cm^{-1}, this being the value of k_F corresponding to a Fermi energy of 4 ev. Because the dependence of R on q is given roughly by $R \propto q^{-4}$, the uncertainty in the choice of q does lead to considerable uncertainty in R. Comparison of the theoretical and empirical values of q for sodium enables one to say that, for sodium at least, our estimate of R is not in error by more than a factor of 5.

By an approximate integration of (1) it is easily shown that the positron energy falls from 4 ev to 1 ev in about 3×10^{-15} sec, from 1 ev to 0.1 ev in about 2×10^{-13} sec and from 0.1 ev to 0.025 ev in about 3×10^{-12} sec. Since positrons are observed[3] to annihilate in metals with a lifetime of about 10^{-10} sec, most of them must be thermalized before annihilation.

The incompatibility of the fundamental assumptions of the time-dependent perturbation method found in reference 5 does not occur in this calculation.

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Electromagnetic Effects of Spin Wave Resonance in Ferromagnetic Metals

W. S. Ament and G. T. Rado
Naval Research Laboratory, Washington, D. C.
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It was shown experimentally by Rado and Weertman that under suitable conditions there is an observable effect of exchange interactions on the ferromagnetic resonance in metals. The present paper provides an electromagnetic theory of this "spin wave resonance" experiment and satisfactorily explains the exchange shift as well as the width and shape of the absorption line. A combined solution is obtained of Maxwell's equations and the equation of motion of the magnetization vector **M**, the latter equation including the exchange term due to the nonuniform orientation of **M** in the skin depth. It is shown that the triple refraction caused by the exchange effect necessitates the introduction of new boundary conditions. The final result, which is checked numerically and by an approximate calculation, is an expression for the measurable surface impedance and the "equivalent isotropic permeability" derived therefrom. This result is discussed and generalized, the properties of thermal spin waves in metals are briefly considered, and previous theories of exchange effects in ferromagnetic resonance are shown to be inadequate.

I. INTRODUCTION

It was recently shown by Rado and Weertman[1,2] (to be referred to as RW) that under suitable conditions the effects of exchange interactions on the ferromagnetic resonance of metals can be observed experimentally. Such effects had not been observed previously but their physical basis has long been known. In the skin depth of a ferromagnetic metal the orientation of the magnetization vector **M** is not uniform so that the effective exchange field is not parallel to **M**. Thus there exists an exchange torque which is, in principle, capable of modifying the motion of **M** and the nature of the ferromagnetic resonance. Following RW, we refer occasionally to such a modified ferromagnetic resonance as "spin wave resonance."

The available theories of exchange effects in ferromagnetic resonance are inadequate in two respects. First, they do not predict satisfactorily under what conditions such effects might actually be observable, so that RW had to choose their experimental conditions largely on the basis of physical considerations. Second, these theories do not provide a reliable quantitative description of the exchange effects, so that their use offers at most a qualitative guidance in the interpretation of the RW experiments.

The present work, which we reported briefly at an earlier date,[3] is an attempt to eliminate the theoretical inadequacies mentioned above by giving a consistent description of spin wave resonance on the basis of electromagnetic theory. Such a description should make it possible to account for the position, width, and shape of the resonance line, as well as to evaluate the important exchange factor A and the spectroscopic splitting factor g from the experimental results.

Basically, the electromagnetic problem treated in the present paper involves a combined solution of the equation of motion of **M** (the so-called "spin wave equation") and Maxwell's equations, the solution being required to satisfy two sets of boundary conditions. The first set represents the usual continuity conditions on the tangential components of **E** and **H**, and the second set represents some new conditions that are imposed by the (semi-classically described) exchange effect on certain derivatives of **M**. Specifically, we consider a ferromagnetic metal, possessing a conductivity σ and a saturation magnetization M_s, which is exposed to a saturating static magnetic field of magnitude H_z and to a microwave field of circular frequency ω. The measurable quantity we calculate is the surface

[1] G. T. Rado and J. R. Weertman, Phys. Rev. **94**, 1386 (1954); the symbol μ^2 appearing in the next to last paragraph of this reference is a misprint and should read μ_2.

[2] G. T. Rado and J. R. Weertman (to be published).

[3] W. S. Ament and G. T. Rado, Phys. Rev. **94**, 1411 (1954).

impedance Z, but we find it convenient to introduce another measurable quantity, called the "equivalent isotropic permeability" μ_{equ}, and to express Z in terms of μ_{equ}. Our final result, which can often be approximated by our Eq. (31), is a theoretical expression for μ_{equ} in terms of the known parameters ω, σ, M_s, and H_z, and the unknown parameters A, g, and λ. The relaxation frequency λ, which is a measure of whatever damping mechanism of unknown origin may exist, has been included in our calculation in order to permit a theoretical comparison of the effects of exchange and relaxation on the observed resonance line. However, in the experiments of RW the relaxation effects prove to be negligibly small, so that in their case there are only two unknown parameters, A and g.

In Sec. II we formulate the problem and carry the solution of the differential equations sufficiently far to obtain a secular equation for the propagation constant k in the metal. Since the existence of the exchange torque causes this equation to be cubic rather than linear in k^2, the calculation of the amplitudes of the newly introduced waves, and hence of the value of Z, requires the additional boundary conditions mentioned above and formulated in Sec. III. Using both sets of boundary conditions and an algebraic "bialternant method" which circumvents the necessity of actually solving the secular equation, we then proceed, in Sec. IV, to calculate Z and to derive our explicit but approximate analytical formula for μ_{equ}. In Sec. V, we discuss certain limiting cases of our final result, and briefly consider the method of curve matching used for extracting A and g from the experimental μ_{equ}. We do not compare our formula in detail with the measured μ_{equ} since such a comparison is given by RW, but we do compare our formula with the results of a rather accurate numerical calculation which we performed on a digital computer. Next, we briefly discuss various generalizations of the result of Sec. IV, including the effect of a curved rather than plane ferromagnetic sample, the case of oblique rather than normal incidence of the microwaves on the sample, and the effect of changing the type of damping term used in the equation of motion. In Sec. VI, we give a critical discussion of the theoretical work of other authors and show that it resulted in some incorrect conclusions.

Since in Sec. IV we omitted the lengthy algebraic details of our general solution, we give in Appendix A an outline of a simple alternative solution of the problem. This method is valid unless the value of H_z is in the immediate vicinity of a certain specified value. The result of this alternative solution verifies our approximate formula and provides additional physical insight into the problem. Finally, in Appendix B, we derive the dispersion law for thermal spin waves from our general secular equation, and show that the usual dispersion law is not modified in any essential way even though Maxwell's equations and the metallic conductivity have been taken into account. We then suggest that in microwave resonance work at not too high temperatures the concept of a magnetization vector is indeed justified even if exchange effects are important, as in the experiments of RW.

II. DIFFERENTIAL EQUATIONS

In the interior of a saturated ferromagnetic metal the propagation of microwaves is determined by Maxwell's equations and the equation of motion of $\mathbf{M}$. Putting $\mathbf{B}=\mathbf{H}+4\pi\mathbf{M}$, and using Gaussian units, we write Maxwell's equations in the form

$$\nabla\times\mathbf{E}=-(1/c)\partial(\mathbf{H}+4\pi\mathbf{M})/\partial t, \tag{1a}$$

$$\nabla\times\mathbf{H}=(4\pi\sigma/c)\mathbf{E}, \tag{1b}$$

because the high values of σ obtaining in metals permit us to neglect the displacement current compared to the conduction current at microwave frequencies. The equation of motion of $\mathbf{M}$, sometimes known as the spin wave equation, can be written in the form

$$(1/\gamma)\partial\mathbf{M}/\partial t=\mathbf{M}\times[\mathbf{H}+(2A/M_s^2)\nabla^2\mathbf{M} -(\lambda/\gamma M_s^2)\mathbf{M}\times\mathbf{H}], \tag{2}$$

where γ is given by $ge/2mc$, and the other quantities have been introduced in Sec. I. The quantity of the brackets of Eq. (2) is an effective magnetic field and contains the following contributions. The first term, $\mathbf{H}$, is the actual magnetic field and includes all demagnetizing fields; this term, as well as the effect of anisotropy, will be discussed later in this Section. The second term, which is proportional to the exchange factor (or "exchange stiffness constant") A, is the effective exchange field due to the nonuniformity in the orientation of $\mathbf{M}$. The third term, which is proportional to the relaxation frequency λ, is an effective field that represents phenomenologically the influence of any unknown damping mechanism. It may be noted that the $\mathbf{H}$ occurring in this particular term ought to be replaced by $[\mathbf{H}+(2A/M_s^2)\nabla^2\mathbf{M}]$ because the damping describes the approach of $\mathbf{M}$ to the *total* field. However, this correction is easily shown to be a second order effect in most cases so that we shall use the simpler expression given in Eq. (2).

Equation (2) was first obtained, in a slightly different form, by Landau and Lifshitz[4] and used for the study of domain wall motion. Neither their paper nor the more recent work explains the physical origin of the λ term, and it is not clear whether even the form of this term correctly represents all the observed relaxation phenomena. As mentioned in Sec. I, we include this term primarily to permit comparisons with the exchange term, and we postpone to Sec. V a discussion of the effect of an alternative (Bloch type) damping term. Concerning the nature of the exchange term, however,

[4] L. Landau and E. Lifshitz, Physik. Z. Sowjetunion 8, 153 (1935).

the situation is more satisfactory. Various authors[5] derived this term from the atomic model of a ferromagnet, expressing A in terms of the Weiss molecular field coefficient or the Bloch $T^{\frac{3}{2}}$-law coefficient, and Herring[6] estimated A on the basis of the energy band model of ferromagnetism. Since all the theoretical treatments of the A-term involve several uncertainties, it is well to realize that the form of this term follows from symmetry considerations and that the magnitude of A can be obtained from suitable experiments. It is for this latter purpose, of course, that the spin wave resonance experiment of RW and the present electromagnetic calculation were undertaken.

Returning to Eqs. (1) and (2), we now solve them for the case of a ferromagnetic metal in the form of a plane sample parallel to the xz plane, the air-metal boundary being at $y=0$. The static magnetic field H_z is taken to be along the z axis, because for reasons discussed by RW the case of a static field normal to the plane of the sample is not well suited for detecting exchange effects. The applied microwaves are assumed to be plane waves normally incident upon the xz plane, the tangential component of their magnetic vector being along the x axis. Some generalizations of this physical situation will be discussed in Sec. V.

We now decompose the fields into a static component and a microwave component, so that

$$\mathbf{M}=M_s\mathbf{i}_z+\mathbf{m}, \tag{3a}$$

$$\mathbf{H}=H_z\mathbf{i}_z+\mathbf{h}, \tag{3b}$$

$$\mathbf{E}=\mathbf{e}, \tag{3c}$$

where $\mathbf{i}_z$ is a unit vector along the z axis, and the microwave components $\mathbf{m}$, $\mathbf{h}$, and $\mathbf{e}$ are understood to be proportional to $\exp(i\omega t-ky)$ in the metal. We further assume that $|\mathbf{m}|/M_s$ and $|\mathbf{h}|/H_z$ are small compared to unity. As to demagnetizing effects, a static demagnetizing correction due to the shape of the sample is assumed to have been applied so that H_z is the static field inside the sample. Dynamic demagnetizing corrections, on the other hand, must not be applied explicitly because they will emerge from the solution given below. It should also be noted that in certain simple cases the effect of anisotropy can easily be taken into account, as is well known.[5] For a single crystal with a direction of easy magnetization along the z axis, for example, one simply adds to H_z the value $2|K_1|/M_s$, or $4|K_1|/3M_s$, depending on whether the first order anisotropy constant K_1 is positive or negative. In the first experiments of RW, of course, this problem does not arise because in their case K_1 is approximately zero.

[5] For a recent treatment and references to earlier work see C. Kittel, Revs. Modern Phys. **21**, 541 (1949); C. Herring and C. Kittel, Phys. Rev. **81**, 869 (1951).

[6] C. Herring, Phys. Rev. **85**, 1003 (1952); **87**, 60 (1952). See also the comments by E. P. Wohlfarth, Proc. Phys. Soc. (London) **A65**, 1053 (1952).

Eliminating $\mathbf{e}$ from Eqs. (1) in the usual way by taking the curl of Eq. (1b), we obtain with the use of Eqs. (3) and the above mentioned assumptions

$$(i\delta^2k^2/2)h_y\mathbf{i}_y-[1+(i\delta^2k^2/2)]\mathbf{h}=4\pi\mathbf{m}, \tag{4}$$

or in component form

$$[1+(i\delta^2k^2/2)]h_x+4\pi m_x=0, \tag{5a}$$

$$h_y+4\pi m_y=0, \tag{5b}$$

$$[1+(i\delta^2k^2/2)]h_z+4\pi m_z=0, \tag{5c}$$

where h_x, h_y, h_z, and m_x, m_y, m_z, are the scalar components of $\mathbf{h}$ and $\mathbf{m}$, respectively, $\mathbf{i}_y$ is a unit vector along the y direction, and

$$\delta=(c^2/2\pi\omega\sigma)^{\frac{1}{2}} \tag{6}$$

is the classical skin depth for permeability unity. Equation (5b) is seen to express the Kittel[7] demagnetizing effect without the explicit introduction of a demagnetizing factor.

Next we combine Eqs. (2) and (3), obtaining with the same assumptions

$$(i\omega/\gamma)\mathbf{m}=\mathbf{i}_z\times\{M_s\mathbf{h}-[H_z-(2Ak^2/M_s)]\mathbf{m}\} \\ -(\lambda/\gamma M_s)(H_z\mathbf{m}_{x,y}-M_s\mathbf{h}_{x,y}), \tag{7}$$

where $\mathbf{h}_{x,y}$ and $\mathbf{m}_{x,y}$ are the vector components of $\mathbf{h}$ and $\mathbf{m}$, respectively, in the x, y plane. In component form, Eq. (7) becomes

$$(i\omega/\gamma)m_x=-M_sh_y+[H_z-(2Ak^2/M_s)]m_y \\ -(\lambda/\gamma M_s)(H_zm_x-M_sh_x), \tag{8a}$$

$$(i\omega/\gamma)m_y=M_sh_x-[H_z-(2Ak^2/M_s)]m_x \\ -(\lambda/\gamma M_s)(H_zm_y-M_sh_y), \tag{8b}$$

$$m_z=0. \tag{8c}$$

It is seen that Eqs. (5) and (8) have a solution for which $h_x=h_y=\mathbf{m}=0$, and $h_z\neq 0$. This wave is not excited by the assumed incident field, and we shall henceforth be concerned only with those waves for which $h_z=0$. If h_y is eliminated by means of Eq. (5b), then Eqs. (8a), (8b), and (5a) constitute a system of three linear homogeneous equations for the unknowns m_y, m_x, and h_x. Introducing the dimensionless parameters,

$$\eta=H_z/(4\pi M_s), \tag{9a}$$

$$\Omega=\omega/(4\pi M_s\gamma), \tag{9b}$$

$$L=\lambda/(M_s\gamma), \tag{9c}$$

$$\epsilon^2=A/(2\pi M_s^2\delta^2), \tag{9d}$$

$$K=k\epsilon\delta, \tag{9e}$$

into these three equations, we obtain

$$(K^2-1-\eta)m_y+(i\Omega+L\eta)m_x-(L/4\pi)h_x=0, \tag{10a}$$

$$-(i\Omega+L\eta+L)m_y+(K^2-\eta)m_x+(1/4\pi)h_x=0, \tag{10b}$$

$$-8\pi i\epsilon^2m_x+(K^2-2i\epsilon^2)h_x=0. \tag{10c}$$

[7] C. Kittel, Phys. Rev. **71**, 270 (1947).

In order that Eqs. (10) possess a nonvanishing solution, the determinant of the coefficients must vanish. This requirement leads to the secular equation

$$K^6 - c_1K^4 + c_2K^2 - c_3 = 0, \tag{11}$$

where c_1, c_2, c_3 are given by

$$c_1 = 1 + 2\eta + 2i\epsilon^2, \tag{12a}$$

$$c_2 = \eta - \Omega^2 + i\Omega L + \eta L(2i\Omega + L) + \eta^2(1+L^2) + 4i\epsilon^2(1+\eta), \tag{12b}$$

$$c_3 = 2i\epsilon^2\{(1+\eta)^2 - \Omega^2 + L(1+\eta)[2i\Omega + L(1+\eta)]\}. \tag{12c}$$

Since the secular equation is cubic in K^2, there are three propagation constants k_1, k_2, k_3 (obtained from K_1, K_2, K_3) whose real part is positive; the corresponding three waves, propagating along $\mathbf{i}_y$, represent energy flow into the metal. (The other three waves, propagating along $-\mathbf{i}_y$, are of no physical interest because the metal sample is assumed to be very thick compared to the penetration depth of the microwaves, so that there are no reflected waves inside the metal.) For each of the three waves (K_1,K_2,K_3) the field components can be expressed in terms of h_x provided Eq. (11) is solved. The resulting relations can be written in compact form by affixing the subscript n $(n=1, 2, 3)$ to specify which of the three K-values is referred to. Using any two of the Eqs. (10) [although Eqs. (10a) and (10b) prove to be the most convenient], we thus obtain

$$m_{nx} = u_n h_{nx}, \tag{13}$$

$$m_{ny} = v_n h_{nx}, \tag{14}$$

where u_n and v_n, being abbreviations for certain known functions of the coefficients of Eqs. (10), evidently depend on K_n^2. Similarly, Eq. (1b) leads to

$$e_{nz} = (cK_n/4\pi\sigma\epsilon\delta)h_{nx}, \tag{15}$$

and Eqs. (5b) and (14) give

$$h_{ny} = -4\pi v_n h_{nx}. \tag{16}$$

III. BOUNDARY CONDITIONS

The observable electromagnetic properties of a metal are fully determined by specifying, in the case of linear polarization, the ratio of the tangential components of $\mathbf{e}$ and $\mathbf{h}$ at the air-metal boundary. We call this ratio the "surface impedance" Z, so that for our field configuration

$$Z = (e_z/h_x)_{y=0}. \tag{17}$$

The Z defined by Eq. (17) is dimensionless and would have to be multiplied by a factor having the dimensions of (velocity)$^{-1}$, such as $(4\pi/c)$, in order to have the dimensions of an impedance. However, the simple definition of Z given by Eq. (17) is adequate when used consistently.

In the absence of exchange effects the value of Z and the amplitude of the single wave existing in the metal can be calculated, as is well known, by using the boundary conditions satisfied by the tangential components of $\mathbf{e}$ and $\mathbf{h}$. For the present case these conditions are

$$h_x \text{ continuous at } y=0, \tag{18}$$

$$e_z \text{ continuous at } y=0. \tag{19}$$

In the presence of exchange effects, however, two additional waves exist in the metal (see Sec. II), so that the calculation of Z and of the amplitudes of the three waves requires two new boundary conditions. These new conditions, formulated below, are evidently a consequence of exchange effects.

As discussed in connection with Eq. (2), the effective exchange field is

$$\mathbf{H}_{\text{ex}} = (2A/M_s^2)\nabla^2\mathbf{M} = (2A/M_s^2)\partial^2\mathbf{m}/\partial y^2,$$

an expression which is usually derived[5] from the Dirac cosine coupling between neighboring spins. Since this coupling implies that the torque exerted by the ith spin on the jth spin is equal and opposite to the torque exerted by the jth spin on the ith spin, it follows that the total exchange torque inside the specimen vanishes. Consequently, the total exchange torque per unit area of air-metal boundary is given by

$$\int_0^\infty \mathbf{M}\times\mathbf{H}_{\text{ex}}dy = 0,$$

where the upper limit of integration is taken as infinity since the sample is assumed to be much thicker than the skin depth. Integrating by parts, we thus obtain

$$(2A/M_s^2)[\mathbf{M}\times\partial\mathbf{m}/\partial y]_0^\infty = 0,$$

which leads (with $|\mathbf{m}|\ll M_s$) to

$$(2A/M_s^2)M_s\mathbf{i}_z\times(\partial\mathbf{m}/\partial y)_{y=0} = 0,$$

because $(\partial\mathbf{m}/\partial y)_{y=\infty}$ evidently vanishes. Since A is not zero, and $\mathbf{m}$ is always perpendicular to $\mathbf{i}_z$, we obtain the new boundary conditions

$$(\partial m_x/\partial y) = 0 \quad \text{at} \quad y=0, \tag{20}$$

$$(\partial m_y/\partial y) = 0 \quad \text{at} \quad y=0, \tag{21}$$

which we shall use for calculating Z.

It should be noted that in the arguments leading up to Eqs. (20) and (21), we have omitted any explicit consideration of the exchange field at the air-metal boundary. But at this boundary the exchange field is not equal to the $\mathbf{H}_{\text{ex}}$ used above because at $y=0$ the $-y$ direction is not equivalent to the $+y$ direction. In fact, we find by extending the usual derivation[5] of the effective exchange field that at the boundary

$$(\mathbf{H}_{\text{ex}})_{y=0} = (2A/M_s^2)[(f/a)(\partial\mathbf{m}/\partial y) + (\partial^2\mathbf{m}/\partial y^2)]_{y=0}, \tag{22a}$$

where f is a numerical factor of order unity that depends on the lattice type, and a is the lattice spacing. Thus the total exchange torque per unit area of air-metal boundary is given by

$$(2A/M_s^2)\left[\mathbf{M}\times(f/a)(a/f')(\partial\mathbf{m}/\partial y)_{y=0}+\int_0^\infty \mathbf{M}\times(\partial^2\mathbf{m}/\partial y^2)dy\right]=0, \quad (22b)$$

where f' is a numerical factor of order unity which is defined in such a way that (a/f') is half the separation between neighboring spins along the y direction. (For a b.c.c. lattice, $f=f'=4$.) While Eq. (22b) evidently leads to the same boundary conditions, Eqs. (20) and (21), which we derived above, it should be noted that the first term in the brackets involves the assumption that the exchange torque in a slab of area unity and thickness (a/f') may be expressed in terms of the mean exchange torque density in the slab. This assumption is admittedly questionable, but we believe it to be equivalent to the usual "continuum hypothesis" which is implied whenever $\mathbf{H}_{\text{ex}}$ is described by a differential expression. This continuum hypothesis asserts that the point lattice of electron spins envisaged in Heisenberg's model of ferromagnetism may legitimately be replaced by a continuum for the purpose of calculating $\mathbf{H}_{\text{ex}}$. As long as the effective skin depth is large compared to the lattice spacing a, the continuum hypothesis as well as our boundary condition is probably a good approximation.

Using the values of the microwave components given by Eqs. (13), (14), and (15), we can now write down the boundary condition equations (denoted by primes) which correspond to Eqs. (18), (19), (20), and (21). Introducing the abbreviation

$$Z'=(4\pi\sigma\epsilon\delta/c)Z, \quad (23)$$

and taking into account all three waves in the metal, we thus obtain

$$h_{1x}+h_{2x}+h_{3x}=h_{0x}, \quad (18)'$$

$$K_1h_{1x}+K_2h_{2x}+K_3h_{3x}=Z'h_{0x}, \quad (19)'$$

$$u_1K_1h_{1x}+u_2K_2h_{2x}+u_3K_3h_{3x}=0, \quad (20)'$$

$$v_1K_1h_{1x}+v_2K_2h_{2x}+v_3K_3h_{3x}=0, \quad (21)'$$

where h_{0x} denotes the value of h_x in air. This system of four linear homogeneous equations possesses a nonvanishing solution provided the condition

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ Z' & K_1 & K_2 & K_3 \\ 0 & u_1K_1 & u_2K_2 & u_3K_3 \\ 0 & v_1K_1 & v_2K_2 & v_3K_3 \end{vmatrix}=0 \quad (24)$$

is satisfied. Equation (24) will be used in the following section to calculate Z' and hence Z.

IV. THE SURFACE IMPEDANCE AND THE EQUIVALENT ISOTROPIC PERMEABILITY

Using Eq. (24), we could now express Z' in terms of known quantities if the secular equation (11) had actually been solved. While Eq. (11) could, in principle, be solved in closed form, such a solution would be rather cumbersome and not very useful. We shall circumvent this difficulty by making use of the fact that Z' can be represented in terms of certain symmetric functions of the roots (K_1,K_2,K_3) of Eq. (11). To see this, notice that the coefficients of the elements 1, Z' of the first column of Eq. (24) are *antisymmetric* polynomial functions of K_1, K_2, K_3. All such functions have a common antisymmetric factor, the remaining symmetric factors being directly expressible, through the theory of bialternants,[8] in terms of certain symmetric functions P, Q, R, which will be defined later. This fact so simplifies the algebra that we shall use the term "bialternant method" to denote the present procedure of bypassing the explicit solution of the secular polynomial.

Solving Eq. (24) for Z' and inserting the explicit expressions (which we have not written down) for the u_n and v_n in terms of the K_n^2, we obtain after a lengthy calculation

$$Z'=\frac{R(QP-R)}{RP+R(1+2\eta)+Qd}, \quad (25)$$

where P, Q, R denote the symmetric functions

$$P=K_1K_2+K_2K_3+K_3K_1, \quad (26a)$$

$$Q=K_1+K_2+K_3, \quad (26b)$$

$$R=K_1K_2K_3, \quad (26c)$$

and d is an abbreviation for the quantity

$$d=\eta-\Omega^2+i\Omega L+\eta(\eta+2i\Omega L+L^2+L^2\eta). \quad (27)$$

Next we make use of the well-known fact that the roots of the cubic equation (11) are related to its coefficients by the equations

$$K_1^2+K_2^2+K_3^2=c_1, \quad (28a)$$

$$K_1^2K_2^2+K_2^2K_3^2+K_3^2K_1^2=c_2, \quad (28b)$$

$$K_1^2K_2^2K_3^2=c_3. \quad (28c)$$

From Eqs. (26) and (28), we now obtain

$$Q^2=c_1+2P, \quad (29a)$$

$$P^2=c_2+2c_3^{\frac{1}{2}}Q, \quad (29b)$$

$$R=c_3^{\frac{1}{2}}, \quad (29c)$$

where the sign of $c_3^{\frac{1}{2}}$ must be taken as positive because we are only interested in waves whose propagation constant has a positive real part. Finally, we could (in

[8] A. C. Aitken, *Determinants and Matrices* (Oliver and Boyd, Edinburgh and London, 1951); see especially p. 117.

principle) solve the simultaneous quadratic equations (29a) and (29b) for P and Q, substitute the resulting values [together with the R from Eq. (29c)] into Eq. (25), and thus calculate Z' or Z. However, we shall not carry out these steps in full generality because it is more convenient to use the analytic approximation discussed below.

Once Z is calculated, the comparison with experimental results is most easily carried out by introducing the concept of "equivalent isotropic permeability," denoted by μ_{equ}. Following RW, we define μ_{equ} to be that isotropic complex permeability ($\mu_{\text{equ}}=\mu_1-i\mu_2$) which gives rise to the same surface impedance as the actual relation obtaining between the vectors $\mathbf{b}$ and $\mathbf{h}$. Since it is easily shown that in an isotropic situation, characterized by $\mathbf{b}=\mu_{\text{equ}}\mathbf{h}$, Maxwell's equations for a metal lead to $Z=(\mu_{\text{equ}}/\epsilon_{\text{eff}})^{\frac{1}{2}}$, where $\epsilon_{\text{eff}}=-4\pi i\sigma/\omega$ is the effective dielectric constant, it follows that

$$\mu_{\text{equ}}=-2i(cZ/\omega\delta)^2=-(i/2)(Z'/\epsilon)^2, \tag{30}$$

where the quantity ϵ, defined by Eq. (9d), should not be confused with ϵ_{eff}. The μ_{equ} calculated from Eq. (30), which we may call $(\mu_{\text{equ}})_{\text{calc}}$, can be compared with the results of resonance experiments by deducing from the latter a value for $(\mu_{\text{equ}})_{\text{exper}}$. To do this, one simply interprets the measured quality factor and resonance frequency of a cavity (or attenuation factor and phase velocity of a transmission line) on the basis of Maxwell's equations by proceeding as if $\mathbf{b}$ and $\mathbf{h}$ were related by $\mathbf{b}=(\mu_{\text{equ}})_{\text{exper}}\mathbf{h}$, and then compares $(\mu_{\text{equ}})_{\text{exper}}$ with $(\mu_{\text{equ}})_{\text{calc}}$.

Returning to the problem of actually calculating Z', and hence μ_{equ}, by the method outlined above, we now restrict ourselves to the case where each of the quantities η, Ω^2, L^2, ΩL, and ϵ is negligible compared to unity. With these approximations, which are valid in the RW experiments, the "coupling constant" $c_3^{\frac{1}{2}}$ of Eqs. (29a) and (29b) is quite small and permits an approximate solution of these simultaneous equations. Equations (29), (25), and (30) then lead to our final result:

$$\mu_{\text{equ}}=\frac{\eta-\Omega^2+i\Omega L+2\epsilon(1+i)}{[\eta-\Omega^2+i\Omega L+\epsilon(1+i)]^2}, \tag{31}$$

where η, Ω, L, and ϵ are defined by Eqs. (9a) through (9d).

V. DISCUSSION AND GENERALIZATION OF THE RESULT

If we disregard the exchange effect, so that A and ϵ vanish, then Eq. (31) gives

$$\mu_{\text{equ}}=1/(\eta-\Omega^2+i\Omega L), \tag{31a}$$

a special result that can be derived without using the methods of the present paper. To do this, one simply ignores Maxwell's equations [except for the demagnetizing condition (5b)], solves the equation of motion [Eq. (2)] subject to the approximations noted in the last paragraph, calculates the quantity $\mu_x=b_x/h_x$, and identifies μ_x with μ_{equ}. The result thus obtained, which is identical with Eq. (31a), shows that in this special case the line width and shape are essentially determined by λ, and that the resonance condition is $\eta=\Omega^2$. Since in our work $\eta=H_z/(4\pi M_s)$ is neglected compared to unity, the condition $\eta=\Omega^2$ is evidently equivalent to Kittel's formula,[7] $\omega=\gamma(B_zH_z)^{\frac{1}{2}}$, for the resonance condition in the absence of exchange.

If, on the other hand, we disregard the phenomenological damping effect, so that λ and L vanish, then Eq. (31) gives

$$\mu_{\text{equ}}=\frac{\eta-\Omega^2+2\epsilon(1+i)}{[\eta-\Omega^2+\epsilon(1+i)]^2}. \tag{31b}$$

Equation (31b) shows that the line width and shape are essentially determined by σ and A, i.e., by the combined effect of eddy current dissipation and exchange. The resonance field, defined as the H_z corresponding to $\mu_1=0$, is now given by $\eta\approx\Omega^2-(0.7044)\epsilon$, thus being shifted to a value smaller than that predicted by Eq. (31a). We note that Eq. (31b) proves to be a fairly good representation of the experimental results of RW, who observed a fractional exchange shift of the resonance field amounting to 20 or 30 percent, and we refer to their papers for numerical evaluations of Eq. (31b) and for a detailed comparison between theory and experiment.

Next we outline the method of curve matching used to extract A and g from the experimental μ_{equ}. Introducing the quantity

$$N=(\epsilon+\eta-\Omega^2)/\epsilon, \tag{32}$$

we write Eq. (31b) in the form

$$\epsilon\mu_{\text{equ}}=(N+1+2i)/(N+i)^2, \tag{33}$$

so that if we regard the real number N as a parameter, we can construct a universal curve by plotting the imaginary part of the right-hand side of Eq. (33) as a function of the real part. The resulting curve is "egg-shaped" and contains points corresponding to all possible values of N. Next we multiply all the experimental μ_{equ} by a scale factor, to be identified with ϵ, which is chosen in such a way that the product $\epsilon(\mu_{\text{equ}})_{\text{exper}}$, when plotted with its imaginary part as a function of its real part, matches the theoretical egg-shaped curve described above. This value of ϵ determines A [see Eq. (9d)] because M_s is known and δ can be calculated from ω and σ. Finally, we obtain g from any convenient point on the egg-shaped curve. To do this, we simply choose some value of H_z, compute the corresponding $\epsilon(\mu_{\text{equ}})_{\text{exper}}$ from the ϵ determined above and the experimental data, and then ascertain from the egg-shaped curve the value of N corresponding to this particular $\epsilon(\mu_{\text{equ}})_{\text{exper}}$. Knowing N and H_z, we then compute Ω from Eq. (32), and γ (and hence g) from Eq. (9b).

It is worth noting in this connection that if exchange effects are absent, then Eq. (31a) shows that a plot of the imaginary part of μ_{equ} as a function of its real part should be a circle tangent to the real axis at the origin. Thus the appearance of an egg-shaped curve in the complex-plane representation of $(\mu_{\text{equ}})_{\text{exper}}$ immediately suggests that the "zero-exchange" formula (31a) is inadequate, so that Eqs. (31b) or (31) must be considered in interpreting the experiments.

To assess the error caused by the analytical approximation involved in our final result, Eq. (31), we checked this equation by two methods. In the first method, we used a digital computer, the NAREC electronic computer at the Naval Research Laboratory, to solve Eqs. (29a) and (29b) numerically by a cyclic procedure of successive approximations. The numerical values of Ω, L, and ϵ chosen for this purpose were typical of those encountered in the RW experiments, and H_z was regarded as a parameter. The values of P and Q thus obtained, together with the R from Eq. (29c), were then substituted into Eqs. (25) and (30) to yield computed values of μ_{equ} as a function of H_z. When plotted in the complex plane representation described above, these computed values of μ_{equ} led to an egg-shaped curve whose ordinates and abscissas agreed to better than five percent with the prediction of Eq. (31). In the second method, presented in Appendix A, we used a power series expansion which contains more stringent approximations than those involved in the derivation of Eq. (31). However, this method permits an analytic solution of the secular equation (11) and leads to a final result which is invalid for a narrow range of H_z values but agrees otherwise with Eq. (31).

Since our final result, Eq. (31), was derived for the somewhat specialized physical conditions assumed in Sec. II, we shall now discuss three ways of generalizing these conditions (to apply to the experimental situation of RW) without altering the validity of Eq. (31).

(1) Equation (31) presumes a plane sample. However, it is easily shown that this equation is equally valid for a curved sample provided $|1/k|$ is negligible compared to the radius of curvature r. This condition means roughly that the "effective" skin depth is small compared to r so that the space dependence of the waves in the metal is exponential. Thus Eq. (31) is valid, for example, in the case of the cylindrical geometry used in the experiments of RW. In that case the proof involves the replacement of the Bessel functions appearing in the problem by their asymptotic values, as in the isotropic[9] situation.

(2) Equation (31) presumes normal incidence of the microwaves upon the sample. However, in the RW experiments (and in most ferromagnetic resonance experiments) the propagation vector possesses a component parallel to the surface of the sample, and it is in fact from measured effects resulting from this parallel component, such as the change due to H_z of the quality factor and resonance frequency of a cavity resonator, that the surface impedance and hence μ_{equ} is determined experimentally. We must therefore investigate whether the oblique incidence of the microwaves upon the sample modifies Eq. (31). To do this, we assume that the microwave components $\mathbf{m}$, $\mathbf{h}$, and $\mathbf{e}$ are proportional to $\exp(i\omega t-ky-pz)$, thus adding a z-dependence, and repeat the calculation leading to Eqs. (5) and (8). We find that if we make the approximation (generally valid at microwave frequencies) that $|p^2|$ is negligible compared to $|k^2|$, and $|i\delta^2p^2/2|$ is negligible compared to unity, then Eqs. (5a), (5b), and (8) remain unchanged. Equation (5c), however, no longer predicts $h_z=0$, but leads instead to the relation

$$h_z=\frac{(i\delta^2k^2/2)}{1+(i\delta^2k^2/2)}\frac{p}{k}h_y. \tag{34}$$

Since the absolute magnitude of the first factor on the right-hand side of Eq. (34) is at most of order unity, we obtain $|h_z|\lesssim|h_yp/k|$, showing that h_z is negligibly small compared to h_y. But since h_y is of the same order as the other "old" field components, we have effectively $h_z\approx 0$. This result is a consequence of the fact that the wavelength in the metal, which is of the order of the "effective" skin depth $|1/k|$, is negligibly small compared to the wavelength in the airspace bounded by the metal, which is of the order of $|1/p|$ or $\sim 2\pi c/\omega$. Thus we see that in the approximation considered here, all of the Eqs. (5) and (8) are unchanged, and consequently the remaining theory and the final result in the case of oblique incidence is the same as in the case of normal incidence.

(3) Equation (31) presumes that the unknown damping mechanism is described by the phenomenological Landau-Lifshitz damping term. As mentioned in Sec. II, however, the validity of this term is by no means assured by existing experimental results, and some ferromagnetic resonance experiments[10] indicate, in fact, that the Bloch-type phenomenological damping term is sometimes preferable. To investigate the effect of Bloch-type damping on our result, we simply replace the last term in Eq. (7) (which is proportional to λ) by the simpler term $-\mathbf{m}_{x,y}/(\gamma T_2)$, where the quantity T_2, known as the transverse relaxation time, is (like λ) a phenomenological constant. Thus we omit throughout our calculation all terms arising from the term $\lambda\mathbf{h}_{x,y}/\gamma$ of Eq. (7), and replace λ by $M_s/(H_zT_2)$ in the remaining terms. The final result obtained in this way turns out to be identical with Eq. (31b), so that the replacement of the Landau-Lifshitz damping by the Bloch damping leads to a replacement of Eq. (31) by Eq. (31b). This means that in our approximation the Bloch damping can be described by putting $\lambda=0$ in Eq. (31). Since the

[9] M. H. Johnson and G. T. Rado, Phys. Rev. **75**, 841 (1949).

[10] J. A. Young, Jr., and E. A. Uehling, Phys. Rev. **90**, 990 (1953); **94**, 544 (1954).

experiments of RW can be interpreted on the basis of $\lambda=0$, i.e., on the basis of Eq. (31b), the question of Landau-Lifshitz damping *versus* Bloch damping does not arise in their case.

VI. DISCUSSION OF PREVIOUS THEORIES

Kittel and Herring[11] were the first to calculate an explicit magnitude for the exchange shift of the resonance. However, they used a perturbation method and did not take Maxwell's equations properly into account. As discussed by RW, the formula of Kittel and Herring can be used for rough qualitative purposes but not for a quantitative prediction of the exchange shift. This is due to the fact that the Kittel-Herring formula contains the unknown factor $(\mu_2)_{\mathrm{unp}}$, the imaginary part of the "unperturbed" permeability. It is clearly a task of the theory to predict $(\mu_2)_{\mathrm{unp}}$ from the fundamental constants of the material, and in the absence of such a prediction $(\mu_2)_{\mathrm{unp}}$ is not known *a priori* unless it is taken from experimental results. But since the exchange effect cannot be "switched off," the experimental μ_2 automatically includes the exchange effect, and is therefore a "perturbed" μ_2, so that the Kittel-Herring perturbation treatment is not strictly valid unless the exchange effect is so small that it is experimentally uninteresting. We derived the Kittel-Herring result from our final equation (31) by a power series expansion, and investigated under what condition this derivation is valid. We found that to obtain their formula we had to assume that the quantity $|\epsilon/(\eta-\Omega^2+i\Omega L)|$ is small compared to unity (which is generally not a permissible assumption), and that we had to carry the expansion to the second approximation in this quantity. Furthermore, we had to assume that λ is not zero, an assumption which is particularly serious because the exchange effect must evidently be calculable for $\lambda=0$, as shown by our Eq. (31b). It should also be noted that Kittel and Herring have not calculated the width and shape of the resonance line, so that their theory cannot be used to decide whether the width of an observed resonance line is due to eddy current losses (caused by the exchange effect) or relaxation phenomena. Finally, Kittel and Herring concluded that at microwave frequencies the exchange effects in ferromagnetic resonance are not likely to be of importance in pure metals at room temperature, or in alloys at any temperature.

In connection with a general discussion of internal fields in ferromagnetics, Macdonald[12] briefly refers to his unpublished calculations on exchange effects in ferromagnetic resonance and states that he agrees with the conclusion of Kittel and Herring mentioned above. Since this conclusion was contradicted by the RW experiments, we undertook the calculations of the present paper and found that the conclusion of Kittel and Herring, and of Macdonald, is not justified. After the oral presentation[3] of our work, Dr. Macdonald kindly lent us his thesis[13] which contains his calculations, and pointed out that he had independently arrived at similar methods and the same new boundary conditions [our Eqs. (20) and (21)] as we did. We therefore believe that Macdonald's conclusion concerning the inappreciable magnitude of the exchange effect at room temperature is probably due to the complicated nature of his implicit final formulas, and to the fact that his numerical computations were limited to those conditions, such as the resonance in nickel at $\sim$30 000 Mc/sec, where the exchange effect is indeed very small. Our final result [Eq. (31)], on the other hand, admittedly lacks generality, but it is an explicit and useful formula that permits simple predictions within its range of applicability. Furthermore, the applicability of our Eq. (31) extends to just those situations in which physical considerations, discussed by RW, lead one to expect that the exchange effect is actually appreciable.

[11] C. Kittel and C. Herring, Phys. Rev. **77**, 725 (1950).

[12] J. R. Macdonald, Proc. Phys. Soc. (London) **A64**, 968 (1951).

APPENDIX A. AN APPROXIMATE METHOD OF SOLUTION

As in Sec. IV, in the paragraph preceding Eq. (31), we again assume that each of the quantities η, Ω^2, L^2, and ΩL is negligible compared to unity. However, instead of assuming that ϵ is negligible compared to unity, we now make the more stringent assumption that $\epsilon^2/|\eta-\Omega^2+i\Omega L|$ is negligible compared to unity. With these approximations, the secular equation (11) becomes

$$K^6-K^4+(\eta-\Omega^2+i\Omega L)K^2-2i\epsilon^2=0, \tag{A1}$$

and yields the approximate roots

$$K_{1,2}{}^2=\tfrac{1}{2}\{(\eta-\Omega^2+i\Omega L)\pm[(\eta-\Omega^2+i\Omega L)^2-8i\epsilon^2]^{\frac{1}{2}}\}, \tag{A2a}$$

$$K_3{}^2=1. \tag{A2b}$$

From Eq. (10c), we now obtain (with $n=1, 2, 3$)

$$m_{nx}=(K_n{}^2/8\pi i\epsilon^2)h_{nx}, \tag{A3}$$

and from Eqs. (10b) and (A3)

$$m_{ny}=\frac{(K_n{}^2-\eta)K_n{}^2+2i\epsilon^2}{8\pi i\epsilon^2(i\Omega+L)}h_{nx}. \tag{A4}$$

If Eqs. (A3) and (A4), which correspond to Eqs. (13) and (14), are substituted into Eq. (10a), the latter yields Eq. (A1) and is therefore identically satisfied. In a similar way, we obtain an approximate expression for e_{nz}. Next we write down the boundary condition determinant (24), substitute for the u_n and v_n from Eqs. (A3) and (A4), add a certain multiple of the fourth row to a certain multiple of the third row, and

[13] J. R. Macdonald, Ph.D. Thesis, Oxford, 1950 (unpublished).

obtain

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ Z' & K_1 & K_2 & K_3 \\ 0 & (K_1^4+2i\epsilon^2)K_1 & (K_2^4+2i\epsilon^2)K_2 & (K_3^4+2i\epsilon^2)K_3 \\ 0 & K_1^3 & K_2^3 & K_3^3 \end{vmatrix} = 0. \tag{A5}$$

Since in our approximation we can write

$$K_1^4+2i\epsilon^2+1\approx 1,$$
$$K_2^4+2i\epsilon^2+1\approx 1,$$
$$K_3^4+2i\epsilon^2+1\approx 2,$$

Eq. (A5) becomes

$$\begin{vmatrix} 1 & 1 & 1 \\ Z' & K_1 & K_2 \\ 0 & K_1^3 & K_2^3 \end{vmatrix} = 0. \tag{A6}$$

If we now disregard the case $K_1=K_2$, which will be discussed later, Eq. (A6) yields

$$Z'=\frac{K_1K_2(K_1+K_2)}{K_1^2+K_1K_2+K_2^2}, \tag{A7}$$

which can be combined with Eqs. (30) and (A2a) to give the final result:

$$\mu_{\text{equ}}=\frac{\eta-\Omega^2+i\Omega L+2\epsilon(1+i)}{[\eta-\Omega^2+i\Omega L+\epsilon(1+i)]^2}, \tag{A8}$$

in agreement with Eq. (31).

To analyze the limitations of this simple derivation we now distinguish two cases. (1) If $\lambda=0$, then $K_1\neq K_2$, but the quantity $\epsilon^2/|\eta-\Omega^2+i\Omega L|$ becomes very large if H_z is such that η satisfies $\eta\approx\Omega^2$, so that Eq. (A8) is not valid in the immediate vicinity of this H_z. (2) If $\lambda\neq 0$, with λ being very small, then the same limitation exists as in case (1). But if $\lambda\neq 0$, with λ being arbitrary, then we have the additional limitation that Eq. (A8) is invalid if $K_1=K_2$. The latter situation arises if ω is such that Ω satisfies simultaneously the conditions $\eta-\Omega^2=2\epsilon$ and $L\Omega=2\epsilon$ obtained from Eq. (A2a).

The method given in this Appendix is useful because it leads to explicit expressions for the three propagation constants and because it permits a simple derivation of our final result for μ_{equ}. All the limitations described in the previous paragraph apply, of course, to the *method* used to derive Eq. (A8) and not to this equation itself, since the same result, Eq. (31), was derived without these limitations by using the bialternant method of Sec. IV.

APPENDIX B. THE DISPERSION LAW FOR THERMAL SPIN WAVES

In considering thermally excited spin waves we can restrict ourselves to frequencies which are sufficiently high to satisfy the relation

$$\omega\sim k_BT/\hbar, \tag{B1}$$

where k_B, T, and $\hbar$ denote Boltzmann's constant, the absolute temperature, and (Planck's constant/2π), respectively. We consider, moreover, only the type of linear polarization assumed in Sec. II. Equation (B1) shows that Ω is now very large and thus permits us to write the secular equation (11) in the simplified form

$$K^6-K^4-\Omega^2K^2+2i\epsilon^2\Omega^2=0 \tag{B2}$$

for any typical ferromagnetic metal at all but extremely low temperatures. The approximate solutions of Eq. (B2) are

$$K_1^2=2i\epsilon^2, \tag{B3}$$

$$K_{2,3}^2=\pm\Omega. \tag{B4}$$

From the definition of K, Eq. (9e), it is seen that Eq. (B3) gives

$$k^2=2i/\delta^2, \tag{B5}$$

which is precisely what one obtains from Maxwell's equations for permeability unity. Consequently the spin wave corresponding to this k is "nonmagnetic," being characterized by $m_x=m_y=0$, so that it cannot give rise to a deviation of M_s from M_0, the value of M_s at $T=0$.

Equation (B4), on the other hand, shows that the wave corresponding to the plus sign is attenuated while that corresponding to the minus sign is not, and that k for the latter wave is given by

$$\omega=-(2A\gamma/M_s)k^2, \tag{B6}$$

which agrees with the usual dispersion law[14] for thermal spin waves and thus leads to the Bloch $T^{\frac{3}{2}}$-law for $(M_0-M_s)/M_0$. It is rather satisfying that Eq. (B6) agrees with the result of the standard treatments because the latter ignore the conductivity of the metal, and either neglect the magnetic interactions between the spins or treat them by magnetostatics rather than by Maxwell's equations. We further note that the thermal spin waves, which give rise to the difference between M_0 and M_s, are on the whole much shorter than the microwaves employed in resonance experiments. The wavelengths of the latter may be estimated from Eq. (A2a), which shows that in most situations, including the RW spin wave resonance experiments, the wavelengths of the microwaves in the metal will not be much smaller than 10^{-5} cm. Thus we suggest that at temperatures sufficiently far below the Curie point, where the spin wave picture is at least approximately valid, the concept of a saturation magnetization vector may legitimately be used not only in ferromagnetic resonance but even in spin wave resonance.

[14] See Herring and Kittel, reference 5.