Boson-Fermion Scattering in the Heisenberg Representation*

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It is shown that the S-matrix for boson-fermion scattering can be simply expressed in the Heisenberg representation. By performing a time integration one obtains the S-matrix in the Schrödinger representation, which has the same form as the conventional perturbation theory sum over states. Suitably limiting the nature of the intermediate states entering into this sum leads to integral equations for certain matrix elements which are equal to the S-matrix elements on the energy shell. These equations appear in a completely renormalized form. For example, in the fixed source limit, the four pion-nucleon scattering states satisfy the same equation (with different numerical coefficients). The equations are nonlinear, but involve only the scattering phase shifts. The equivalent equation for photopion production is linear, and in the fixed source limit can be written down from a knowledge of the experimental scattering phase shifts. The zero-pion-mass theorems of Gell-Mann and Goldberger (concerning the isotopic spin independence of the zero-energy S-wave scattering) and of Kroll and Ruderman, [Phys. Rev. **93**, 233 (1954)] follow simply from the formalism.

I. EQUATION FOR THE S-MATRIX

We shall for simplicity derive our results for only one process: the scattering of symmetric pseudoscalar mesons by a nucleon. The method is easily generalizable to other processes of interest: in Sec. II we state without proof the results for Compton scattering and photopion production.

We start from Dyson's definition[1] of the S-matrix for the scattering of an ith meson in momentum state q to a jth meson in state q', the scattering nucleon going from p to p'[2]:

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dt_1\cdots dt_n(\Phi_{p'},a_j(q')\times P[H_I(t_1)\cdots H_I(t_n)]a_i^*(q)\Phi_p). \quad (1.1)$$

Here P is Dyson's time-ordering operator and a^* and a are, respectively, creation and annihilation operators for single mesons. Also

$$H_I=ig\int\bar{\psi}(x)\gamma_5\tau_i\psi(x)\phi_i(x)d\tau_x-\delta m\int\bar{\psi}(x)\psi(x)d\tau_x-\tfrac{1}{2}\delta\mu^2\int\phi_i(x)\phi_i(x)d\tau_x+\tfrac{1}{4}\lambda\int[\phi_i(x)\phi_i(x)]^2d\tau_x. \quad (1.2)$$

We proceed by commuting $a_i^*(q)$ through to the left and $a_j(q')$ through to the right in Eq. (1.1). Since

$$[\phi_k(x),a_i^*(q)]=\delta_{ki}e^{iqx}/(2q_0)^{\frac{1}{2}} \quad (1.3)$$

and

$$[a_j(q'),\phi_k(x)]=\delta_{kj}e^{-iq'x}/(2q_0)^{\frac{1}{2}}, \quad (1.4)$$

we obtain (for $q\neq q'$ and $p\neq p'$):

$$S=-i\int\frac{dx e^{iqx}}{(2q_0)^{\frac{1}{2}}}\bigg(\Phi_{p'},a_j(q')\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dt_1\cdots dt_n\times P[H_I(t_1),\cdots H_I(t_n),O_i(x)]\Phi_p\bigg), \quad (1.5)$$

$$S=(-i)^2\int\frac{dxdy}{(4q_0q_0')^{\frac{1}{2}}}e^{iqx}e^{-iq'y}\times\bigg(\Phi_{p'},\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dt_1\cdots dt_n\times P[H_I(t_1),\cdots,H_I(t_n),O_j(y),O_i(x)]\Phi_p\bigg)-i\lambda\int\frac{dxe^{i(q-q')x}}{(4q_0q_0')^{\frac{1}{2}}}\bigg(\Phi_{p'},\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dt_1\cdots dt_n\times P[H_I(t_1),\cdots H_I(t_n),\phi_k(x)\phi_k(x)\delta_{ij}+2\phi_i(x)\phi_j(x)]\Phi_p\bigg), \quad (1.6)$$

where

$$O_i(x)=ig\bar{\psi}(x)\gamma_5\tau_i\psi(x)-\delta\mu^2\phi_i(x)+\lambda\phi_i(x)\phi_k(x)\phi_k(x).$$

Equation (1.6) can now be transformed by recognizing that an expression of the form

$$\bigg(\Phi_{p'},\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dt_1\cdots dt_n\times P[H_I(t_1),\cdots,H_I(t_n),A_1(x),A_2(y),\cdots]\Phi_p\bigg), \quad (1.7)$$

(where the A_α's are functions of the field variables at the

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[1] F. J. Dyson, Phys. Rev. **75**, 1737 (1949).

[2] Throughout this section we shall distinguish Heisenberg operators from interaction representation operators by using boldface type (**A**) for the former. State vectors are represented by capital Greek letters. Φ_p is an eigenstate of the unperturbed Hamiltonian of momentum p_i and energy $p_0=(p_i^2+m)^{\frac{1}{2}}$, where m is the observed Fermion mass. Ψ_p is an eigenstate of the total Hamiltonian with the same momentum and energy. A four-vector inner product is written $px=p_\mu x_\mu=p_ix_i-p_0x_0=\mathbf{p}\cdot\mathbf{x}-p_0x_0$. The subscripts i, j, k, etc., refer to the space-like components of a four-vector as well as to isotopic spin indices. Integrations over three-dimensional volumes are written $\int d\tau_x$, over four-dimensional volumes $\int dx$.

points x, y, etc., but not of their time derivatives) can be written equally well

$$(\Psi_{p'}, P[\mathbf{A}_1(x), \mathbf{A}_2(y)]\Psi_p), \tag{1.8}$$

where the operators $\mathbf{A}_\alpha(x)$ now have Heisenberg representation time dependence and the Ψ_p's are exact single-particle energy-momentum eigenstates of the entire Hamiltonian.

The identity of (1.7) and (1.8) may be proved in two ways. The first consists in calculating the scattering of a free nucleon by an arbitrary external symmetric pseudoscalar field, $\phi_i^{(e)}(x)$, both in the Heisenberg and interaction representations and equating the coefficients of $\phi_i^{(e)}(x)\phi_j^{(e)}(y)$ under the integral in the two forms of the S-matrix. The second is a direct calculation of the type used by Gell-Mann and the author.[3] Their proof was given for vacuum expectations of ordered products, but applies to other eigenstates provided the self-mass is properly subtracted from the interaction Hamiltonian so that there is no self-energy of these states.

We may therefore rewrite Eq. (1.6):

$$S=\int \frac{e^{iqx}e^{-iq'y}}{(4q_0q_0')^{\frac{1}{2}}}dxdy\{-(\Psi_{p'}, P[\mathbf{O}_j(y), \mathbf{O}_i(x)]\Psi_p) -i\delta(x-y)\lambda(\Psi_{p'}, [\delta_{ij}\boldsymbol{\phi}_k(x)\boldsymbol{\phi}_k(x) +2\boldsymbol{\phi}_i(x)\boldsymbol{\phi}_j(x)]\Psi_p)\}. \tag{1.9}$$

If it were not for renormalization terms, the first-matrix element in Eq. (1.9) would be simply the time-ordered product of the interaction densities. With a $\delta\mu^2$ and a λ term, however, it is actually the combination that appears there that is simple, since the meson field satisfies the equation of motion:

$$(\Box^2-\mu_0^2)\boldsymbol{\phi}_i(x)=ig\bar{\psi}(x)\gamma_5\tau_i\psi(x)+\lambda\boldsymbol{\phi}_i(x)\boldsymbol{\phi}_k(x)\boldsymbol{\phi}_k(x),$$

or

$$(\Box^2-\mu^2)\boldsymbol{\phi}_i(x)=ig\bar{\psi}(x)\gamma_5\tau_i\psi(x) +\lambda\boldsymbol{\phi}_i(x)\boldsymbol{\phi}_k(x)\boldsymbol{\phi}_k(x)-\delta\mu^2\boldsymbol{\phi}_i(x) \equiv\mathbf{O}_i(x), \tag{1.10}$$

where μ is the renormalized meson mass. Our final formula is thus

$$S=-\int \frac{e^{iqx}e^{-iq'y}}{(4q_0q_0')^{\frac{1}{2}}}dxdy \times(\Psi_{p'}, \{P[(\Box^2-\mu^2)\boldsymbol{\phi}_j(y), (\Box^2-\mu^2)\boldsymbol{\phi}_i(x)] +i\lambda\delta(x-y)[\delta_{ij}\boldsymbol{\phi}_k(x)\boldsymbol{\phi}_k(x) +2\boldsymbol{\phi}_i(x)\boldsymbol{\phi}_j(x)]\}\Psi_p).^4 \tag{1.11}$$

[3] M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).

[4] The author has learned from Professor Goldberger and Professor Nambu that they have separately obtained results substantially equivalent to this equation.

The theorem of Gell-Mann and Goldgerger follows simply from Eq. (1.11). Ignoring the trivial normalization factor $(4q_0q_0')^{\frac{1}{2}}$, it is obvious that

$$\langle q', j|S|q, i\rangle=\langle -q, i|S|-q', j\rangle,$$

so that for $q'=-q=0$ (which is only possible for $\mu=0$) $\langle j|S|i\rangle=\langle i|S|j\rangle$ and the scattering is a symmetric function of i and j. It must therefore be a multiple of δ_{ij} and independent of isotopic spin to this approximation.

2. DISCUSSION OF THE RESULT

Before further discussion of our result we list similar formulas for two other scattering processes. Their derivation is completely analogous to that given in Sec. I for meson-nucleon scattering.

A. Compton scattering (by a nucleon interacting with pions)

$$S=\frac{e_i'e_j}{(4k_0k_0')^{\frac{1}{2}}}\left[-2ie^2\delta_{ij}\int e^{i(k-k')x}(\Psi_{p'}, \boldsymbol{\phi}^*(x)\boldsymbol{\phi}(x)\Psi_p) -\int dxdye^{ikx}e^{-ik'y}(\Psi_{p'}, P[\mathbf{j}_i(y), \mathbf{j}_j(x)]\Psi_p)\right]. \tag{2.1}$$

Here k_μ' and e_i' are the final photon four-momentum and polarization, k_μ and e_i the initial photon four-momentum and polarization. $\boldsymbol{\phi}(x)$ and $\boldsymbol{\phi}^*(x)$ are the charged meson field operators, and $\mathbf{j}_i(x)$ is the current density operator.

B. Photopion production (to lowest order in e)

$$S\begin{Bmatrix}0\\+\\-\end{Bmatrix}=\frac{ie_j}{(4q_0k_0)^{\frac{1}{2}}}\int dxe^{ikx} \times\left[e^{-iqx}2eq_j\left(\Psi_{p'}, \begin{Bmatrix}0\\ \boldsymbol{\phi}(x)\\ -\boldsymbol{\phi}^*(x)\end{Bmatrix}\Psi_p\right) -i\int dye^{-iqy}\left(\Psi_{p'}, P\left[(\Box^2-\mu^2) \times\begin{Bmatrix}\phi_3(y)\\ \boldsymbol{\phi}(y)\\ \boldsymbol{\phi}^*(y)\end{Bmatrix}, \mathbf{j}_j(x)\right]\Psi_p\right)\right]. \tag{2.2}$$

Here q_μ is the meson four-momentum, k_μ and e_j the photon four-momentum and polarization. The curly brackets represent the cases {neutral, positive, negative} production. Also $\mathbf{j}_j(x)$ and $\boldsymbol{\phi}$, $\boldsymbol{\phi}^*$ have the same meaning as in Eq. (2.1).

As to renormalization: the two terms in Eq. (1.11) are not separately finite. Part of the λ term must be used to cancel a 4-meson divergence arising from the $[(\Box^2-\mu^2)\phi]^2$ term. Essentially the same is true of Eq. (2.1), where divergences in the two terms cancel against each other. To lowest order in e^2, however, each term of

Eq. (2.1) is finite as it stands if the matrix-elements are expressed in terms of the renormalized coupling constant g_c and the renormalized nucleon and meson masses. In Eq. (2.2) [as well as in Eq. (1.11)], a further renormalization is necessary, one which in any practical calculation can be done with no difficulty. That is, each meson operator $\phi_i(x)$ must be divided by the mesonic wave-function renormalization, $Z_3^{\frac{1}{2}}$. This extra factor is introduced because of the well-known ambiguous limit,

$$A = \lim_{q^2+\mu^2\to 0} (q^2+\mu^2)\Delta_F'(q),$$

which arises in calculating outgoing or incoming meson lines in the S-matrix. Our calculation is such that we first performed the multiplication and then took the limit as $\mu^2+q^2\to 0$. The result of this limiting order is $A=Z_3$. It is well known[2] that A must be set equal to $Z_3^{\frac{1}{2}}$ in order for the renormalization program to work, and plausible arguments can be given for this choice. This problem is not of immediate interest here, since it is already present in the conventional definition of the S-matrix [Eq. (1.1)]. The extra factor $Z_3^{\frac{1}{2}}$ is easily recognized and removed in any calculation.

In order to make practical use of the Eqs. (1.11), (2.1), and (2.2) it is convenient to transform them to the Schrödinger representation. This is accomplished by making explicit use of the Heisenberg representation time dependence

$$\begin{aligned}\mathbf{A}(x) = \mathbf{A}(x_i, x_0) &= e^{iHx_0}\mathbf{A}(x_i, 0)e^{-iHx_0}\\ &= e^{iHx_0}A(\mathbf{x})e^{-iHx_0},\end{aligned}$$

where $A(\mathbf{x}) = \mathbf{A}(x_i, 0)$ is the Schrödinger representation operator function of the space coordinate $x_i = \mathbf{x}$.

Thus an expression of the form

$$\int e^{ikx}dx(\Psi_{p'}, \mathbf{A}(x)\Psi_p) \tag{2.3}$$

is given by

$$\begin{aligned}\int e^{ikx}dx(\Psi_{p'}, \mathbf{A}(x)\Psi_p) &= \int dx e^{ikx}(\Psi_{p'}, e^{iHx_0}A(\mathbf{x})e^{-iHx_0}\Psi_p)\\ &= \int dx_0 e^{-ik_0x_0}e^{ip_0'x_0}e^{-ip_0x_0} \times \int d\tau_x(\Psi_{p'}, A(\mathbf{x})\Psi_p)e^{i\mathbf{k}\cdot\mathbf{x}}\\ &= 2\pi\delta(k_0+p_0-p_0')\int d\tau_x(\Psi_{p'}, A(\mathbf{x})\Psi_p)e^{i\mathbf{k}\cdot\mathbf{x}},\end{aligned} \tag{2.4}$$

whereas one of the form

$$\int dxdy e^{ikx}e^{-iqy}(\Psi_{p'}, P[\mathbf{A}(x), \mathbf{B}(y)]\Psi_p)$$

is given by

$$\begin{aligned}\int dxdy e^{ikx}e^{-iqy}&(\Psi_{p'}, P[\mathbf{A}(x), \mathbf{B}(y)]\Psi_p)\\ &= \frac{2\pi}{i}\delta(k_0-q_0+p_0-p_0')\Bigg(\Psi_{p'}, \Bigg[\int d\tau_x e^{i\mathbf{k}\cdot\mathbf{x}}A(\mathbf{x})\frac{1}{H-p_0+q_0-i\alpha}\int d\tau_y e^{-i\mathbf{q}\cdot\mathbf{y}}B(\mathbf{y})\\ &\qquad + \int d\tau_y e^{-i\mathbf{q}\cdot\mathbf{y}}B(\mathbf{y})\frac{1}{H-p_0-k_0-i\alpha}\int d\tau_x e^{i\mathbf{k}\cdot\mathbf{x}}A(\mathbf{x})\Bigg]\Psi_p\Bigg)\\ &= \frac{2\pi}{i}\delta(k_0-q_0+p_0-p_0')\sum_n\Bigg[\frac{\left(\Psi_{p'}, \int d\tau_x e^{i\mathbf{k}\cdot\mathbf{x}}A(\mathbf{x})\Psi_n\right)\left(\Psi_n, \int d\tau_y e^{-i\mathbf{q}\cdot\mathbf{y}}B(\mathbf{y})\Psi_p\right)}{E_n-p_0+q_0-i\alpha}\\ &\qquad + \frac{\left(\Psi_{p'}, \int d\tau_y e^{-i\mathbf{q}\cdot\mathbf{y}}B(\mathbf{y})\Psi_n\right)\left(\Psi_n, \int d\tau_x e^{i\mathbf{k}\cdot\mathbf{x}}A(\mathbf{x})\Psi_p\right)}{E_n-p_0-k_0-i\alpha}\Bigg].\end{aligned} \tag{2.5}$$

The sum is over the complete set of stationary states Ψ_n of the Hamiltonian.

By means of Eqs. (2.4) and (2.5) we see immediately that the scattering matrix elements occurring in Eq. (1.11), Eq. (2.1), and Eq. (2.2) can be written as sums over states in a manner strongly reminiscent of old-fashioned first- and second-order perturbation theory. There are two essential but not independent differences:

(a) Every possible intermediate state Ψ_n which is allowed by the conservation laws will enter into the sums.

(b) The matrix elements must be calculated between exact eigenstates of the Hamiltonian rather than between first approximations to these eigenstates (as is done in perturbation theory). Since the only property of the intermediate states which we have used is their completeness and orthogonality, they may be chosen in the most convenient way; that is, they may have incoming waves or outgoing waves or any other boundary conditions that leave them all orthogonal.

3. INTEGRAL EQUATIONS FOR PION-NUCLEON SCATTERING

We consider the expression

$$\langle p', q, j|O_i(x)|p\rangle \equiv (\Psi_{p'}^{(-)}(q,j), \mathbf{O}_i(x)\Psi_p) \equiv (\Psi_{p'}^{(-)}(q,j), (\Box^2-\mu^2)\phi_i(x)\Psi_p), \quad (3.1)$$

where $\Psi_{p'}^{(-)}(q,j)$ is an incoming wave scattering state whose plane wave part consists of a nucleon of momentum p' and a jth meson in momentum state q. The charge and spin quantum numbers of the nucleon are suppressed in this notation. Now

$$\Psi_{p'}^{(-)}(q,j)=U^\dagger(\infty,0)a_j^*(q)\Phi_{p'} \quad (3.2)$$

and $\Psi_p=U(0,-\infty)\Phi_p$, where $U(t_1,t_2)$ is the unitary operator describing the time development of a state vector in the interaction representation.[5] Thus

$$\langle p', q, j|O_i(x)|p\rangle = (\Phi_{p'}, a_j(q)U(\infty,0)(\Box^2-\mu^2)\phi_i(x)U(0,-\infty)\Phi_p) = \left(\Phi_{p'}, a_j(q)\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dt_1\cdots dt_n \times P[H_I(t_1),\cdots,H_I(t_n),O_i(x)]\Phi_p\right), \quad (3.3)$$

where, as in Sec. I,

$$O_i(x)=ig\bar{\psi}(x)\gamma_5\tau_i\psi(x)-\delta\mu^2\phi_i(x)+\lambda\phi_i(x)\phi_k(x)\phi_k(x).$$

Using the same technique as in Sec. I, we commute $a_j(q)$ through the P-bracket in Eq. (3.3). We find easily:

$$\langle p', q, j|O_i(x)|p\rangle = \frac{\lambda e^{-iqx}}{(2q_0)^{\frac{1}{2}}}\left(\Phi_{p'}, \sum_{n=0}^{\infty}\frac{(-i)^n}{n!} \times\int dt_1\cdots dt_n P[H_I(t_1),\cdots H_I(t_n), \times\delta_{ij}\phi_k(x)\phi_k(x)+2\phi_i(x)\phi_j(x)]\Phi_p\right) -i\int\frac{dy e^{-iqy}}{(2q_0)^{\frac{1}{2}}}\left(\Phi_{p'}, \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int dt_1\cdots dt_n \times P[H_I(t_1),\cdots,H_I(t_n),O_j(y),O_i(x)]\Phi_p\right), \quad (3.4)$$

$$=\frac{\lambda e^{-iqx}}{(2q_0)^{\frac{1}{2}}}(\Psi_{p'}, [\delta_{ij}\phi_k(x)\phi_k(x)+2\phi_i(x)\phi_j(x)]\Psi_p) -i\int\frac{e^{-iqy}dy}{(2q_0)^{\frac{1}{2}}}(\Psi_{p'}, P[\mathbf{O}_j(y),\mathbf{O}_i(x)]\Psi_p). \quad (3.5)$$

Equation (3.5) follows from Eq. (3.4) by the same arguments as those given in Sec. I [Eqs. (1.7) and (1.8)].

Comparison of Eqs. (3.5) and (1.11) shows that

$$\langle p', q, j|S|p, k, i\rangle = \frac{i}{(2k_0)^{\frac{1}{2}}}\int e^{ikx}dx\langle p', q, j|O_i(x)|p\rangle, \quad (3.6)$$

so that on the energy shell (i.e., when $p_\lambda'+q_\lambda-p_\lambda=k_\lambda$, where $k_\lambda^2+\mu^2=0$) the quantity $\langle p', q, j|O_i(x)|p\rangle$ is essentially the meson-nucleon S-matrix with the delta function of energy and momentum left out. It is not, however, the conventional T-matrix, since in the fixed source limit it depends primarily on the variable q and only trivially on the Fourier transform variable $p_\lambda'+q_\lambda-p_\lambda$, whereas the T-matrix depends nontrivially on both of its variables. It is this fact that makes our integral equation for $\langle p', q, j|O_i|p\rangle$ become, in the fixed source limit, an integral equation for the phase shifts.

We now transform Eq. (3.5) to the Schrödinger representation by performing the y_0 integration. We find, setting $x=0$:

$$\langle p', q, j|O_i(0)|p\rangle = \frac{\lambda}{(2q_0)^{\frac{1}{2}}}(\Psi_{p'}, [\delta_{ij}\phi_k(0)\phi_k(0)+2\phi_i(0)\phi_j(0)]\Psi_p) -\frac{1}{(2q_0)^{\frac{1}{2}}}\sum_{\substack{n\\ \mathbf{p}_n=\mathbf{p}'+\mathbf{q}}}(\Psi_{p'}, O_j(0)\Psi_n) \times(\Psi_n, O_i(0)\Psi_p)/(E_n-p_0'-q_0-i\alpha) -\frac{1}{(2q_0)^{\frac{1}{2}}}\sum_{\substack{n\\ \mathbf{p}_n=\mathbf{p}-\mathbf{q}}}(\Psi_{p'}, O_i(0)\Psi_n) \times(\Psi_n, O_j(0)\Psi_p)/(E_n-p_0+q_0-i\alpha). \quad (3.7)$$

In Eq. (3.7) the momentum $\mathbf{p}_n$ is the total momentum of the state Ψ_n. The sum is again over all the stationary states of the Hamiltonian.

Equation (3.7) is the first of a series of coupled integral equations which may be derived in a completely analogous manner to the derivation of Eq. (3.7). The iteration procedure is almost obvious for p-wave scattering. (As far as S-wave scattering is concerned this formalism does not appear particularly useful.) We retain, as a first approximation, only the single-nucleon and the single-nucleon plus single-meson intermediate states (and of course drop the λ term). The single-nucleon intermediate states provide an inhomogeneous term; the nucleon-plus-meson states involve the same

[5] M. Gell-Mann and M. Goldberger, Phys. Rev. **91**, 398 (1953).

matrix element that appears on the left-hand side of Eq. (3.7).

Now

$$(\Psi_{p'}, O_i(0)\Psi_p)=(\Psi_{p'}, (\Box^2-\mu^2)\phi_i(x)\Psi_p)|_{x=0}$$
$$=-[(p_\lambda'-p_\lambda)^2+\mu^2]$$
$$\times\Delta_{Fc}(p'-p)ig_c\Gamma_{5c}(p',p)\tau_i Z_3^{\frac{1}{2}}, \quad (3.8)$$

where Δ_{Fc} and Γ_{5c} are the renormalized meson-propagation function and nucleon-meson vertex operator, respectively, and g_c is the renormalized coupling constant. In accordance with the discussion of Sec. I we may drop the factor $Z_3^{\frac{1}{2}}$. Furthermore, for $(\Delta p)^2/m^2\ll 1$, $[(\Delta p)^2+\mu^2]\Delta_{Fc}(\Delta p)\approx 1$, and $\Gamma_{5c}(p',p)\approx(\bar{u}(p'),\gamma_5 u(p))$ so that in the nonrelativistic region we will have

$$(\Psi_{p'}, O_i(0)\Psi_p)\approx -ig_c(\bar{u}(p'),\gamma_5\tau_i u(p))$$
$$\approx -\frac{ig_c}{2m}\tau_i\boldsymbol{\sigma}\cdot\Delta\mathbf{p}.$$

Also,

$$(\Psi_{p'}, O_i(0)\Psi_n)=(\Psi_n, O_i(0)\Psi_{p'})^*,$$

since O_i is a self-adjoint operator. Equation (3.7) thus becomes

$$\langle p',q,j|O_i(0)|p\rangle=-\frac{1}{(2q_0)^{\frac{1}{2}}}\sum_{\sigma,\tau}\Bigg[\frac{(\Psi_{p'}, O_j(0)\Psi_{\mathbf{p}'+\mathbf{q}}^{(\sigma,\tau)})(\Psi_{\mathbf{p}'+\mathbf{q}}^{(\sigma,\tau)}O_i(0)\Psi_p)}{E(\mathbf{p}'+\mathbf{q})-E(\mathbf{p}')-q_0}+\frac{(\Psi_{p'}, O_i(0)\Psi_{\mathbf{p}-\mathbf{q}}^{(\sigma,\tau)})(\Psi_{\mathbf{p}-\mathbf{q}}^{(\sigma,\tau)}, O_j(0)\Psi_p)}{E(\mathbf{p}-\mathbf{q})-E(\mathbf{p})+q_0}\Bigg]$$
$$-\frac{1}{(2q_0)^{\frac{1}{2}}}\sum_{k=1}^{3}\int\frac{d\tau_{q'}d\tau_{p''}}{(2\pi)^3}\Bigg[\frac{\langle p'',q',k|O_j(0)|p'\rangle^*\langle p'',q',k|O_i(0)|p\rangle\delta(\mathbf{p}''+\mathbf{q}'-\mathbf{p}'-\mathbf{q})}{p_0''+q_0'-p_0'-q_0-i\alpha}+\frac{\langle p'',q',k|O_i(0)|p'\rangle^*\langle p'',q',k|O_j(0)|p\rangle\delta(\mathbf{p}''+\mathbf{q}'-\mathbf{p}+\mathbf{q})}{p_0''+q_0'-p_0+q_0}\Bigg], \quad (3.9)$$

where the inhomogeneous term is essentially known, at least for small momenta.

The motivation for including only one-meson intermediate states is twofold. (1) It simplifies the problem. (2) Since the experimental P-wave scattering shows a low-energy resonant behavior, one might hope that the single-meson terms would dominate the sum in Eq. (3.7) for small external meson energy.

It must unfortunately be emphasized that in a theory with strong coupling there is no compelling reason for thinking that this approximation is a good one, as compared, say, with the Tamm-Dancoff scheme, where the successive approximations follow noninteracting states.

Higher approximations can be included in either of two ways: (1) Higher terms in the series (3.7) can be calculated in Born approximation and lumped with the inhomogeneous term in Eq. (3.9), or (2) An analogous calculation to the one leading to Eq. (3.7) can be performed for amplitudes of the form

$$(\Psi_{p'}^{(-)}(q_1,q_2,j,k), (\Box^2-\mu^2)\phi_i(x)\Psi_p)$$

and

$$(\Psi_{p'}^{(-)}(q_1,j), (\Box^2-\mu^2)\phi_i(x)\Psi_p^{(+)}(q_2,k)),$$

etc., leading to integral equations which will be coupled to Eq. (3.9). Arbitrary numbers of pions can be included in this way, although the complexity of the problem increases tremendously with each order. The point we would like to emphasize in favor of our scheme is that, as long as the meson-meson interaction is omitted, together with intermediate states involving pairs, these equations are automatically expressed in terms of the renormalized masses and coupling constant so that no new divergences will appear. This is the main advantage of the present method as contrasted with the Dancoff or Bethe-Salpeter approximation scheme.

In the fixed-source limit, Eq. (3.9) simplifies considerably. The appropriate matrix-element is $\langle q,j,\tau'|O_i(k)|\tau\rangle$, (where $O_i(k)=i(f/\mu)\boldsymbol{\sigma}\cdot\mathbf{k}\tau_i$ and $|\tau\rangle$ is a one-nucleon state), and satisfies the equation

$$\langle q,j,\tau'|O_i(k)|\tau\rangle$$
$$=\frac{1}{(2q_0)^{\frac{1}{2}}}\Bigg[\left(\frac{f}{\mu}\right)^2(\tau_j\tau_i\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}-\tau_i\tau_j\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{q})\Bigg]$$
$$-\sum_{l=1}^{3}\sum_{\tau''=1}^{2}\int\frac{d\tau_p}{(2\pi)^3}$$
$$\times\left\{\frac{\langle p,l,\tau''|O_j(q)|\tau'\rangle^*\langle p,l,\tau''|O_i(k)|\tau\rangle}{p_0-q_0-i\alpha}+\frac{\langle p,l,\tau''|O_i(k)|\tau'\rangle^*\langle p,l,\tau''|O_j(q)|\tau\rangle}{p_0+q_0}\right\}. \quad (3.10)$$

Here f is the rationalized, renormalized, pseudovector coupling constant.

Equation (3.10) immediately yields an equation for the phase-shifts. We define $g_i(q)=e^{i\delta(q)}\sin\delta(q)$, where $i=1$ stands for the $(\frac{1}{2},\frac{1}{2})$ state, $i=2$ for the $(\frac{1}{2},\frac{3}{2})$ and

$(\frac{3}{2},\frac{1}{2})$ states and $i=3$ for the $(\frac{3}{2},\frac{3}{2})$ state. Then

$$g_1(q)=-\frac{8}{3}\frac{f^2}{4\pi}\frac{q^3}{\mu^2\omega_q}+\frac{q^3}{\pi}\int\frac{d\omega_p}{p^3}\left[\frac{|g_1(p)|^2}{\omega_p-\omega_q-i\alpha}+\frac{|g_1(p)|^2-8|g_2(p)|^2+16|g_3(p)|^2}{9(\omega_p+\omega_q)}\right],$$

$$g_2(q)=-\frac{2}{3}\frac{f^2}{4\pi}\frac{q^3}{\mu^2\omega_q}+\frac{q^3}{\pi}\int\frac{d\omega_p}{p^3}\left[\frac{|g_2(p)|^2}{\omega_p-\omega_q-i\alpha}+\frac{7|g_2|^2-2|g_1|^2+4|g_3|^2}{9(\omega_p+\omega_q)}\right],\qquad(3.11)$$

$$g_3(q)=\frac{4}{3}\frac{f^2}{4\pi}\frac{q^3}{\mu^2\omega_q}+\frac{q^3}{\pi}\int\frac{d\omega_p}{p^3}\left[\frac{|g_3(p)|^2}{\omega_p-\omega_q-i\alpha}+\frac{4|g_1|^2+4|g_2|^2+|g_3|^2}{9(\omega_p+\omega_q)}\right].$$

Here $\omega_p=p_0=(p^2+\mu^2)^{\frac{1}{2}}$.

The coupling of the various amplitudes is not in contradiction with the over-all conservation laws; for example, the $|g_1|^2$ term in the g_3 equation takes into account a self-energy process modifying the $(\frac{3}{2},\frac{3}{2})$ scattering.

We finally remark that a linear integral equation may be derived for the γ–π production matrix element by methods that are completely analogous to those we have used to obtain Eq. (3.9). In this case we must consider the matrix element $(\Psi_{p'}(q,j),\mathbf{j}(x)\Psi_p)$, where $\mathbf{j}(x)$ is the current density. The inhomogeneous term now consists not only of the contribution of single nucleon states to the sum over states but of the direct production term [essentially the first term in Eq. (2.2)]. The kernel of the equation (if the sum is restricted to single meson states) is just the matrix element $O_i(x)$ which turned up in the meson-scattering problem, and which for the fixed source P-wave theory is trivially related to the scattering phase shift.

The resulting equation for the fixed source theory follows:

$$\langle q,j,\tau'|j_{k\lambda}|\tau\rangle(2q_0)^{\frac{1}{2}}=\frac{ef}{\mu}\Big\langle\tau'\Big|(\delta_{j2}\tau_1-\delta_{j1}\tau_2)\left(\boldsymbol{\sigma}\cdot\mathbf{e}_\lambda+\frac{2\mathbf{q}\cdot\mathbf{e}_\lambda\boldsymbol{\sigma}\cdot(\mathbf{k}-\mathbf{q})}{(\mathbf{q}-\mathbf{k})^2+\mu^2}\right)+\frac{f}{\mu q_0}\Big\{[\tau_j\boldsymbol{\sigma}\cdot\mathbf{q},\boldsymbol{\mu}\times\mathbf{k}\cdot\mathbf{e}_\lambda]-\frac{e(1+\tau_3)}{4m}\mathbf{q}\cdot\mathbf{e}_\lambda i\boldsymbol{\sigma}\cdot\mathbf{q}\tau_j\Big\}\Big|\tau\Big\rangle-\sum_{\tau'',l}\int\frac{d\tau_p}{(2\pi)^3}\times\left\{\frac{\langle p,l,\tau''|O_j(q)|\tau'\rangle^*\langle p,l,\tau''|j_{k\lambda}|\tau\rangle}{\omega_p-\omega_q-i\alpha}+\frac{\langle p,l,\tau''|j_{k\lambda}|\tau'\rangle^*\langle p,l,\tau''|O_j(q)|\tau\rangle}{\omega_p+\omega_q}\right\},\qquad(3.12)$$

where $j_{k\lambda}=\int e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{j}(x)\cdot\mathbf{e}_\lambda$, $\boldsymbol{\mu}$ is the nucleon magnetic moment operator and $\langle p,l,\tau''|O_j(q)|\tau\rangle$ is the meson-nucleon scattering matrix element that satisfies Eq. (3.10).

4. KROLL-RUDERMAN THEOREM

We shall now show how Eq. (2.2) immediately leads to the Kroll-Ruderman[6] theorem on threshold photoproduction.

At threshold $\mathbf{q}=0$ so that the first term in Eq. (2.2) does not contribute. The S-matrix is thus given by

$$S\begin{Bmatrix}0\\+\\-\end{Bmatrix}=\frac{e_j}{(4q_0k_0)^{\frac{1}{2}}}\int dxdye^{ikx}e^{-iqy}\times\left(\Psi_{p'},P\left[(\Box^2-\mu^2)\begin{Bmatrix}\phi_3(y)\\\phi(y)\\\phi^*(y)\end{Bmatrix},\mathbf{j}_j(x)\right]\Psi_p\right),\qquad(4.1)$$

with $q_i=0$, $q_0=\mu$, $k_0+E(k_i)=\mu+m$, $p_i'=0$, $p_0'=m$.

The coefficient of e_j in Eq. (4.1) is a pseudovector, and must be of the form $s_j=(4q_0k_0)^{-\frac{1}{2}}[A\sigma_j+B\boldsymbol{\sigma}\cdot\mathbf{k}k_j]$. Since we are going to let $k_i\to0$, we neglect the second term, and attempt to calculate the coefficient A. We multiply s_i by k_i. Thus, if we call

$$(\Box^2-\mu^2)\begin{Bmatrix}\phi_3(y)\\\phi(y)\\\phi^*(y)\end{Bmatrix}=\mathbf{O}(y),$$

we find

$$\begin{aligned}A\sigma_ik_i&=\int dxdyk_ie^{ikx}e^{-iqy}(\Psi_{p'},P[\mathbf{O}(y),\mathbf{j}_i(x)]\Psi_p)\\&=i\int dxdye^{ikx}e^{-iqy}\times\left(\Psi_{p'},P\left[\mathbf{O}(y),\frac{\partial}{\partial x_i}\mathbf{j}_i(x)\right]\Psi_p\right)\\&=-i\int dxdye^{ikx}e^{-iqy}\times(\Psi_{p'},P[\mathbf{O}(y),\partial\rho(x)/\partial x_0]\Psi_p)\\&=-i\int dxdye^{ikx}e^{-iqy}\left(\Psi_{p'},\left\{\frac{\partial}{\partial x_0}P[\mathbf{O}(y),\rho(x)]+\delta(x_0-y_0)[\rho(x),\mathbf{O}(y)]\right\}\Psi_p\right)\\&=k_0\int dxdye^{ikx}e^{-iqy}(\Psi_{p'},P[\rho(x),\mathbf{O}(y)]\Psi_p)\\&\quad-i\int dxdye^{ikx}e^{-iqy}\delta(x_0-y_0)\times(\Psi_{p'},[\rho(x),\mathbf{O}(y)]\Psi_p).\qquad(4.2)\end{aligned}$$

The first term in Eq. (4.2) vanishes with k like $k_0\cdot k_i$ and is thus a contribution to A of order $k_0/m\cong\mu/m$. This can

[6] N. M. Kroll and M. A. Ruderman, Phys. Rev. 93, 233 (1954).

be seen most easily by expanding it into a sum over states such as the one in Eq. (2.5). Then the quantity $\rho(x)$ enters only in the form $\int e^{i\mathbf{k}\cdot\mathbf{x}}\rho(\mathbf{x})d\tau_x$ which, as $k_i\rightarrow 0$, is a c-number, the total charge, and hence has no off-diagonal matrix elements. That part of the sum having a free nucleon as intermediate state vanishes identically, since $\mathbf{q}=0$ and $(\Psi_p, \int \mathbf{O}(y)d\tau_y\Psi_p)=0$. Thus we need keep only the second term of (3.2) as $\mathbf{k}\rightarrow 0$, so that we may write

$$A\sigma_i k_i \cong -i\int dxdye^{ikx}e^{-iqy}\delta(x_0-y_0) \times(\Psi_{p'}, [\rho(x), \mathbf{O}(y)]\Psi_p). \quad (4.3)$$

The commutator in Eq. (4.3) can be explicitly calculated, since the two operators are to be taken at the same time.

Since

$$(\square^2-\mu^2)\begin{Bmatrix}\phi_3(y)\\ \phi(y)\\ \phi^*(y)\end{Bmatrix}=ig\bar{\psi}(y)\begin{Bmatrix}\tau_3\\ \sqrt{2}\tau_-\\ \sqrt{2}\tau_+\end{Bmatrix}\gamma_5\psi(y) -\delta\mu^2\begin{Bmatrix}\phi_3(y)\\ \phi(y)\\ \phi^*(y)\end{Bmatrix}+\lambda\phi_k(y)\phi_k(y)\begin{Bmatrix}\phi_3(y)\\ \phi(y)\\ \phi^*(y)\end{Bmatrix}$$

and

$$\rho(x)=ie[\pi^*(x)\phi^*(x)-\pi(x)\phi(x)] +e\psi^*(x)\left(\frac{1+\tau_3}{2}\right)\psi(x),$$

we find, for $x_0=y_0$,

$$[\rho(x), \mathbf{O}(y)]=e\delta(\mathbf{x}-\mathbf{y})(\square^2-\mu^2)\begin{Bmatrix}0\\ -\phi(x)\\ \phi^*(x)\end{Bmatrix}, \quad (4.4)$$

so that

$$A\sigma_i k_i=-ie\int dxe^{i(k-q)x} \times\left(\Psi_{p'}, (\square^2-\mu^2)\begin{Bmatrix}0\\ -\phi(x)\\ \phi^*(x)\end{Bmatrix}\Psi_p\right). \quad (4.5)$$

The integrated matrix element on the right of Eq. (4.5) can be evaluated exactly. It is

$$-i(2\pi)^4\delta(k_\lambda-q_\lambda+p_\lambda-p_\lambda')[-(k-q)_\nu^2-\mu^2] \times\Delta_{Fc}((k-q)_\eta^2)ieg_c\Gamma_{5c}(p', p)\begin{Bmatrix}0\\ -\sqrt{2}\\ \sqrt{2}\end{Bmatrix}Z_3^{\frac{1}{2}}, \quad (4.6)$$

where the delta function is 4 dimensional and Δ_{Fc} and Γ_{5c} are, respectively, the renormalized meson propagation function and meson-nucleon 3-vertex operator. g_c is the renormalized mesonic coupling constant.

Now $\Gamma_{5c}(p', p)=\gamma_5 g((p'-p)_\lambda^2)$, so that, to order k^2, $\Gamma_{5c}(p', p)\approx\gamma_5 g(0)=\gamma_5$. Similarly $\Delta_{Fc}(\Delta p)=1/[(\Delta p)^2+\mu^2]+\sim 1/m^2$ so that $[\mu^2+(\Delta p)^2]\Delta_{Fc}(\Delta p)\approx 1+\sim\mu^2/m^2$. We find, therefore,

$$A\sigma_i k_i\cong(2\pi)^4 i\delta(k-q+p-p') \times\begin{Bmatrix}0\\ -\sqrt{2}\\ \sqrt{2}\end{Bmatrix}\bar{u}(p')ieg_c\gamma_5 u(p), \quad (4.7)$$

where we have dropped the factor $Z_3^{\frac{1}{2}}$ as discussed in Sec. I. In the c.m. system, $\mathbf{p}=-\mathbf{k}$, $\mathbf{p}'=0$. $\bar{u}(p')ig_c\gamma_5 u(p)$ is thus given by

$$(u(0), ig_c\beta\gamma_5 u(-\mathbf{k}))=ig_c\boldsymbol{\sigma}\cdot\mathbf{k}/2m(1+\sim k^2/m^2), \quad (4.8)$$

and

$$A=-(2\pi)^4\delta(k-q+p-p')\frac{eg_c}{2m}\begin{Bmatrix}0\\ -\sqrt{2}\\ \sqrt{2}\end{Bmatrix}. \quad (4.9)$$

The S-matrix is given by

$$S=-\frac{(2\pi)^4}{(4q_0k_0)^{\frac{1}{2}}}\delta(k-q+p-p')\mathbf{e}_\lambda\cdot\boldsymbol{\sigma}\frac{eg_c}{2m} \times\left[\begin{Bmatrix}0\\ -\sqrt{2}\\ \sqrt{2}\end{Bmatrix}+\sim\frac{\mu}{m}\right] \quad (4.10)$$

for neutral, positive, and negative threshold production, respectively. Equation (4.10) is equivalent to the result of Kroll and Ruderman.