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Fundamental Wave Functions in an Unbounded Magneto-Hydrodynamic Field. I. General Theory*

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This is the first of two papers dealing with a systematic study of the linearized, unbounded medium problems in magneto-hydrodynamics of incompressible and compressible fluids. Part I deals with the fundamental equations which are set up quite generally for an ideal, homogeneous and isotropic, conducting fluid devoid of viscosity and expansive friction, subject only to the initial assumption that the externally applied field of magnetic induction be constant and uniform. The energy and momentum balance in a magneto-hydrodynamic field is verified with the aid of the exact fundamental equations and the conservation laws of energy and momentum, for a rigid volume fixed in the (stationary) observer's inertial frame of reference, are displayed in differential and in integral form. By successive eliminations there is obtained a single partial differential equation in the particle velocity from which the unwanted second-order terms are merely dropped in a linearized small amplitude theory, a process which is fully justified by considering the special case of infinite conductivity, zero displacement current, and incompressible fluids. Also, assuming that a particular solution of the linearized magneto-hydrodynamic wave equation has been obtained, it is shown how to compute quite generally, from the linearized Maxwellian set, the accompanying electromagnetic field vectors expressed in terms of the assumed velocity field. These computations are carried out for plane homogeneous waves and for time-harmonic cylindrical waves. The actual determination of particular wave functions appropriate for incompressible and compressible fluids, together with the computation of the corresponding wave numbers, is reserved for the sequel to this paper, Part II.

1. INTRODUCTION

THE field of magneto-hydrodynamics, like hydrodynamics itself, is essentially nonlinear, for the interaction between a moving conducting fluid and the electromagnetic field also contains nonlinear terms. The importance of the new field, especially in cosmic physics, has been attested by a score of papers on various subjects such as solar physics, cosmic radiation, stellar oscillations, geomagnetism, propagation in an ionized atmosphere, etc., in which nonlinear phenomena are very much in evidence. A brief account of these researches, together with a complete bibliography, has been given by Lundquist[1] in an excellent review paper. In this paper, except where noted, we confine our attention to the important class of linearized problems which give rise to time harmonic magneto-hydrodynamic waves in compressible and incompressible fluids.

The underlying fundamental notions in the theory of magneto-hydrodynamic waves in an incompressible fluid were originally given by Alfvén[2] in the course of his researches on the theory of sun-spots. The theory of magneto-hydrodynamic waves was first considered in some detail by Walén[3] who set up the magneto-hydrodynamic equations starting with the principle of conservation of energy. Laboratory experiments in magneto-hydrodynamic waves in mercury have been reported by Lundquist,[4] and more recently using liquid sodium by Lehnert.[5]

Waves in an ionized gas in the presence of a magnetic field have been considered by Åström[6] and magneto-hydrodynamic waves in a compressible fluid of infinite conductivity have been studied by Herlofson.[7] A more systematic account of plane magneto-hydrodynamic waves including the effects of finite conductivity, viscosity, and compressibility of the medium is found in a paper by van de Hulst.[8] Time harmonic cylindrical waves in compressible and incompressible fluids have been considered by Lundquist.[9] Also, in a recent paper, Hines[10] develops some generalized magneto-hydrodynamic formulas, using an extension of the magneto-ionic approach, which are applicable where a purely macroscopic point of view is no longer tenable.

However, nowhere do we find a complete account of the fundamental equations without simplifying assumptions injected from the outset, nor a systematic analysis of the linearized, unbounded media, and boundary value problems in the field of magneto-hydrodynamics of incompressible and compressible fluids. In this paper, Part I, we propose to fulfill this need by first giving a detailed discussion of the general linearized theory of magneto-hydrodynamic phenomena in an unbounded, homogeneous and isotropic, conducting fluid embedded in a constant and uniform field of magnetic induction, and then examining in general the structure of plane homogeneous waves and of time-harmonic cylindrical waves. The application of the theory to the specific cases of incompressible and compressible fluids and the actual determination of the fundamental wave functions corresponding to all possible modes of propagation is reserved for the sequel to this paper, Part II.

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[1] S. Lundquist, Arkiv Fysik **5**, 297 (1952).
[2] H. Alfvén, Nature **150**, 405 (1942); Arkiv Mat. Astron. Fysik **B29**, No. 2 (1942). See also H. Alfvén, *Cosmical Electrodynamics* (Oxford University Press, London, 1950), Chap. 4.
[3] C. Walén, Arkiv. Mat. Astron. Fysik **30A**, No. 15 (1944).
[4] S. Lundquist, Phys. Rev. **76**, 1805 (1949).
[5] B. Lehnert, Phys. Rev. **94**, 815 (1954).
[6] E. Åström, Nature **165**, 1019 (1950).
[7] N. Herlofson, Nature **165**, 1020 (1950).
[8] H. C. van de Hulst, *Problems of Cosmical Aereodynamics* (Central Air Documents Office, Dayton, 1951), Chap. 6.
[9] Reference 1, Sec. C.
[10] C. O. Hines, Proc. Cambridge Phil. Soc. **49**, Part 2, 299–307 (1953).

2. FUNDAMENTAL EQUATIONS

The systematic study of the linearized, unbounded medium problems in magneto-hydrodynamics requires the simultaneous solution of the Maxwellian equations for a moving medium and the Eulerian equations of motion of the fluid in the presence of the ponderomotive force density of electromagnetic origin. To simplify the analysis from the outset, we assume a homogeneous and isotropic conducting fluid of infinite extent embedded in a uniform magnetic field. Furthermore, we consider only ideal fluids devoid of viscosity and expansive friction. Finally, since our approach is purely macroscopic, we do not consider phenomena in ionized gases at low densities.

2.1 Maxwellian Equations

First, we assume that the homogeneous and isotropic medium is characterized, in rationalized mks units, by the rigorously constant macroscopic parameters[11] μ, ϵ, and σ. Furthermore, we adopt here Minkowski's relativistic electrodynamics of moving bodies, according to which the Maxwellian set becomes[12]

$$\begin{array}{llll} \text{(I)} & \nabla\times\mathbf{e}+\mu(\partial\mathbf{h}/\partial t)=0, & \text{(III)} & \nabla\cdot\mathbf{h}=0, \\ \text{(II)} & \nabla\times\mathbf{h}-\epsilon(\partial\mathbf{e}/\partial t)=\mathbf{j}, & \text{(IV)} & \nabla\cdot\mathbf{e}=\eta/\epsilon, \\ & \text{(V)} \quad \mathbf{j}=\eta\mathbf{v}+\sigma(\mathbf{e}+\mathbf{v}\times\mathbf{B}), & & \\ & \text{(VI)} \quad \partial\eta/\partial t+\nabla\cdot\mathbf{j}=0, & & \end{array} \tag{1}$$

where $\mathbf{e}$ and $\mathbf{h}$ represent the electric and magnetic intensities of the induced field, $\mathbf{j}$ the current density in the medium, $\mathbf{v}$ the velocity of the fluid, and $\mathbf{B}$ the total magnetic induction prevailing at a point and representing the sum, $\mathbf{B}=\mathbf{B}_0+\mu\mathbf{h}$, of the externally applied field $\mathbf{B}_0$ (which is assumed constant and uniform throughout this investigation) and the induced field $\mu\mathbf{h}$. Equation (IV) defines the electric charge density by the unconventional symbol η, and (V) exhibits the current density vector as the sum of the convection current $\eta\mathbf{v}$ plus the conduction current $\sigma(\mathbf{e}+\mathbf{v}\times\mathbf{B})$. Finally, (VI) expresses the principle of conservation of charge (equation of continuity) that governs the behavior of the charge and current densities.

It must be clearly understood that, in (I–VI), all field vectors are referred to the (stationary) observer's inertial frame of reference in which the fluid is moving with the instantaneous velocity $\mathbf{v}=\mathbf{v}(\mathbf{r},t)$, where $\mathbf{r}$ is the position vector of the point of observation referred to the said inertial frame. Admittedly, the set (I–VI) applies rigorously only to uniformly moving bodies and its application to more complicated kinds of motion may be regarded at least as a first approximation. Fortunately, however, if the fluid motion is nonrelativistic ($v\ll c$) and if the accelerations produced by the electromagnetic forces are small, which is the case in the present instance, the set (I–VI) adequately describes the electromagnetic field associated with (slow moving) accelerated bodies.[13]

2.2 Eulerian Equations

For an ideal conducting fluid devoid of viscosity and expansive friction, the hydrodynamic equations of motion and the equation of continuity (conservation of mass) are

$$\begin{array}{ll} \text{(VII)} & \rho(d\mathbf{v}/dt)+\nabla p=\mathbf{f}; \\ \text{(VIII)} & \partial\rho/\partial t+\nabla\cdot(\rho\mathbf{v})=0, \end{array} \tag{2}$$

where ρ is the density of the fluid, p the hydrodynamic pressure, and $\mathbf{f}$ is the ponderomotive force density which, in the assumed absence of gravity, must be equated to the Lorentz force density acting on the charge and current distribution, i.e.,

$$\mathbf{f}=\eta\mathbf{e}+\mathbf{j}\times\mathbf{B}, \tag{3}$$

in which $\mathbf{e}$ is the electric intensity and $\mathbf{B}$ the *total* field of magnetic induction. In Euler's Eq. (VII), the total time derivative of the velocity is given by

$$d\mathbf{v}/dt=\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}=\partial\mathbf{v}/\partial t+\tfrac{1}{2}\nabla v^2+(\nabla\times\mathbf{v})\times\mathbf{v}, \tag{4}$$

in which the latter form is invariant.

2.3 Energy and Momentum Balance

The system of equations (I–VIII), together with Eq. (3), are sufficient to describe completely the behavior of a magneto-hydrodynamic field. In particular, it is instructive to verify from these equations, which of course apply to the complete nonlinear theory, that the phenomenon is governed by the laws of conservation of energy and momentum. To this end we consider first the power per unit of volume developed by the ponderomotive force density of electromagnetic origin; that is, introducing the conduction current density

$$\mathbf{J}=\mathbf{j}-\eta\mathbf{v}=\sigma(\mathbf{e}+\mathbf{v}\times\mathbf{B}), \tag{5}$$

we compute from (3) the power per unit of volume

$$\mathbf{f}\cdot\mathbf{v}=\eta(\mathbf{e}\cdot\mathbf{v})-\mathbf{j}\cdot(\mathbf{v}\times\mathbf{B})=-J^2/\sigma+\mathbf{j}\cdot\mathbf{e}, \tag{6}$$

in which the second form is obtained from the first by eliminating $\mathbf{v}\times\mathbf{B}$ with the aid of (5). Next, we replace $\mathbf{j}$ in the last form of (6) by the left member of (II) and, making use of (I), we obtain finally

$$\mathbf{f}\cdot\mathbf{v}=-J^2/\sigma-(\partial/\partial t)[\tfrac{1}{2}\epsilon e^2+\tfrac{1}{2}\mu H^2]-\nabla\cdot(\mathbf{e}\times\mathbf{H}), \tag{7}$$

which expresses the power per unit of volume in terms of the conduction current density $\mathbf{J}$ and of the electromagnetic field vectors $\mathbf{e}$ and $\mathbf{H}$, where $\mathbf{H}$ is the *total* magnetic intensity, $\mathbf{H}=\mathbf{H}_0+\mathbf{h}$.

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[11] In this paper we assume a fluid with the permeability and dielectric constant of vacuum, i.e., $\mu\epsilon=c^{-2}$. The extension of the theory to media having more general electric and magnetic properties can be readily made, but is not considered here.

[12] See, for example, R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, London, 1934), Sec. 52; C. Møller, *The Theory of Relativity* (Oxford University Press, London, 1952), Sec. 73.

[13] R. C. Tolman, reference 12, p. 101; C. Møller, reference 12, p. 200.

Proceeding similarly, we can also express the Lorentz force density itself in terms of the electromagnetic field vectors. Introducing for the purpose the *total* Maxwell's electromagnetic stress tensor[14]

$$\mathfrak{T}_t=\epsilon[\mathbf{ee}-\tfrac{1}{2}e^2\mathfrak{J}]+\mu[\mathbf{HH}-\tfrac{1}{2}H^2\mathfrak{J}], \tag{8}$$

where $\mathfrak{J}$ denotes the *idemfactor*, and taking its (tensor) divergence we have

$$\nabla\cdot\mathfrak{T}_t=\epsilon[\mathbf{e}\nabla\cdot\mathbf{e}+(\nabla\times\mathbf{e})\times\mathbf{e}] +\mu[\mathbf{H}\nabla\cdot\mathbf{H}+(\nabla\times\mathbf{H})\times\mathbf{H}], \tag{9}$$

with the aid of which we obtain directly from (3), making use of (II) and (IV) to eliminate respectively the current and charge densities, the desired expression for the Lorentz force density

$$\mathbf{f}=\nabla\cdot\mathfrak{T}_t-\mu\epsilon(\partial/\partial t)(\mathbf{e}\times\mathbf{H}), \tag{10}$$

in which there appear only the electromagnetic field vectors $\mathbf{e}$ and $\mathbf{H}$.

Energy Balance

To verify the energy balance, we take the scalar product of $\mathbf{v}$ and the vectors on both sides of (VII) to obtain

$$\rho\mathbf{v}\cdot(d\mathbf{v}/dt)+\mathbf{v}\cdot\nabla p=\mathbf{f}\cdot\mathbf{v}. \tag{11}$$

Next, making use of (4), we note that the first term above can be written as

$$\begin{aligned}\rho\mathbf{v}\cdot(d\mathbf{v}/dt)&=\rho\mathbf{v}\cdot(\partial\mathbf{v}/\partial t)+\tfrac{1}{2}\rho\mathbf{v}\cdot\nabla v^2\\&=(\partial/\partial t)(\tfrac{1}{2}\rho v^2)+\nabla\cdot(\tfrac{1}{2}\rho v^2\mathbf{v}),\end{aligned} \tag{12}$$

in which the latter form is deduced from (VIII). Similarly, the second term may be written as

$$\mathbf{v}\cdot\nabla p=\nabla\cdot(p\mathbf{v})-p\nabla\cdot\mathbf{v}=\nabla\cdot(p\mathbf{v})+(p/\rho)(d\rho/dt), \tag{13}$$

in which again the latter form is deduced with the aid of the equation of continuity.

Substituting into (11) the forms (12), (13), and (7) and transposing terms, we obtain finally the equation

$$\begin{aligned}(\partial/\partial t)(\tfrac{1}{2}\rho v^2)+(\partial/\partial t)(\tfrac{1}{2}\epsilon e^2+\tfrac{1}{2}\mu H^2)&=-J^2/\sigma-(p/\rho)(d\rho/dt)\\&\quad-\nabla\cdot[\mathbf{e}\times\mathbf{H}+(\tfrac{1}{2}\rho v^2+p)\mathbf{v}],\end{aligned} \tag{14}$$

which represents in differential form the conservation of energy in a magneto-hydrodynamic field per unit volume *fixed* in the observer's inertial frame of reference. In fact, the terms on the left side of (14) represent the time rate of increase of the total energy (kinetic plus electromagnetic) stored per unit volume, and this rate of increase must be accounted for by the terms on the right side. Thus, the term $-J^2/\sigma$ represents the rate of Joule heat *loss* per unit volume, which is an irreversible process, whereas the term $-(p/\rho)(d\rho/dt)$ may be interpreted as the (reversible) rate of doing work per unit volume associated with pressure fluctuations. And, finally, the divergence term on the right side of (14) is readily interpreted as the rate at which energy flows through the walls *into* the unit volume, the flow consisting of electromagnetic energy and total mechanical energy (kinetic plus potential). Equation (14) was used by Walén[15] as the starting point for his derivation of the magneto-hydrodynamic equations, except that Walén's equation, as written, is correct only for incompressible fluids ($\nabla\cdot\mathbf{v}=0$) and only if one replaces the total time derivative of the electromagnetic energy density by the partial derivative.

To clarify the above interpretation of the energy balance, suppose we multiply both sides of (14) by the element of volume $d\tau$ and integrate throughout a rigid volume fixed in the observer's inertial frame of reference. In this way, making use of the divergence theorem, we obtain

$$\begin{aligned}\frac{d}{dt}\int_V[\tfrac{1}{2}\rho v^2+(\tfrac{1}{2}\epsilon e^2+\tfrac{1}{2}\mu H^2)]d\tau&=-\int_V\left[\frac{J^2}{\sigma}+\frac{p}{\rho}\left(\frac{d\rho}{dt}\right)\right]d\tau\\&\quad-\int_S\mathbf{n}\cdot[\mathbf{e}\times\mathbf{H}+(\tfrac{1}{2}\rho v^2+p)\mathbf{v}]da,\end{aligned} \tag{15}$$

which expresses in integral form the conservation of energy for a fixed volume in a magneto-hydrodynamic field. Thus, the volume integral on the left represents the time rate of increase of the total (kinetic plus electromagnetic) energy stored within the fixed volume, and the volume integral on the right accounts, respectively, for the irreversible Joule heat loss throughout the volume and for the reversible rate of doing work associated with pressure fluctuations. And, finally, the surface integral on the right measures the time rate of influx of electromagnetic and total mechanical energy through the walls of the fixed volume.

Momentum Balance

To establish the momentum balance we need only refer to (VII), noting that the first term on the left may be written as

$$\begin{aligned}\rho(d\mathbf{v}/dt)&=\rho(\partial\mathbf{v}/\partial t)+\rho(\mathbf{v}\cdot\nabla)\mathbf{v}\\&=(\partial/\partial t)(\rho\mathbf{v})+\nabla\cdot(\rho\mathbf{vv}),\end{aligned} \tag{16}$$

where the latter form is deduced with the aid of (VIII). Then, replacing ∇p by $\nabla\cdot(p\mathfrak{J})$, where $\mathfrak{J}$ is the *idemfactor*, and substituting into (VII) the forms (16) and (10), we obtain after transposing terms

$$(\partial/\partial t)(\rho\mathbf{v})+\mu\epsilon(\partial/\partial t)(\mathbf{e}\times\mathbf{H}) =\nabla\cdot[\mathfrak{T}_t-(\rho\mathbf{v})\mathbf{v}-p\mathfrak{J}], \tag{17}$$

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[14] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1944), second edition, p. 7; see also J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), Secs. 2.5 and 2.6.

[15] C. Walén, Arkiv Mat. Astron. Fysik **A30**, No. 15, 2 (1944).

which represents in differential form the conservation of momentum in a magneto-hydrodynamic field per unit volume *fixed* in the observer's inertial frame of reference. In fact, the terms on the left side of (17) express the time rate of change of the total mechanical plus electromagnetic momentum contained in a unit volume and this must be equal to the force acting on the matter and the electromagnetic field within the unit volume as accounted for by the divergence term on the right. Thus, this force which acts through the walls of the unit volume is seen to consist of three terms: the electromagnetic stresses, the influx of matter carrying momentum, and the net force due to the pressure acting at right angles to the walls of the unit volume.

To gain further insight into the momentum balance, it is instructive to integrate both sides of (17) throughout a rigid volume fixed in the observer's inertial frame of reference. Thus, making use of the (tensor) divergence theorem, we obtain

$$\frac{d}{dt}\int_V [\rho\mathbf{v}+\mu\epsilon(\mathbf{e}\times\mathbf{H})]d\tau = \int_S \mathbf{n}\cdot\mathfrak{T}_t da - \int_S (\mathbf{n}\cdot\mathbf{v})\rho\mathbf{v}da - \int_S \mathbf{n}p da, \quad (18)$$

which expresses in integral form the conservation of momentum for a fixed volume in a magneto-hydrodynamic field. The volume integral on the left represents the time rate of change of the total mechanical plus electromagnetic momentum contained within the fixed volume and therefore must be equal to the total force acting on the matter and the electromagnetic field within the volume. This force is fully accounted for by the three surface integrals on the right of (18). The first surface integral denotes the force acting on the fixed volume which arises from the electromagnetic stresses across the bounding surface; the second surface integral accounts for the influx of matter carrying momentum across the walls of the fixed volume and may be interpreted as the force resulting from the impact of the moving fluid on the bounding surface; and the third surface integral is merely the net force acting on the fixed volume by virtue of the normal pressure on the walls of the volume.

2.4 Reduction to One Fundamental Equation

In order to solve a given magneto-hydrodynamic problem we would like to eliminate from the system (I–VIII) all but one of the dependent vector variables, but this is impossible in general because of the complexity of the equations and because of nonlinearity. However, it proves possible to obtain quite generally a single vector partial differential equation in the fluid velocity $\mathbf{v}$ in which there remain only a number of unwanted second-order terms that one eventually ignores in a linearized theory. To this end we first take the curl of (I) to obtain, making use of (II),

$$[\nabla^2-\mu\epsilon(\partial^2/\partial t^2)]\mathbf{e}=\mu(\partial\mathbf{j}/\partial t)+\nabla\nabla\cdot\mathbf{e}, \quad (19)$$

from which, making use of (V) to eliminate $\nabla\cdot\mathbf{e}$ and multiplying vectorially into the *constant* vector $\mathbf{B}_0$ both sides of the resulting equation, we obtain

$$\begin{aligned}&[\sigma+\epsilon(\partial/\partial t)][\nabla^2-\mu\epsilon(\partial^2/\partial t^2)](\mathbf{e}\times\mathbf{B}_0)\\&\quad+\sigma[\nabla\nabla\cdot(\mathbf{v}\times\mathbf{B})]\times\mathbf{B}_0\\&\qquad=\mu(\partial/\partial t)[\sigma+\epsilon(\partial/\partial t)](\mathbf{j}\times\mathbf{B}_0)\\&\qquad\qquad-[\nabla\nabla\cdot(\eta\mathbf{v})]\times\mathbf{B}_0. \quad (20)\end{aligned}$$

Next, to proceed with the elimination, we rewrite the Eulerian equation (VII) in the form

$$\mathbf{F}=\mathbf{f},\quad \mathbf{F}\equiv\rho(d\mathbf{v}/dt)+\nabla p, \quad (21)$$

from which, introducing the Maxwell's electromagnetic stress tensor for the *induced* field,

$$\mathfrak{T}=\epsilon[\mathbf{ee}-\tfrac{1}{2}e^2\mathfrak{J}]+\mu[\mathbf{hh}-\tfrac{1}{2}h^2\mathfrak{J}], \quad (22)$$

and its (tensor) divergence

$$\nabla\cdot\mathfrak{T}=\epsilon[\mathbf{e}\nabla\cdot\mathbf{e}+(\nabla\times\mathbf{e})\times\mathbf{e}]+\mu[\mathbf{h}\nabla\cdot\mathbf{h}+(\nabla\times\mathbf{h})\times\mathbf{h}], \quad (23)$$

we obtain, making use of (3),

$$\mathbf{j}\times\mathbf{B}_0=\mathbf{F}-\nabla\cdot\mathfrak{T}+\mu\epsilon(\partial/\partial t)(\mathbf{e}\times\mathbf{h}), \quad (24)$$

where the vector $\mathbf{F}$ denotes the hydrodynamic term

$$\mathbf{F}=\rho(d\mathbf{v}/dt)+\nabla p=\rho(\partial\mathbf{v}/\partial t)+\tfrac{1}{2}\rho\nabla v^2+\rho(\nabla\times\mathbf{v})\times\mathbf{v}+\nabla p. \quad (25)$$

Similarly, making use of (V) and (24), we obtain

$$\mathbf{e}\times\mathbf{B}_0=\sigma^{-1}[\mathbf{F}-\nabla\cdot\mathfrak{T}+\mu\epsilon(\partial/\partial t)(\mathbf{e}\times\mathbf{h})]-(\eta/\sigma)(\mathbf{v}\times\mathbf{B}_0)-(\mathbf{v}\times\mathbf{B})\times\mathbf{B}_0, \quad (26)$$

with which we have completed the elimination of the electromagnetic field vectors, except for second-order terms, from the vectors $\mathbf{j}\times\mathbf{B}_0$ and $\mathbf{e}\times\mathbf{B}_0$ which still remain in (20).

Thus, finally, substituting into (20) the expressions (24) and (26), we obtain the complete and exact magnetohydrodynamic equation in the fluid velocity, namely

$$\begin{aligned}&[\sigma+\epsilon(\partial/\partial t)][\nabla^2-\mu\epsilon(\partial^2/\partial t^2)]\\&\quad\times\{\sigma^{-1}[\mathbf{F}-\nabla\cdot\mathfrak{T}+\mu\epsilon(\partial/\partial t)(\mathbf{e}\times\mathbf{h})]\\&\quad-(\eta/\sigma)(\mathbf{v}\times\mathbf{B}_0)-(\mathbf{v}\times\mathbf{B})\times\mathbf{B}_0\}+\sigma[\nabla\nabla\cdot(\mathbf{v}\times\mathbf{B})]\times\mathbf{B}_0\\&\qquad=\mu(\partial/\partial t)[\sigma+\epsilon(\partial/\partial t)][\mathbf{F}-\nabla\cdot\mathfrak{T}+\mu\epsilon(\partial/\partial t)(\mathbf{e}\times\mathbf{h})]\\&\qquad\qquad-[\nabla\nabla\cdot(\eta\mathbf{v})]\times\mathbf{B}_0, \quad (27)\end{aligned}$$

the notable feature of which being the fact that all the troublesome terms which render its solution in the present form completely intractable appear only as quadratic terms. Therefore, it is suggested that in a linearized theory we merely drop the unwanted second-order terms. However, to justify this procedure more fully, we consider next the special case of infinite conductivity and zero displacement current which serves as a guide post.

2.5 Infinite Conductivity and Zero Displacement Current

It has been claimed by Walén[16] that, in the limit of infinite conductivity and zero displacement current, the magneto-hydrodynamic equations for an incompressible fluid become linear. We propose to show that Walén's statement is true only in the following restricted sense: that in this limiting case there exists a class of solutions (magneto-hydrodynamic waves) of the linearized equations which also satisfy the nonlinear system except for an uninteresting quadratic term in the fluid velocity. To this end, let us examine the limiting form of (27) as $\sigma\rightarrow\infty$ and $\epsilon\rightarrow 0$. In this limit, we have merely

$$\{\nabla\times[\nabla\times(\mathbf{v}\times\mathbf{B})]\}\times\mathbf{B}_0 = \mu(\partial/\partial t)[\rho(\partial\mathbf{v}/\partial t)+\tfrac{1}{2}\rho\nabla v^2 + \rho(\nabla\times\mathbf{v})\times\mathbf{v}+\nabla p-\mu(\nabla\times\mathbf{h})\times\mathbf{h}], \quad (28)$$

where $\mathbf{B}=\mathbf{B}_0+\mu\mathbf{h}$ and in which quadratic terms are still very much in evidence. Dropping these terms outright, however, we obtain the much simpler linear equation

$$[\nabla\times\nabla\times(\mathbf{v}\times\mathbf{B}_0)]\times\mathbf{B}_0=\mu\rho(\partial^2\mathbf{v}/\partial t^2)+\mu\nabla(\partial p/\partial t), \quad (29)$$

which can be solved exactly.

Thus, putting $\mathbf{B}_0=\hat{e}_zB_0$ and assuming $\nabla\cdot\mathbf{v}=0$, we obtain for the left side of (29)

$$[\nabla\times\nabla\times(\mathbf{v}\times\mathbf{B}_0)]\times\mathbf{B}_0=B_0^2[\partial^2\mathbf{v}/\partial z^2-\nabla(\partial v_z/\partial z)], \quad (30)$$

and introducing the phase velocity

$$V_a=\pm B_0(\mu\rho)^{-\frac{1}{2}}, \quad (31)$$

which we will henceforth refer to as *Alfvén's phase velocity* in honor of its discoverer,[17] we re-write (29) in the form

$$(\partial^2\mathbf{v}/\partial z^2)-V_a^{-2}(\partial^2\mathbf{v}/\partial t^2) = \nabla[(\mu B_0^{-2})(\partial p/\partial t)+(\partial v_z/\partial z)]. \quad (32)$$

Taking the divergence of the vectors on both sides of (32) and recalling that $\nabla\cdot\mathbf{v}=0$ has been assumed we obtain

$$\nabla^2[(\mu B_0^{-2})(\partial p/\partial t)+(\partial v_z/\partial z)]=0, \quad (33)$$

i.e., the expression within the bracket must be a solution of Laplace's equation everywhere. Therefore, we must have

$$(\mu B_0^{-2})(\partial p/\partial t)+(\partial v_z/\partial z)=\text{constant}, \quad (34)$$

which allows the computation of the (excess) pressure in terms of the z component of velocity. Furthermore, substituting (34) into (32) we obtain the one-dimensional vector wave equation

$$(\partial^2\mathbf{v}/\partial z^2)-V_a^{-2}(\partial^2\mathbf{v}/\partial t^2)=0, \quad (35)$$

whose most general solution may be written as

$$\mathbf{v}(x,y,z,t)=\mathbf{v}_+(x,y)f(z-V_at)+\mathbf{v}_-(x,y)g(z+V_at), \quad (36)$$

where f and g are arbitrary, dimensionless, single-valued, finite, continuous, and differentiable functions of their respective arguments and where $\mathbf{v}_+$ and $\mathbf{v}_-$ are arbitrary velocity amplitude vectors independent of z and t.

We propose to return to the infinite conductivity case in more detail in the sequel to this paper, Part II. Here, we merely wish to use (36) to solve for the induced magnetic field $\mathbf{h}$. Thus, using the linearized form of (V), we obtain in the limit of infinite conductivity:

$$\mathbf{e}=-\mathbf{v}\times\mathbf{B}_0, \quad (37)$$

from which, making use of (II), we have

$$\mu(\partial\mathbf{h}/\partial t)=\nabla\times(\mathbf{v}\times\mathbf{B}_0)=B_0(\partial\mathbf{v}/\partial z), \quad (\nabla\cdot\mathbf{v}=0). \quad (38)$$

Noting that, as a consequence of the form of the solution (36), $(\partial\mathbf{v}/\partial t)=-V_a(\partial\mathbf{v}/\partial z)$, we obtain from (38) the important result:

$$\mathbf{h}/H_0=-\mathbf{v}/V_a, \quad (39)$$

where $H_0=B_0/\mu$ and V_a is given by (31), the choice of sign depending on the direction of propagation. Equation (39) may be rewritten, making use of (31), in the symmetric form

$$(\mu)^{\frac{1}{2}}\mathbf{h}=\mp(\rho)^{\frac{1}{2}}\mathbf{v}, \quad (40)$$

which expresses the fact that, in this special case, the vectors $\mathbf{v}$ and $\mathbf{h}$ are everywhere parallel or antiparallel provided that the fluid is incompressible. Thus, finally, making use of (40) in the original nonlinearized wave equation (28), we obtain exactly

$$\{\nabla\times[\nabla\times(\mathbf{v}\times\mathbf{B}_0)]\}\times\mathbf{B}_0 = \mu(\partial/\partial t)[\rho(\partial\mathbf{v}/\partial t)+\tfrac{1}{2}\rho\nabla v^2+\nabla p], \quad (41)$$

which differs from the linearized form (29) only in the presence of the quadratic term $\frac{1}{2}\rho\nabla v^2$ in the bracket to the right, hence proving our original contention.

2.6 Linearized Form of the Fundamental Equations

Although the exact magneto-hydrodynamic equation (27) does not reduce strictly to linear form, even in the special limiting case considered above, the particular solution (39) does suggest an absolute criterion, independent of the conductivity, for the applicability of small amplitude linear theory. Thus, we need only assume that the fluid velocity always remains small in comparison with Alfvén's phase velocity, $v\ll V_a$, in which case, by virtue of (39), the induced magnetic intensity will always remain small in comparison with the externally applied magnetic intensity, $h\ll H_0$. If this is true, then all second-order terms appearing in (27) can be safely neglected in an approximate linearized theory. Thus, neglecting $\mu\mathbf{h}$ in comparison with $\mathbf{B}_0$, i.e., replacing $\mathbf{B}$ by $\mathbf{B}_0$ wherever it appears in the fundamental equations and dropping all second-order terms from (27), we

[16] Reference 3, Eqs. (9) and (10), p. 5.
[17] H. Alfvén, Arkiv Mat. Astron. Fysik **29B**, No. 2 (1942).

obtain the linearized form of the magneto-hydrodynamic wave equation, namely,

$$[\sigma+\epsilon(\partial/\partial t)][\nabla^2-\mu\epsilon(\partial^2/\partial t^2)][\sigma^{-1}\mathbf{F}-(\mathbf{v}\times\mathbf{B}_0)\times\mathbf{B}_0]$$
$$=\mu[\sigma+\epsilon(\partial/\partial t)](\partial\mathbf{F}/\partial t)-\sigma[\nabla\nabla\cdot(\mathbf{v}\times\mathbf{B}_0)]\times\mathbf{B}_0, \quad (42)$$

where $\mathbf{F}$ denotes here the linearized form of the hydrodynamic term (25), which now reduces to

$$\mathbf{F}=\rho(\partial\mathbf{v}/\partial t)+\nabla p. \quad (43)$$

The important feature of Eq. (42) is that it exhibits the fluid velocity $\mathbf{v}$ as the sole dependent variable. That is, we have successfully eliminated the electromagnetic field vectors in a linearized theory. This possibility apparently had been overlooked in the literature even for the special case in which the electric displacement current is altogether neglected. Once in possession of the fluid velocity for a given case, as determined from (42), the computation of the accompanying electromagnetic field vectors is readily effected by making use of the original (linearized) Maxwellian equations.

In the important theoretical case of infinite conductivity we readily obtain from (42), letting $\sigma\rightarrow\infty$, the much simpler equation

$$\{\nabla\times[\nabla\times(\mathbf{v}\times\mathbf{B}_0)]+\mu\epsilon(\partial^2/\partial t^2)(\mathbf{v}\times\mathbf{B}_0)\}\times\mathbf{B}_0$$
$$=\mu(\partial\mathbf{F}/\partial t), \quad (44)$$

which should be compared with (29). As we have seen, considerable simplification ensues in (42) and (44) if we neglect altogether the electric displacement current, as commonly done by most writers on the subject, which we can do here by merely putting $\epsilon=0$. However, we do not propose to make this approximation now, for it obscures some of the essential features of the resulting wave phenomenon, although we do intend to examine in the end the limiting form of the general results as one neglects the electric displacement current.

Depending on the exact nature of the hydrodynamic term $\mathbf{F}$, which appears in Eqs. (42) and (44), we recognize two distinct classes of *linearized* problems:

I. *Incompressible Fluids*

For an ideal incompressible fluid the condition of incompressibility demands that $d\rho/dt=0$, whence the equation of continuity (VIII) reduces to

$$\nabla\cdot\mathbf{v}=0, \quad (45)$$

which means that the velocity field must be solenoidal. In this case, therefore, we must seek solutions of the magneto-hydrodynamic wave equations (42) or (44), with $\mathbf{F}$ as in Eq. (43), subject to the divergence condition (45). The pressure p then remains in $\mathbf{F}$ and, therefore, must be determined in the course of solving for $\mathbf{v}$ from (42) or (44). We find later, Part II, that magneto-hydrodynamic waves in an incompressible fluid can be of two types: devoid of pressure fluctuations and accompanied by a pressure wave.

II. *Compressible Fluids* (*Magneto-Acoustics*)

For an ideal compressible fluid, devoid of viscosity and expansive friction, we have in addition to the equation of continuity (VIII) an equation of state which yields the functional dependence between the pressure and the density. For example, for an ideal gas subject to adiabatic processes we have

$$p/p_0=(\rho/\rho_0)^\gamma, \quad (46)$$

where γ is the ratio of specific heats and p_0 is the pressure corresponding to the equilibrium density ρ_0. Quite generally, however, if we have available an equation of state between p and ρ, then (linearizing)

$$\nabla p=(dp/d\rho)_0\nabla\rho=V_s^2\nabla\rho;\quad V_s=(dp/d\rho)_0^{\frac{1}{2}}, \quad (47)$$

where V_s is the velocity of sound in the medium. In particular, if the adiabatic condition (46) holds, we obtain

$$V_s=(\gamma p_0/\rho_0)^{\frac{1}{2}}. \quad (48)$$

Inserting (47) into (43) and making use of the linearized form of the equation of continuity,

$$\partial\rho/\partial t+\rho_0(\nabla\cdot\mathbf{v})=0, \quad (49)$$

we can eliminate the pressure and the density, obtaining

$$\partial\mathbf{F}/\partial t=\rho_0(\partial^2\mathbf{v}/\partial t^2)-\rho_0V_s^2\nabla\nabla\cdot\mathbf{v}. \quad (50)$$

Therefore, we must now seek solutions of the magneto-hydrodynamic wave equations (42) or (44) after inserting for $\partial\mathbf{F}/\partial t$ the expression on the right of (50). It is shown later, Part II, that magneto-hydrodynamic waves in a perfect compressible fluid can be of two types: devoid of pressure fluctuations (as in the case of incompressible fluids), and accompanied by a pressure wave (magneto-acoustic waves) of which there are two distinct modes.

3. PLANE WAVES

At the outset we take the constant externally applied field of magnetic induction parallel to the z-axis, i.e., $\mathbf{B}_0=\hat{e}_zB_0$; and we make the assumption that the pressure and the Cartesian components of the field vectors exhibit the common space-time dependence characterized by the *dimensionless* factor

$$\psi(\mathbf{r},t)=\exp\{i(\mathbf{k}\cdot\mathbf{r}-\omega t)\}, \quad (51)$$

where ω is the fixed angular frequency of the time harmonic oscillations and $\mathbf{k}$ is the vector propagation constant, which in general turns out to be a complex vector. The function $\psi(\mathbf{r},t)$ satisfies the three-dimensional scalar Helmholtz equation

$$(\nabla^2+k^2)\psi=0, \quad (52)$$

where $k^2=\mathbf{k}\cdot\mathbf{k}$ is in general complex and must be determined, for a particular solution, from the magneto-hydrodynamic wave equations (42) or (44).

Further, since the direction of the constant vector $\mathbf{B}_0$

constitutes an obvious axis of symmetry of the problem, we postulate that the vector propagation constant $\mathbf{k}$, which can assume an arbitrary direction with respect to the externally applied magnetic field, can be written quite generally as

$$\mathbf{k}=\mathbf{n}k=\hat{e}_x k_x+\hat{e}_z k_z, \tag{53}$$

where $\mathbf{n}$ is a unit vector in the direction of propagation and k_x and k_z represent, respectively, the transverse and longitudinal wave numbers. We find in Part II that, only in the case of infinite conductivity, does the system sustain plane homogeneous waves in which $\mathbf{n}$ is a *real* unit vector and the wave number k is also real. When the conductivity is finite, the resulting plane waves are still homogeneous, but now the wave number k is complex. In both cases, it is possible to set up plane wave solutions which are nonhomogeneous, i.e., equiphase and equiamplitude planes no longer coinciding, which means then that $\mathbf{n}$ is a *complex* unit vector, but we have not found these solutions of practical interest.

Introducing the substitutions $\nabla=i\mathbf{k}$ and $\partial/\partial t=-i\omega$, which are a consequence of (51), we obtain, instead of (43) and (45),

$$\mathbf{F}=-i\omega\rho\mathbf{v}+ip\mathbf{k};\quad \mathbf{k}\cdot\mathbf{v}=0, \tag{54}$$

in the case of incompressible fluids, Class I problems; and, instead of (50),

$$-i\omega\mathbf{F}=-\omega^2\rho_0\mathbf{v}+\rho_0 V_s^2(\mathbf{k}\cdot\mathbf{v})\mathbf{k}, \tag{55}$$

where V_s is defined by (47), in the case of compressible fluids (magneto-acoustics), Class II problems.

To deduce the linearized equation in the fluid velocity $\mathbf{v}$ that results from the elimination of the electromagnetic field vectors $\mathbf{e}$, $\mathbf{h}$, and $\mathbf{j}$, we need only make similar substitutions in the magneto-hydrodynamic wave equation (42), obtaining

$$\begin{aligned}(\omega\epsilon+i\sigma)(k^2-\omega^2\mu\epsilon)[\sigma^{-1}\mathbf{F}-B_0^2(\mathbf{v}\times\hat{e}_z)\times\hat{e}_z]\\ =i\omega\mu(\omega\epsilon+i\sigma)\mathbf{F}-i\sigma B_0^2[\hat{e}_z\cdot(\mathbf{k}\times\mathbf{v})](\mathbf{k}\times\hat{e}_z),\end{aligned} \tag{56}$$

which is valid for finite conductivity. And, either from (44) or else letting $\sigma\rightarrow\infty$ in (56), we obtain

$$\begin{aligned}B_0^2\{(k^2-\omega^2\mu\epsilon)(\mathbf{v}\times\hat{e}_z)\\ -[\hat{e}_z\cdot(\mathbf{k}\times\mathbf{v})]\mathbf{k}\}\times\hat{e}_z=-i\omega\mu\mathbf{F},\end{aligned} \tag{57}$$

which applies in the case of infinite conductivity. In both cases, the vector $\mathbf{F}$ is given by (54) or (55) depending on the class of problems being discussed.

In either class of problems, it is clear from the postulate of plane waves as given by (51) and (53) that the elementary solutions of the vector equations (56) or (57) must be of one or more of the following three forms:

$$\mathbf{v}_1=\hat{e}_y v_0\psi;\quad \mathbf{v}_2=\mathbf{n}\times\mathbf{v}_1=(\mathbf{n}\times\hat{e}_y)v_0\psi;\quad \mathbf{v}_3=\mathbf{n}v_0\psi, \tag{58}$$

where ψ is given by (51) and v_0 is an arbitrary velocity amplitude which, according to the conditions imposed by a linearized theory (Sec. 2.6), must be much smaller than Alfvén's phase velocity (31), that is, $v_0\ll V_a$. As illustrated in Fig. 1, which is drawn for a *real* propagation vector $\mathbf{k}$, the first two proposed solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ are solenoidal, $\mathbf{k}\cdot\mathbf{v}=0$, while the third one is irrotational, $\mathbf{k}\times\mathbf{v}_3=0$. It is clear from (54) that only the first two solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ are admissible in the case of incompressible fluids, whereas the solution $\mathbf{v}_3$ must necessarily be present in the case of magneto-acoustics, at least whenever pressure fluctuations accompany the wave phenomenon. Finally, it is seen from (52) that all three velocity vectors (58) are linearly independent solutions of the three-dimensional vector Helmholtz equation,

$$(\nabla^2+k^2)\mathbf{v}=0. \tag{59}$$

The actual selection of a particular solution (58) and the determination of the corresponding wave number k, from either (56) or (57), will be found in Part II, where we discuss the application of the present theory to incompressible and compressible fluids.

Assuming that an appropriate particular solution of the vector equations (56) or (57) has been selected from (58), we can proceed quite generally from the linearized form of (1) to the computation of the electromagnetic field vectors $\mathbf{e}$, $\mathbf{h}$, and $\mathbf{j}$ in terms of the known velocity $\mathbf{v}$. And, at every stage of the analysis, it proves extremely useful to examine the limiting form of the results as the conductivity becomes infinite and as we neglect the electric displacement current.

Thus, making use of the linearized form of the Maxwellian set (I–VI) we obtain, by successive eliminations, the field vectors

$$\begin{aligned}\mathbf{h}&=\frac{i\sigma\mathbf{k}\times(\mathbf{v}\times\mathbf{B}_0)}{k^2-\omega^2\mu\epsilon-i\omega\mu\sigma}\xrightarrow[\sigma\rightarrow\infty]{}-\frac{\mathbf{k}\times(\mathbf{v}\times\mathbf{B}_0)}{\omega\mu}=\frac{\mathbf{k}\times\mathbf{e}}{\omega\mu};\\ \mathbf{e}&=\frac{i\sigma\{(\omega^2\mu\epsilon+i\omega\mu\sigma)(\mathbf{v}\times\mathbf{B}_0)-[\mathbf{k}\cdot(\mathbf{v}\times\mathbf{B}_0)]\mathbf{k}\}}{(\omega\epsilon+i\sigma)(k^2-\omega^2\mu\epsilon-i\omega\mu\sigma)}\\ &\qquad\xrightarrow[\sigma\rightarrow\infty]{}-\mathbf{v}\times\mathbf{B}_0;\\ \mathbf{j}&=\frac{\sigma(\omega\epsilon+i\sigma)(k^2-\omega^2\mu\epsilon)(\mathbf{v}\times\mathbf{B}_0)-i\sigma^2[\mathbf{k}\cdot(\mathbf{v}\times\mathbf{B}_0)]\mathbf{k}}{(\omega\epsilon+i\sigma)(k^2-\omega^2\mu\epsilon-i\omega\mu\sigma)}\\ &\qquad\xrightarrow[\sigma\rightarrow\infty]{}(i/\omega\mu)\{(k^2-\omega^2\mu\epsilon)(\mathbf{v}\times\mathbf{B}_0)-[\mathbf{k}\cdot(\mathbf{v}\times\mathbf{B}_0)]\mathbf{k}\}.\end{aligned} \tag{60}$$

And, in case we neglect the electric displacement current, we can obtain from (60) the corresponding limiting forms by merely putting $\epsilon=0$. We then find that the forms of $\mathbf{e}$ and $\mathbf{h}$ for the case of infinite conductivity remain unaltered whether we retain or neglect the electric displacement current, but that such is not the case for the current density $\mathbf{j}$.

4. CYLINDRICAL WAVES

We postulate again that the externally applied field of magnetic induction is parallel to the z-axis, $\mathbf{B}_0=\hat{e}_z B_0$,

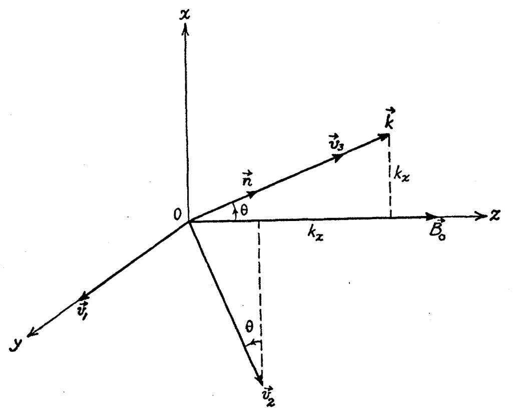

FIG. 1. Elementary plane wave solutions of the vector equation in the particle velocity.

and we then assume that the pressure and all the field vectors can be deduced from the scalar function of position and time

$$\psi(\mathbf{r},t)=A\phi(\boldsymbol{\varrho})e^{i(kz-\omega t)}, \qquad (61)$$

where $\mathbf{r}=\boldsymbol{\varrho}+\hat{e}_z z$ is the position vector of the point of observation and A is a constant of dimensions meter^2 sec^{-1}. The dimensionless factor $\phi(\boldsymbol{\varrho})$, which is a function of the transverse coordinates only, is assumed to satisfy the two-dimensional scalar Helmholtz equation

$$(\nabla_t^2+\gamma^2)\phi=0, \qquad (62)$$

where ∇_t^2 is the transverse part of the Laplacian operator, $\nabla^2=\nabla_t^2+(\partial/\partial z)^2$, and γ is the transverse wave number. As a consequence of (62) the space-time function (61) satisfies the three-dimensional scalar Helmholtz equation:

$$(\nabla^2+K^2)\psi=0, \quad K^2=\gamma^2+k^2, \qquad (63)$$

where k is the longitudinal wave number. In general, γ^2 is chosen as a positive definite quantity whose actual value, for a given mode of propagation, is dictated by boundary conditions on a cylindrical coordinate surface with generators parallel to the applied field, whereas k^2 turns out to be in general complex.

Making the substitution $\partial/\partial t=-i\omega$, in accordance with (61), we have, instead of (43) and (45),

$$\mathbf{F}=-i\omega\rho\mathbf{v}+\nabla p; \quad \nabla\cdot\mathbf{v}=0, \qquad (64)$$

in the case of incompressible fluids; and, instead of (50),

$$i\omega\mathbf{F}=\omega^2\rho_0\mathbf{v}+\rho_0 V_s^2\nabla\nabla\cdot\mathbf{v}, \qquad (65)$$

in the case of magneto-acoustics.

Next, resolving every vector into its transverse and longitudinal components, i.e., $\mathbf{v}=\mathbf{v}_t+\hat{e}_z v_z$, $\nabla=\nabla_t+\hat{e}_z(\partial/\partial z)$, we deduce from (42) the form of the magneto-hydrodynamic wave equation which applies to cylindrical waves in the case of finite conductivity, namely

$$(\omega\epsilon+i\sigma)(\nabla^2+\omega^2\mu\epsilon)[\sigma^{-1}\mathbf{F}+B_0^2\mathbf{v}_t] =-i\omega\mu(\omega\epsilon+i\sigma)\mathbf{F}+i\sigma B_0^2[\nabla_t^2\mathbf{v}_t-\nabla_t\nabla_t\cdot\mathbf{v}_t], \qquad (66)$$

and proceeding similarly with (44) we obtain the corresponding wave equation which abides in the case of infinite conductivity,

$$B_0^2[(\partial^2/\partial z^2+\omega^2\mu\epsilon)\mathbf{v}_t+\nabla_t\nabla_t\cdot\mathbf{v}_t]=-i\omega\mu\mathbf{F}. \qquad (67)$$

In both cases the vector $\mathbf{F}$ assumes the form (64) for incompressible fluids or the form (65) for magneto-acoustics.

To obtain the velocity field corresponding to a particular situation we note, in complete analogy with the above discussion for plane waves, that the elementary solutions of the magneto-hydrodynamic wave equations (66) or (67) must be of one or more the following three forms[18]:

$$\begin{aligned}\mathbf{v}_1&=\nabla\times(\hat{e}_z\psi)=\nabla\psi\times\hat{e}_z,\\ \mathbf{v}_2&=(ik)^{-1}\nabla\times\mathbf{v}_1=(ik)^{-1}\nabla\times\nabla\times(\hat{e}_z\psi),\\ \mathbf{v}_3&=\nabla\psi,\end{aligned} \qquad (68)$$

where ψ is defined by (61) and k is the longitudinal wave number. The first two proposed solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ are solenoidal, $\nabla\cdot\mathbf{v}=0$, whereas the third one is irrotational, $\nabla\times\mathbf{v}_3=0$. As in the case of plane waves, it is clear from (64) that only the solenoidal solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ are admissible in the case of incompressible fluids, whereas the irrotational solution $\mathbf{v}_3$ must of necessity appear in the case of magneto-acoustics, at least when the wave phenomenon is accompanied by pressure fluctuations. Finally, it is seen from (63) that all three velocity vectors (68) are linearly independent solutions of the vector Helmholtz equation:

$$(\nabla^2+K^2)\mathbf{v}=0, \quad K^2=\gamma^2+k^2, \qquad (69)$$

where γ and k are the transverse and longitudinal wave numbers, respectively. As a consequence of (69) it is readily shown from the linearized form of the Maxwellian set (I–VI) that the electromagnetic field vectors $\mathbf{e}$, $\mathbf{h}$, and $\mathbf{j}$ are themselves solutions of the vector Helmholtz equation with K^2 as in (69).

The actual selection from (68) of a particular solution or a linear combination thereof for a given case, and the computation of the longitudinal wave number k in terms of a preassigned transverse wave number γ are effected in the course of solving the magneto-hydrodynamic wave equations (66) or (67). The details of these computations, as they apply to incompressible and compressible fluids, will be found in the sequel to this paper, Part II.

Once in possession of an appropriate particular solution of the magneto-hydrodynamic wave equations (66) or (67), we can proceed quite generally from the linearized form of the Maxwellian set (I–VI), as in the case of plane waves, to the computation of the electromagnetic field vectors $\mathbf{e}$, $\mathbf{h}$, and $\mathbf{j}$. Making use of the fact that these vectors satisfy the vector Helmholtz equation

[18] The actual proof that the velocity vectors (68), or linear combinations thereof, constitute elementary solutions of the magneto-hydrodynamic wave equations (66) or (67) will be given in Part II.

(69), we obtain by successive eliminations:

$$\mathbf{h}=\frac{\sigma\nabla\times(\mathbf{v}\times\mathbf{B}_0)}{K^2-\omega^2\mu\epsilon-i\omega\mu\sigma}\xrightarrow[\sigma\to\infty]{}\frac{i\nabla\times(\mathbf{v}\times\mathbf{B}_0)}{\omega\mu}=-\frac{i\nabla\times\mathbf{e}}{\omega\mu},$$

$$\mathbf{e}=\frac{i\sigma[(\omega^2\mu\epsilon+i\omega\mu\sigma)(\mathbf{v}\times\mathbf{B}_0)+\nabla\nabla\cdot(\mathbf{v}\times\mathbf{B}_0)]}{(\omega\epsilon+i\sigma)(K^2-\omega^2\mu\epsilon-i\omega\mu\sigma)}\xrightarrow[\sigma\to\infty]{}-\mathbf{v}\times\mathbf{B}_0, \quad (70)$$

$$\mathbf{j}=\frac{\sigma(\omega\epsilon+i\sigma)(K^2-\omega^2\mu\epsilon)(\mathbf{v}\times\mathbf{B}_0)+i\sigma^2\nabla\nabla\cdot(\mathbf{v}\times\mathbf{B}_0)}{(\omega\epsilon+i\sigma)(K^2-\omega^2\mu\epsilon-i\omega\mu\sigma)}\xrightarrow[\sigma\to\infty]{}(i/\omega\mu)[(K^2-\omega^2\mu\epsilon)(\mathbf{v}\times\mathbf{B}_0)+\nabla\nabla\cdot(\mathbf{v}\times\mathbf{B}_0)],$$

which are seen to agree with (60), corresponding to plane waves, if we merely replace ∇ by $i\mathbf{k}$ and write k^2 instead of K^2.

In case we neglect the electric displacement current, we need merely put $\epsilon=0$ in (70) to obtain the corresponding limiting forms. It is noteworthy to point out once more that, in the case of infinite conductivity, the forms for $\mathbf{e}$ and $\mathbf{h}$ remain unaltered whether we retain or neglect the electric displacement current, but that such is not the case for the current density $\mathbf{j}$.

In conclusion, the author wishes to express his sincere appreciation to Professors David S. Saxon and Leon Knopoff, of the Physics Department and Institute of Geophysics, respectively, for many illuminating discussions that proved extremely fruitful in the course of these studies.

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Particle Transport, Electric Currents, and Pressure Balance in a Magnetically Immobilized Plasma

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An analysis of a plasma immobilized by a magnetic field shows that each kind of charged particle has a general drift perpendicular to the gradient of the field, but that there is no corresponding electric current density. There is an exact cancellation arising from the gradient of the Larmor radius. The current density which is present arises exclusively from the particle density gradient and is not associated with any drift of matter.

IN an infinite, completely ionized plasma in a magnetic field where the pertinent variables density and magnetic field are each a function of one coordinate only which lies perpendicular to the field, it is generally accepted that

$$B_1^2-B_2^2=8\pi(n_2-n_1)kT, \quad (1)$$

where B_1 and B_2 are the field strengths at points where the total particle concentrations are n_1 and n_2, respectively, and T is the absolute temperature of the plasma. This equation is usually derived by applying magnetohydrodynamic principles through the equations:

$$\nabla p=\mathbf{j}\times\mathbf{B},\quad 4\pi\mathbf{j}=\nabla\times\mathbf{B},$$
$$4\pi\nabla p=(\nabla\times\mathbf{B})\times\mathbf{B}=(\mathbf{B}\cdot\nabla)\mathbf{B}-\tfrac{1}{2}\nabla B^2,$$
$$(\mathbf{B}\cdot\nabla)\mathbf{B}=0 \text{ under symmetry assumed},$$
$$p=nkT.$$

Equation (1) should also be derivable from analysis based on the microscopic structure of such a plasma, and it is worth while to do this because the derivation brings to light a peculiar and possibly important property of the plasma. An analysis along somewhat similar lines but using large volume elements instead of small ones has already been given by Spitzer,[1] but added insight is gained by the present method.

We adopt a local right-handed Cartesian coordinate system and x- and y-axes in the plane of the paper and magnetic field normal to it. We orient the system so that at the origin, which is the point of interest, the magnetic field, B, and plasma density, n, are functions of x only, and the space variations of n and B are assumed to be small in the span of the average orbit diameter. We consider a volume element $dxdy$ (being unity along z) at the origin. This is illustrated in Fig. 1.

The current density at the origin will then be

$$\mathbf{j}=(e\textstyle\sum_p \mathbf{v}_p-e\sum_e \mathbf{v}_e)/dxdy, \quad (2)$$

where the summations are over all positive ions and all electrons in the element $dxdy$ at any instant. It will suffice (a) to make the detailed analysis for one kind of particle, and we choose ions, (b) to omit z-components of velocity, and (c) to neglect collisions, which lead to diffusion effects which may be superimposed on those

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[1] L. Spitzer, Jr., Astrophys. J. 116, 299 (1952).