\langle U\rangle_{\text{Av}}-\langle U^*\rangle_{\text{Av}}$;
total elastic,	$\langle\sigma_e\rangle_{\text{Av}}=\langle\|1-U\|^2\rangle_{\text{Av}}$;
potential elastic,	$\langle\sigma_{pe}\rangle_{\text{Av}}=\|1-\langle U\rangle_{\text{Av}}\|^2$;
compound elastic,	$\langle\sigma_{ce}\rangle_{\text{Av}}=\langle\|U\|^2\rangle_{\text{Av}}-\|\langle U\rangle_{\text{Av}}\|^2$;
absorption,	$\langle\sigma_a\rangle_{\text{Av}}=1-\langle\|U\|^2\rangle_{\text{Av}}$;
compound-nucleus formation,	$\langle\sigma_C\rangle_{\text{Av}}=1-\|\langle U\rangle_{\text{Av}}\|^2$. (42)

U is the diagonal component of the collision matrix which refers to the incident beam. The cross section for compound-nucleus formation is by definition the sum of the absorption and compound elastic scattering cross sections, and it may be verified that the total elastic scattering is the sum of the potential elastic and compound elastic cross sections.

In view of the fact that all of the poles of the collision function are situated in the lower half of the complex energy plane, with the exception of those on the real axis associated with bound states,[28] the path of integration involved in the averaging of U may be displaced, without crossing poles, far enough upwards so that R' becomes essentially equal to $i\pi s(E)$, and therefore $\mathfrak{R}=\bar{R}$ according to (37). If the average is with respect to an energy interval I such that $D\ll I\ll s(ds/dE)^{-1}$, it may be presumed that the contributions from the connecting sides of the contour effectively cancel and that $\bar{R}$ is nearly constant on the displaced path, in which case

$$\langle U\rangle_{\text{Av}}=\bar{U}, \tag{43}$$

the quantity $\bar{U}$ being given by (38a). The interval I must also be small enough so that the external functions S, P, and ϕ may be considered as constants. It is then evident that $\langle\sigma_0\rangle_{\text{Av}}$, $\langle\sigma_{pe}\rangle_{\text{Av}}$, and $\langle\sigma_C\rangle_{\text{Av}}$ are obtained by simply substituting $\bar{U}$ for $\langle U\rangle_{\text{Av}}$ in (42). This result has already been noted by Feshbach *et al.*[9]

No such simple results of a general nature can be obtained for the quantity $\langle|U|^2\rangle_{\text{Av}}$, which also appears in (42), because its poles are situated in both halves of the $\mathcal{E}$ plane. However, by assuming that the $\gamma_{\lambda'}{}^2$ of R' are all equal and the levels $E_{\lambda'}$ uniformly spaced by an amount D so that

$$R'=\pi s\tan(\pi\mathcal{E}/D), \tag{44a}$$

then it is not difficult to include the residues of the poles of $|U|^2$ in the contour used to derive (43). The result is that

$$(1-|\langle U\rangle_{\text{Av}}|^2)^{-1}=(1-\langle|U|^2\rangle_{\text{Av}})^{-1}+(1-w^{-1})^{-1}, \tag{44b}$$

where

$$w=\exp(-2\pi\Gamma/D),$$

Γ being the total width for absorption, and therefore the remaining cross sections are

$$\begin{aligned}\langle\sigma_a\rangle_{\text{Av}}&=T[(1-w)/(1-w+wT)],\\ \langle\sigma_{ce}\rangle_{\text{Av}}&=T[wT/(1-w+wT)],\end{aligned} \tag{45}$$

where

$$T=\langle\sigma_C\rangle_{\text{Av}}=1-|\bar{U}|^2 \tag{45a}$$

is the transmission factor in the form proposed by Weisskopf.[11] This result has already been obtained in a somewhat different manner by Snowden and Whitehead.[29]

According to (45) the relative probabilities of compound elastic scattering and absorption are wT and

[28] W. Schützer and J. Tiomno, Phys. Rev. **83**, 249 (1951); Marcos Moshinsky, Anais acad. brasil. ciênc. **25**, 343 (1953).

[29] S. C. Snowden and W. D. Whitehead, Phys. Rev. **94**, 1267 (1954).

$(1-w)$, respectively. The following interpretation of these probabilities has been suggested by Porter:[30] According to Weisskopf[11,31] a period $2\pi\hbar/D$ may be attributed to the compound nucleus. Since the decay rate for absorption is $\Gamma/\hbar$, the probability for absorption in this period is $(1-w)$ while that of no absorption is w. At the end of each period the system is presumed to be in a configuration for decay into the entrance channel from which it was formed, and the probability for penetration through this channel is the transmission factor T, a quantity which by definition cannot exceed unity. Therefore, the chance of the occurrence of compound elastic scattering in one period is wT. The periodic motion of the compound nucleus is repeated until there is decay either one way or the other.

Both the detailed calculation by means of (44a) and the interpretation involving the attribution of a period to the compound nucleus depend upon the assumption of a uniform level spacing. This assumption may be questioned because the alternative view that the behavior of the compound nucleus is "chaotic" would imply that as in the familiar one-level resonance formula the relative probabilities for these decays are T for the entrance channel and the corresponding transmission factor $2\pi\Gamma/D$ for absorption. In this case, the probability for compound elastic scattering is relatively larger than wT, especially when $w\ll 1$ or equivalently $2\pi\Gamma/D\gg 1$. This alternative view may correspond to something like a random distribution for the level spacings. Unfortunately there is scant experimental information regarding this distribution, and the true behavior may lie anywhere between these extremes.

Reactions

The average reaction cross sections may be obtained from (40) in the same manner as (22) was obtained from (19), the only difference in the result being that the transmission factors are now $2\pi/D$ times the level averages of the $\Gamma_{\lambda c}$ of (41):

$$T_c=4\bar{R}^{\mathrm{Im}}P_c/|1-\bar{R}_cL_c|^2. \tag{46}$$

It may be verified that this T is the same as that of (45a). In other words, the channel transmission factors are just $k^2/\pi(2l+1)$ times the corresponding cross sections for compound-nucleus formation. This is the well known conclusion arrived at by Weisskopf[11] by considering the quantity defined as $\langle\sigma_C\rangle_{\mathrm{Av}}$ in (42) as the cross section for the actual formation of a compound nucleus and by applying the reciprocity theorem to the hypothesis of independent decay. In the present deliberations we have attempted to show in what circumstances these considerations are valid.

As a conclusion to this subsection, an application is mentioned of the collision matrices (19) or (40) in which the signs of the $\gamma_{\lambda c}$ are uncorrelated. If the reader will refer to the general expressions (4.6) or (4.7) given by Blatt and Biedenharn[32] for the differential cross sections of nuclear reaction products, he will notice that there occur sums of products in which $l_1=l_2$, $l_1'=l_2'$, $J_1=J_2$ and products in which one or more of these equalities are not satisfied. If these cross sections are averaged over an energy interval which is large compared with $\langle\Gamma_\lambda\rangle_{\mathrm{Av}}$, then the arguments used to derive (22) from (20) can be applied to show that the latter products are negligible compared with the former. As noted by Blatt and Biedenharn, the resulting expression with just the former products is the basis of the treatment by Hauser and Feshbach[33] of neutron inelastic scattering.

IX. THE POLE STRENGTH FUNCTION

According to the results of the last two sections, the pole strength function $s(E)$ and its Stieltjes transform (37a) are decisive in the determination of the behaviors of the various average reaction and scattering cross sections. The expressions for $s(E)$ and its transform are discussed in this section for various interaction representations of current interest.

1. The Strong-Coupling Theory

In this theory, the strength function may be approximately represented by the monotonically decreasing function[17]

$$\begin{aligned} s(E)&=(\pi Ka)^{-1}, \quad E>-B;\\ &=0, \qquad\quad E<-B, \end{aligned} \tag{47}$$

where

$$K=[2M(E+T+B)/\hbar^2]^{\frac{1}{2}}$$

is the wave number characteristic of the motion of a nucleon of mass M in a nucleus excited by the capture of a nucleon of energy E; T is the corresponding kinetic energy for an unexcited nucleus; and B is the binding energy of the nucleon to the nucleus. The Stieltjes transform of (47) is

$$\begin{aligned} &\bar{R}(E-\Delta+\tfrac{1}{2}i\Gamma)\\ &\quad=(\hbar^2/2Ma^2T)^{\frac{1}{2}}(x-iy)\left\{\frac{1}{2\pi}\log\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right.\\ &\qquad\left.+i\left[1-\frac{1}{\pi}\tan^{-1}\frac{y}{1+x}-\frac{1}{\pi}\tan^{-1}\frac{y}{1-x}\right]\right\}, \end{aligned} \tag{48}$$

where

$$\begin{aligned} x&=[\tfrac{1}{2}T/H^2+\tfrac{1}{4}\Gamma^2]^{\frac{1}{2}}[(H^2+\tfrac{1}{4}\Gamma^2)^{\frac{1}{2}}+H]^{\frac{1}{2}},\\ y&=[\tfrac{1}{2}T/H^2+\tfrac{1}{4}\Gamma^2]^{\frac{1}{2}}[(H^2+\tfrac{1}{4}\Gamma^2)^{\frac{1}{2}}-H]^{\frac{1}{2}},\\ H&=E-\Delta+T+B, \end{aligned}$$

the inverse tangents being evaluated in the first quadrant. According to the estimates of Appendix B, the

[30] C. E. Porter (private communication).
[31] Victor F. Weisskopf, Helv. Phys. Acta **23**, 187 (1950).
[32] J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. **24**, 258 (1952), in particular reference 13.
[33] W. Hauser and H. Feshbach, Phys. Rev. **87**, 366 (1952).

absorption width Γ is generally much less than the characteristic kinetic energy T, which is expected to be about 20 Mev, so that the above expression may accurately be approximated by

$$\bar{R}(E-\Delta+\tfrac{1}{2}i\Gamma)=s(E-\Delta)\log[(H^{\frac{1}{2}}+T^{\frac{1}{2}})/(H^{\frac{1}{2}}-T^{\frac{1}{2}})]+i\pi s(E-\Delta), \quad (48a)$$

which is independent of Γ. In typical cases, the real and imaginary parts of this transform are of about the same magnitude. It should be emphasized that in this theory the potential scattering is not given by the hard-sphere formula but by that which is obtained from the phase shift $-\phi'$ of (38b).[17]

In the Feshbach-Weisskopf continuum theory,[12] it is assumed that $\bar{R}=i/Ka$, which corresponds to constant K and constant strength s extending in energy from $-\infty$ to ∞, and to a wave function of the form $\exp(-iKr)$ at the nuclear surface.[1] A more realistic function would be of the form $\exp(K'-iK)r$ with a damping coefficient K' of about the same magnitude as K.[9] This modified wave function corresponds to an $\bar{R}$ of the form (48a).

2. The Complex Square Well Representation

As monotonic strength functions such as (47) do not appear to be in accord with the recent experimental results on total neutron cross sections,[8] it is tempting to consider nonmonotonic functions such as

$$s(E)=(1/2\pi)\sum_p \zeta_p{}^2 W_p/[(E_p-E)^2+\tfrac{1}{4}W_p{}^2], \quad (49a)$$

with transforms

$$\bar{R}(\mathcal{E})=\sum_p \zeta_p{}^2/(E_p-\mathcal{E}-\tfrac{1}{2}iW_p) \quad (49b)$$

which are similar to ordinary R functions with levels p and reduced widths $\zeta_p{}^2$. The calculations of Bohr and Mottelson[34] provide some justification for the general form of (49a) on the basis of a shell model with level positions E_p and core couplings proportional to W_p. This analysis also indicates that

$$\zeta_p{}^2\approx\hbar^2/Ma^2, \quad (49c)$$

which is the characteristic single-particle reduced width. This magnitude is to be expected since in the limit of no coupling, (49b) should reduce to a single-particle R function and $s(E)$ to a sum of delta functions each of total strength $\hbar^2/Ma^2$ at the positions E_p of the single-particle levels. Moreover, if the coupling is large compared with the single-particle level spacing $\pi\hbar^2K/Ma$, it may be shown by approximating the sum in (49a) by an integration that the strong-coupling value (47) for the strength is approached. It is also evident that if the W_p are sufficiently small, the corresponding expansion parameters $\pi\langle\Gamma_{\lambda c}\rangle_{Av}/2D=\pi s_c P_c$ of Sec. V can exceed unity, especially in the vicinity of the single-particle levels p.

[34] Aage Bohr and Ben R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, 16 (1953).

It has been shown by Wigner[35] that the reciprocal logarithmic derivative of any real continuous potential of finite range may be expanded as an R function. The R function for the complex square well potential should therefore be a special case of the expansion (49b). In fact by setting the $\zeta_p{}^2$ and W_p equal to constants with respective values $\hbar^2/Ma^2$ and W, and by choosing the level positions to be $E_p=(\hbar^2/2Ma^2)(p-\frac{1}{2})^2\pi^2-V_0$ with p ranging from 1 to ∞ and V_0 an arbitrary energy, the expansion is obtained for $R=Z^{-1}\tan Z$, where

$$Z^2=(2Ma^2/\hbar^2)[(E-\Delta+V_0)+\tfrac{1}{2}i(\Gamma+W)],$$

which is just the expression for the reciprocal logarithmic derivative of a complex square well wave function of zero relative orbital angular momentum. It may also be shown that the corresponding expansions for complex square well wave functions of arbitrary angular momentum $\hbar l$ may be obtained with the same $\zeta_p{}^2$ and W if one sets the boundary condition $B=-l$; the level positions, however, occur at the values $E=E_p$ which are associated with the zeros of the Bessel function $F_{l-1}(\rho_{pl})$ where $\rho=(2Ma^2/\hbar^2)^{\frac{1}{2}}(E-\Delta+V_0)^{\frac{1}{2}}$. The relation between $\mathcal{R}$ and the logarithmic derivative f for this boundary condition is given by (31a).

The expansion (49b) for $\bar{R}$ suggests a "giant resonance" interpretation for the average cross-section data. If the width $\Gamma+W_p$ is small compared with the single-particle level spacing, one may over a limited energy range approximate the $\bar{R}$ of (49b) when substituted in the $\bar{U}$ of (38a) by one shell level p plus a real, constant $\bar{R}^0$ contribution from the remaining levels. Resonance formulas are thereby obtained for scattering and absorption in which the entrance channel width is given by the single-particle width $2P\zeta_p{}^2/|1-\bar{R}^0L|^2$ and the absorption width by $\Gamma+W_p$; the level spacing is that characteristic of a single-particle potential. By averaging over these giant resonances, the averages of the strong-coupling theory are obtained. In applications the interpretation is complicated by the fact that there will be one such resonance for each partial wave that is effective.

The complex square well representation for particle-nuclei interactions at intermediate energies has been considered by many physicists.[36] For a recent account of the interpretation of the neutron interaction cross sections in terms of it, the reader is referred to the paper by Feshbach, Porter, and Weisskopf.[9] The main results of the present investigation in this connection are the demonstration of the relation to the R-matrix or resonance theory and the theoretical indication of the dependence of the imaginary part of the potential on the absorption width. Francis and Watson[37] have investigated formally the matter of equivalent two-body

[35] E. P. Wigner, Am. Math. Monthly **59**, 669 (1952).

[36] Ostrofsky, Breit, and Johnson, Phys. Rev. **49**, 22 (1936); H. A. Bethe, Phys. Rev. **57**, 1125 (1940); Robert E. Le Levier and David S. Saxon, Phys. Rev. **87**, 40 (1952); D. M. Chase and F. Rohrlich, Phys. Rev. **94**, 81 (1954).

[37] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).

potentials by considering the multiple scattering of an incident beam of particles traversing the target nucleus.

As an alternative to the introduction of the continuous, nonmonotonic $s(E)$ of (49a), one may suppose that the behavior of the average total neutron cross sections is due to the presence in the R function expansion of strong shell levels in addition to the usual weak levels which are needed to explain the narrow resonances. Such a state of affairs is depicted in Fig. (1a) where the strength s is represented by the height of the vertical lines for the individual levels; the representation of the strength function (49a) is shown in Fig. (1b). The former situation is actually somewhat unrealistic because of the strictly formal nature of the R-matrix levels. A mixing of the weak and strong levels is to be expected, the consequence of which is the dissolution of the strong levels and a modulation of the strengths of the weak levels, as indicated by Fig. (1b). Experimentally it is not easy to distinguish between these two alternatives because when the imaginary part of the potential is relatively small, the average total cross sections depend primarily on the real part of $\bar{R}$, which is essentially the same in the two cases. Direct determinations of s would therefore be desirable. This can be done by studying the behavior of the $\langle\gamma_{\lambda c}^2\rangle_{\text{Av}}$ for individual resonances[38] and by measuring the deviations from the simple exponential dependence of the transmissions involved in the measurements of total cross sections, these deviations resulting from the fluctuations, caused by the presence of many unresolved resonances, of the actual cross section about its average.[39]

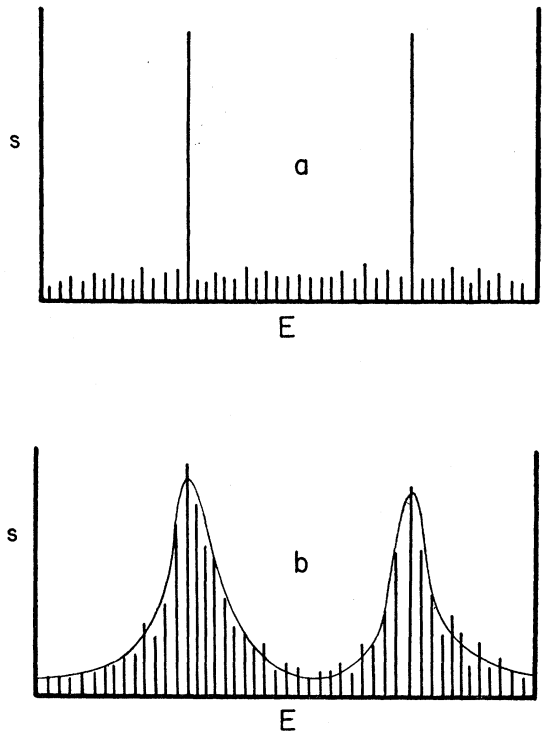

FIG. 1. The strengths $s=\gamma_\lambda^2/D$ of the levels of two nuclear models are represented by the heights of vertical lines at the level positions. a: There are strong, widely spaced levels in addition to weak levels of the strong-coupling type with narrow spacing; b: As a result of mixing of the strong and weak levels of a, the strong levels are weakened and their weak neighbors are strengthened.

3. Surface Absorption Model

A nucleon-nucleus interaction of current interest is one with a finite surface absorption in addition to the volume interaction; this absorption can be represented by a pure imaginary delta function potential of strength $i\alpha\hbar^2/2Ma$ appended to the complex well volume interaction. By integrating the wave equation across this surface, one finds that the logarithmic derivative is changed by a finite amount $-i\alpha/a$. The $\bar{R}$ function just outside of this surface is, therefore,

$$\bar{R}_+=(\bar{R}_-^{-1}-i\alpha)^{-1}, \tag{50a}$$

where $\bar{R}_-$ is the $\bar{R}$ function for the internal potential. According to the inversion formula (37b), the strength function corresponding to (50a) is just $1/\pi$ times the imaginary part of $\bar{R}_+$,

$$s(E-\Delta)=\pi^{-1}(\bar{R}_-{}^{\text{Im}}+\alpha|\bar{R}_-|^2)/(1+\alpha^2|\bar{R}_-|^2). \tag{50b}$$

In the consideration of such a model, Bohr and Mottelson[34] have given an alternative derivation of (50b) for the case where $\bar{R}_-$ is real.

[38] Carter, Harvey, Hughes, and Pilcher, Phys. Rev. **96**, 113 (1954).

[39] R. G. Thomas, to be published.

X. CONCLUDING REMARKS

Although this investigation has primarily been concerned with magnitudes and energy dependences, it may be appropriate to include a remark concerning the angular distributions of the products of the decay of the compound nucleus. It has been a common misconception that the theory of the compound nucleus predicts that these distributions be isotropic. It should be stated that there are no indications from the present treatment that this is necessarily so. This matter has already been thoroughly investigated by Wolfenstein[40] as well as Hauser and Feshbach[33] who show that the theory of the compound nucleus requires only that these distributions be symmetric with respect to a plane perpendicular to the beam direction when averaged over a sufficient number of resonance levels. This symmetry is indeed a consequence of the lack of correlations of the signs of the $\gamma_{\lambda c}$. These authors also show that a sufficient condition for isotropy is that the energy levels of the compound and residual nuclei be sufficiently dense and have a dependence on their respective spins J which is proportional to $2J+1$ for the range of J values that can participate. According to reference 45, significant de-

[40] L. Wolfenstein, Phys. Rev. **82**, 690 (1951).

partures from such a dependence may occur at moderate energies. More general conditions of a necessary nature for isotropy are not known at present.

A concluding remark concerning the non-compound-nucleus processes may also be appropriate. It has been shown that if the signs of the $\gamma_{\lambda c}$ for the nuclear levels are uncorrelated, for any manner of excitation these levels will superpose in such a way as to form a compound nucleus; it has been a common belief that for the region of the configuration space where the nuclear interactions are very strong, these signs will indeed be uncorrelated. However, there is a nuclear surface region where the interactions are not particularly strong and non-compound nucleus processes, such as stripping and pickup, can occur. The R-matrix theory, being general, can account for these processes if the channel radii are extended to the limit of the nuclear interactions, but if this is done, the signs of the $\gamma_{\lambda c}$ would become correlated and some of the deductions in IX concerning the strength functions would lose their validity. A tractable approach to the theory of nuclear reactions may therefore be one where the channel radii are small enough to permit the application of the compound-nucleus theory to the internal, strong-interaction region, the effects of the nuclear interactions in the external region being accounted for by the running-wave Green's-function method.[41] The collision matrix would be expressed as

$$U=U_C+\Delta U, \tag{51}$$

where U_C represents the compound-nucleus contribution and ΔU the additional contribution from the external interactions. As a first approximation the wave functions of the compound nucleus system could be used in the matrix elements for the evaluation of the components of ΔU, rather than plane waves as in the Born approximation.[42] The equivalent two-body representations, such as the complex square well, would be particularly useful for this purpose. According to (51), the resulting collision cross section will contain contributions proportional to $|U_C|^2$ for the compound-nucleus processes, to $|\Delta U|^2$ for the non-compound-nucleus processes, and to the interference terms $U_C\Delta U^*+U_C^*\Delta U$.[43] It is evident that the interference contributions will vanish if averaged over an energy interval containing a sufficient number of levels of the compound nucleus.

[41] The R-matrix theory is based on the *standing-wave* Green's-function method. In fact, the components of the R matrix are the values of the Green's function at the entrances to the various pairs of channels.

[42] See, e.g., E. Gerjuoy, Phys. Rev. **91**, 645 (1953).

[43] The effect of the compound nucleus on the stripping contribution has been considered by N. C. Francis and K. M. Watson, Phys. Rev. **93**, 313 (1954); J. Horowitz and A. M. L. Messiah, J. phys. et radium **14**, 695 (1953); and W. Tobocman, Phys. Rev. **94**, 1655 (1954).

APPENDIX A. DERIVATION OF EQ. (29)

It is necessary to invert the submatrix $(1-R_{ee}L_e)$ in (27). With the separation (2) for the R-matrix, one finds

$$(1-R_{ee}L_e)^{-1}=(1-R^0{}_{ee}L_e)^{-1}(1-R'_{ee}L'_{ee})^{-1}, \tag{A1}$$

with L'_{ee} given in connection with (29). The inversion procedure[3,6] is to assume an expansion

$$(1-R'_{ee}L'_{ee})^{-1}=1+\textstyle\sum_{\mu\nu}(\gamma_{\mu e}\times\beta_{\nu e})A_{\mu\nu}, \tag{A2}$$

with

$$\beta_{\nu e}=L'_{ee}\gamma_{\lambda e}$$

and level coefficients $A_{\mu\nu}$ which are presumably functions of the energy. By multiplying both sides of (A2) by $(1-R'_{ee}L'_{ee})$ one obtains

$$-\sum_{\lambda}\frac{(\gamma_{\lambda e}\times\beta_{\lambda e})}{E_\lambda-E}+\sum_{\mu\nu}(\gamma_{\mu e}\times\beta_{\nu e})A_{\mu\nu}$$
$$-\sum_{\lambda\mu\nu}\frac{(\gamma_{\lambda e}\times\beta_{\nu e})}{E_\lambda-E}A_{\mu\nu}\xi_{\lambda\mu}=0, \tag{A3}$$

after making use of the identities

$$\begin{aligned}M(x\times y)N&=(Mx\times\tilde{N}y),\\(x\times y)(z\times w)&=\zeta(x\times w)\end{aligned} \tag{A4}$$

in which x, y, z, w are arbitrary vectors, M and N are arbitrary matrices, the transpose of the latter being $\tilde{N}$, and $\zeta=(y,z)$ is a scalar product over channels; the $\xi_{\lambda\mu}$ are given by (30b). Equation (A3) may be rearranged to read

$$\sum_{\lambda\nu}(\gamma_{\lambda e}\times\beta_{\nu e})\left[-\frac{\delta_{\lambda\nu}}{E_\lambda-E}+A_{\lambda\nu}-\sum_{\mu}\frac{\xi_{\lambda\mu}A_{\mu\nu}}{E_\lambda-E}\right]=0 \tag{A5a}$$

which is satisfied if for all λ, ν

$$(E_\lambda-E)A_{\lambda\nu}-\textstyle\sum_\mu\xi_{\lambda\mu}A_{\mu\nu}=\delta_{\lambda\nu}. \tag{A5b}$$

The matrix equivalent of this set of equations is (30a). The expansions (A2) and (A1) are then substituted into the expressions (27) for $\mathfrak{R}_{rr}$ and $\mathfrak{R}_{er}$. By repeated applications of (A4) and (A5b) these complicated expressions finally reduce to those of (29).

APPENDIX B. ESTIMATES OF THE TOTAL REACTION WIDTH

The following procedure for estimating the total nuclear decay width is essentially the same as that previously used by Weisskopf.[44] A compound nucleus of excitation energy E_C, total spin J, emits particles of spin i with relative orbital angular momentum hl, in groups of energy ϵ, leaving the residual nucleus with

[44] V. F. Weisskopf, Phys. Rev. **52**, 295 (1937).

total spin I and excitation energy $(E_C-B-\epsilon)$, where B is the binding energy of the emitted particle to the ground state of the residual nucleus; $\mathbf{s}=\mathbf{i}+\mathbf{I}$ is the channel spin. By assuming that the levels in the residual nucleus are sufficiently dense so that the sum over levels can be replaced by an integration, the total width for the decay of a compound-nucleus level of spin J becomes

$$\Gamma^J=\sum_{Il_s}\frac{1}{2\pi}D_C{}^J(E_C)\int_0^{E_C-B}\frac{T_{l\epsilon}d\epsilon}{D_R{}^I(E_C-B-\epsilon)}, \quad \text{(B1)}$$

where $D_C{}^J$ and $D_R{}^I$ are the mean level spacings of the compound and residual nuclei with spins J and I, respectively, and $T_{l\epsilon}$ is the transmission factor for channel $l\epsilon$. It is necessary to make the reasonable assumption that the level densities as a function of spin are proportional to the spin statistical factor,[45] at least over the range of participating spin values:

$$D_C{}^J=D_C/(2J+1), \quad D_R{}^I=D_R/(2I+1), \quad \text{(B2)}$$

D_C and D_R being independent of the spins J and I, respectively. By noting that

$$\sum_{I_s}(2I+1)=(2J+1)(2l+1)(2i+1)$$

and

$$\sigma_C(\epsilon)=(\pi\hbar^2/2M\epsilon)\sum_l(2l+1)T_{l\epsilon}$$

is the cross section for the inverse process of compound nucleus formation when the residual nuclei of excitation $(E_C-B-\epsilon)$ are bombarded by particles of energy ϵ, where M is the reduced mass, one obtains

$$\Gamma=\frac{(2i+1)MD_C(E_C)}{\pi^2\hbar^2}\int_0^{E_C-B}\frac{\epsilon\sigma_C(\epsilon)d\epsilon}{D_R(E_c-B-\epsilon)}, \quad \text{(B3)}$$

which is independent of J and identical to the original expression given by Weisskopf. In order to arrive at an order of magnitude estimate of Γ, it is assumed that $D_C(E)=D_R(E)=c\times\exp(-E/T)$ both functions having the same factor c and temperature T, at least in the region of excitation energies where the main contributions to the integral in (B3) arise; furthermore it is assumed that the energies of the emitted neutrons are all equal to $2T$, their average, that $\sigma_C(\epsilon)=\pi a^2$, being independent of the state of excitation of the residual nucleus, and that $E_c\gg B$, so that one obtains[46]

$$\Gamma\approx 2(2i+1)MT^2a^2\pi^{-1}\hbar^{-2}\exp(-B/T). \quad \text{(B4)}$$

As a typical example, we put $a=8\times10^{-13}$ cm, $T=3$ Mev, for neutron decay, obtaining $\Gamma\approx2.5$ Mev.

The usual statistical-theory assumption that $\sigma_C(\epsilon)$ is independent of the state of excitation of the residual nucleus being bombarded in the inverse process is subject to question because it implies an eventual violation of the Wigner sum rule,[10] $\sum_c\gamma_{\lambda c}{}^2\lesssim 3\hbar^2/2Ma^2$. By considering this rule, an upper-limit estimate for Γ can be made which should not be exceeded by the estimate (B4). If in the alternative expression for the width, $\sum_c 2P_c\gamma_{\lambda c}{}^2$, the barrier factors are replaced by a mean value $(4Ma^2T/\hbar^2)^{\frac{1}{2}}$ corresponding to $l=0$ neutrons of energy $2T$, and the remaining sum replaced by the above sum rule, one concludes that

$$\Gamma\lesssim(36\hbar^2T/Ma^2)^{\frac{1}{2}}. \quad \text{(B5)}$$

For the example cited above, $\Gamma\lesssim 8.5$ Mev. It should probably be required that (B3) be considerably less than the right side of (B5) because the negative-energy channels, which do not contribute to the width $(P=0)$, are to be included in the sum rule. As the temperature T is expected to be a monotonically increasing function of the excitation energy, an energy will ultimately be reached where (B3) exceeds (B5). In the example, this energy would correspond to $T=4.4$ Mev.

In the usual statistical-theory applications, the temperature enters in an expression which refers to the product of the reduced width and the level spacing rather than to the level spacing itself. This is because $\sigma_C(\epsilon)$ is generally assumed to be independent of the state of excitation of the residual nucleus. Although the actual nuclear temperature is expected to increase monotonically with the excitation, the above considerations show that it may preferable to use a more nearly constant temperature to represent this product. There is indeed experimental evidence that the temperature referring to this product actually decreases with increasing excitation indicating that the $\sigma_C(\epsilon)$, or equivalently the $\langle\gamma_{\lambda c}{}^2\rangle_{Av}$, for the bombardment of excited nuclei decrease considerably with the excitation of these nuclei.[47]

[45] Claude Bloch, Phys. Rev. **93**, 1094 (1954). There is also expected to be an additional factor $\exp[-(J+\frac{1}{2})^2/2\sigma^2]$ in the dependence of the level densities on the spins J (and I). According to estimates made of the dispersion coefficient σ using the individual-particle model, Bloch finds significant deviations from the $2J+1$ dependence due to this factor for values of J as low as three and nuclear excitations less than 12 Mev.

[46] By substituting in this formula the estimate $T^2\approx 20E/A$, the energy unit being Mev, and $a=1.4\times10^{-13}A^{\frac{1}{3}}$ cm, D. C. Peaslee obtains $\Gamma\approx(4/3)E_CA^{-\frac{1}{3}}\exp(-B/T)$. Because of a misprint, a different formula was given in a publication by him [Phys. Rev. **86**, 269 (1952)]. As a result the $\bar{\Gamma}$ of his Table I are somewhat overestimated. However, the factors f were also overestimated so that the qualitative conclusions of his paper remain essentially unchanged. [D. C. Peaslee (private communication).]

[47] B. L. Cohen, Phys. Rev. **92**, 1245 (1953).