Functional Analysis and Strong-Coupling Theory

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The relation of functional analysis to the Hamiltonian approach to strongly interacting quantized fields is examined. The propagation of fermions in an external unquantized boson field provides the basis for the solutions, as in the Green function method. The functional method may be useful in studying real nucleons and bound states.

FUNCTIONAL methods have recently been used by Edwards and Peierls[1] to derive an approximation to Green's function for the propagation of a nucleon including radiative corrections from Green's function for the propagation in an external field; their method is a development of the technique invented by Schwinger.[2] Matthews and Salam[3] have shown how the work of Edwards and Peierls may be extended; they derive the complete Green function for nucleon propagation by using functional integrations of the type first used in field theory by Feynman.[4] These functional techniques, which are deduced from the quantum-mechanical form of the action principle, have been developed in the hope that they will make possible approximation methods other than perturbation theory; however, the difficulties in using these methods are considerable, and there is no easy way of estimating the accuracy of any suggested approximate solution.

The functional methods of Schwinger and Feynman have all the advantages of covariant formulation, but they are necessarily difficult to apply to fields which interact strongly. The functional method can be introduced more easily in a simple single-time approach to the problem of strongly interacting fields; in fact the direct approach using the Hamiltonian appears to require the notion of propagation in an external field and a functional procedure. The form of the resulting equations suggests that they might be more suitable for approximations to strongly bound states and to real nucleon states.

As an example we use the theory of nucleons and charged pseudoscalar mesons with a charge independent pseudoscalar coupling. The Hamiltonian $\mathbf{H}$ is given by

$$\mathbf{H}=\mathbf{H}_M+\mathbf{H}_I+\mathbf{H}_N,$$

with

$$\mathbf{H}_M=\tfrac{1}{2}\int d^3\mathbf{x}\{\boldsymbol{\pi}\cdot\boldsymbol{\pi}+c^2(\mathrm{grad}\boldsymbol{\phi}\cdot\mathrm{grad}\boldsymbol{\phi})+c^2\mu^2(\boldsymbol{\phi}\cdot\boldsymbol{\phi})\},$$

$$\mathbf{H}_N=\hbar c\int d^3\mathbf{x}\{\bar{\psi}(\boldsymbol{\gamma}\cdot\mathrm{grad})\psi+\kappa\bar{\psi}\psi\},\tag{1}$$

$$\mathbf{H}_I=\frac{ig}{\sqrt{2}}\int d^3\mathbf{x}\bar{\psi}(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\gamma^5\psi,$$

$\boldsymbol{\phi}=(\phi_1,\phi_2,\phi_3)$, $\boldsymbol{\pi}=(\pi_1,\pi_2,\pi_3)$ are Hermitian vectors in isotopic space; $\boldsymbol{\tau}\times\boldsymbol{\tau}=2i\boldsymbol{\tau}$, $\boldsymbol{\tau}_j^2=I$ $(j=1,2,3)$; and $\bar{\psi}$, ψ are 8-component nucleon variables ($\bar{\psi}=\psi^{\dagger}\gamma^4$). The commutation relations are given at one time t (and all manipulations are carried out at this time):

$$\begin{aligned}[\boldsymbol{\pi}(\mathbf{x},t),\,\boldsymbol{\pi}(\mathbf{x}',t)]_-&=0=[\boldsymbol{\phi}(\mathbf{x},t),\,\boldsymbol{\phi}(\mathbf{x}',t)]_-,\\ [\pi_j(\mathbf{x},t),\,\phi_k(\mathbf{x}',t)]_-&=-i\hbar\delta_{jk}\delta^{(3)}(\mathbf{x}-\mathbf{x}');\\ [\bar{\psi}(\mathbf{x},t),\,\bar{\psi}(\mathbf{x}',t)]_+&=0=[\psi(\mathbf{x},t),\,\psi(\mathbf{x}',t)]_+,\\ [\bar{\psi}_\sigma(\mathbf{x},t),\,\psi_\rho(\mathbf{x}',t)]_+&=\delta_{\sigma\rho}\delta^{(3)}(\mathbf{x}-\mathbf{x}').\end{aligned}\tag{2}$$

$\mathbf{H}_N$ and $\mathbf{H}_I$ could be written in a charge self-conjugate form, rather than (1), but that is not necessary for the discussion.

In perturbation theory the problem of finding the eigenstates Ψ of $\mathbf{H}$ is solved by first diagonalizing $\mathbf{H}_M+\mathbf{H}_N$, while in any strong-coupling theory it is essential to try first to solve the eigenstate problem either for $\mathbf{H}_M+\mathbf{H}_I$ or for $\mathbf{H}_N+\mathbf{H}_I$. The strong-coupling method of Wentzel[5,6] and the intermediate-coupling method of Tomonaga[6–8] start from $\mathbf{H}_M+\mathbf{H}_I$, so that the nucleon recoil has to be added as a perturbation. Apart from the difficulty of allowing for nucleon recoil, these theories become very complicated because of the nature of the nucleon source functions which appear in $\mathbf{H}_I$. Because the three components $\bar{\psi}\tau_j\gamma^5\psi$ do not commute, it is not possible in practice to diagonalize $\mathbf{H}_M+\mathbf{H}_I$ by a single canonical transformation; as a result, any effective manipulation is very difficult. (Scalar or pseudoscalar uncharged meson theory with the simplest coupling has only one component $\bar{\psi}\psi$ or $\bar{\psi}\gamma^5\psi$ occuring in $\mathbf{H}_I$, and a simple canonical transformation solves the problem[9]—apart from nucleon recoil.)

The advantage of first diagonalizing $\mathbf{H}_N+\mathbf{H}_I$ is that the meson variables $\boldsymbol{\phi}(\mathbf{x},t)$ which appear in $\mathbf{H}_I$ commute with each other and with all nucleon variables; the $\boldsymbol{\phi}(\mathbf{x},t)$ can therefore be treated as functions (not operators) in $\mathbf{H}_N+\mathbf{H}_I$, and they will be written $\boldsymbol{\phi}'(\mathbf{x})$ to indicate that they are eigenvalues. The nucleon eigenstates $\Psi_N[\boldsymbol{\phi}']$ and eigenvalues $E[\boldsymbol{\phi}']$ determined

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by

$$(\mathbf{H}_N+\mathbf{H}_I[\boldsymbol{\phi}'])\Psi_N[\boldsymbol{\phi}']=E[\boldsymbol{\phi}']\Psi_N[\boldsymbol{\phi}'] \tag{3}$$

are functionals of $\boldsymbol{\phi}'(\mathbf{x})$. They obviously describe the behavior of the nucleon field in an external meson field $\boldsymbol{\phi}'(\mathbf{x})$.

The solutions of

$$\mathbf{H}\Psi=E\Psi \tag{4}$$

are directly related to the solutions of (3). We use a representation for Ψ specified by some complete set of eigenvalues for the nucleons, and by $\boldsymbol{\phi}'(\mathbf{x})$. $\mathbf{H}_M$ behaves like a number insofar as the nucleon components of Ψ are concerned, but it has nonzero matrix elements between states with different $\boldsymbol{\phi}'(\mathbf{x})$. The solution Ψ can therefore be written

$$\Psi=\sum_{S(\phi')} a[\boldsymbol{\phi}']\Psi_N[\boldsymbol{\phi}'], \tag{5}$$

where $\Psi_N[\boldsymbol{\phi}']$ is a normalized solution of (3). For the moment we relax the condition that $\Psi_N[\boldsymbol{\phi}']$ is normalized and write

$$\Psi=\sum_{S(\phi')} \Psi_N[\boldsymbol{\phi}']. \tag{6}$$

$S(\phi')$ is some set of functions $\boldsymbol{\phi}'(\mathbf{x})$ over which the summation is to be carried out. The set $S(\phi')$ need only be large enough to ensure that (8) below has a solution, or an approximate solution; one advantage of this method over that of Edwards and Peierls[1] is that non-enumerable sums need not appear.

The solution of (4) is thus

$$\sum_{S(\phi')} \mathbf{H}_M\Psi_N[\boldsymbol{\phi}']=\sum_{S(\phi')}(E-E[\boldsymbol{\phi}'])\Psi_N[\boldsymbol{\phi}']. \tag{7}$$

In this representation

$$\boldsymbol{\pi}(\mathbf{x})=\frac{\hbar}{i}\frac{\delta}{\delta\boldsymbol{\phi}(\mathbf{x})},$$

and (7) gives the eigenvalue equation

$$\sum_{S(\phi')}\tfrac{1}{2}\int d^3\mathbf{x}\left\{\left(\frac{\hbar}{i}\right)^2\frac{\delta}{\delta\boldsymbol{\phi}'(\mathbf{x})}\cdot\frac{\delta}{\delta\boldsymbol{\phi}'(\mathbf{x})}+c^2(\operatorname{grad}\boldsymbol{\phi}'(\mathbf{x})\cdot\operatorname{grad}\boldsymbol{\phi}'(\mathbf{x}))+c^2\mu^2\boldsymbol{\phi}'(\mathbf{x})\cdot\boldsymbol{\phi}'(\mathbf{x})\right\}\Psi_N[\boldsymbol{\phi}']=\sum_{S(\phi')}(E-E[\boldsymbol{\phi}'])\Psi_N[\boldsymbol{\phi}'], \tag{8}$$

with the eigenstate Ψ given by (6).

It is easy to derive equations equivalent to (8) for other representations; for example, with operators $\mathbf{q}_\mathbf{k}$, $\mathbf{p}_\mathbf{k}$ given by

$$\boldsymbol{\phi}(\mathbf{x})=\textstyle\sum_\mathbf{k}\mathbf{q}_k e^{i\mathbf{k}\cdot\mathbf{x}},\quad \boldsymbol{\pi}(\mathbf{x})=\sum_\mathbf{k}\mathbf{p}_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}},$$

the eigenvalues $\mathbf{q}_\mathbf{k}'$ can be used to label the boson component, and an equation similar to (8) with $\mathbf{p}_\mathbf{k}$ replaced by $(\hbar/i)(\delta/\delta\mathbf{q}_\mathbf{k})$ is obtained.

Some idea of the techniques which are required may be obtained by looking at the degenerate case in which g vanishes [see (1)]. The solution of (3) is $\Psi_N[\boldsymbol{\phi}']=\Psi_N$, $E[\boldsymbol{\phi}']=E_N$, and (5) becomes

$$\Psi=\sum_{S(\phi')}a[\phi']\Psi_N.$$

In (8) the term $(\operatorname{grad}\boldsymbol{\phi}'(\mathbf{x})\cdot\operatorname{grad}\boldsymbol{\phi}'(\mathbf{x}))$ can be replaced by $-\boldsymbol{\phi}'(\mathbf{x})\cdot\nabla^2\boldsymbol{\phi}'(\mathbf{x})$, and the summation over $\boldsymbol{\phi}'$ can be limited to the set $S_k(\phi')$ of normalized functions $\boldsymbol{\phi}'(\mathbf{x})$ for which

$$\nabla^2\boldsymbol{\phi}'(\mathbf{x})+\mathbf{k}^2\boldsymbol{\phi}'(\mathbf{x})=0,\quad (\mathbf{k}\text{ real}).$$

Equation (8) reduces to

$$\sum_{S_k(\phi')}\tfrac{1}{2}\int d^3\mathbf{x}\left\{-\hbar^2\frac{\delta}{\delta\boldsymbol{\phi}'(\mathbf{x})}\cdot\frac{\delta}{\delta\boldsymbol{\phi}'(\mathbf{x})}+c^2(\mathbf{k}^2+\mu^2)\boldsymbol{\phi}'(\mathbf{x})\cdot\boldsymbol{\phi}'(\mathbf{x})\right\}a[\boldsymbol{\phi}']=E_M\sum_{S_k(\phi')}a[\boldsymbol{\phi}'],$$

where $E_M=E-E_N$. The set $S_k(\phi')$ can be reduced to a single function:

$$a[\boldsymbol{\phi}']=\prod_{j=1}^{3}f(\phi_j'),$$

and a particular solution is

$$f(\phi_j)=\exp(-\tfrac{1}{2}y_j),$$

$$y_j=\left(\frac{c^2(\mathbf{k}^2+\mu^2)}{\hbar^2}\right)^{\frac{1}{2}}\int d^3\mathbf{x}\phi_j^2(\mathbf{x}).$$

In general the choice of the set $S(\phi')$ has to be varied according to the solution which is required; for example, to study real nucleons, functions $\boldsymbol{\phi}'(\mathbf{x})$ which are nonzero only near $\mathbf{x}=0$ are required. The interpretation of the eigenstates Ψ given by (6) and (8) is obtained by classifying the solutions by the eigenvalues of those operators which commute with $\mathbf{H}$. The vacuum state should be the eigenstate of least energy. Further details of the method will be given elsewhere.

Note added in proof.—Equation (9) is separable and its solutions can be studied by assigning $\phi'(\mathbf{x}_s)$ to the sth cell of space. If a cell is of volume $\delta\mathbf{x}$, a solution is $f[\phi_j']=\Pi_s f_s(\phi_j(\mathbf{x}_s))$ with $f_s(\phi_j)=H_N(Z)\ \exp(-\tfrac{1}{2}Z^2)$. $Z=(\delta\mathbf{x}\cdot c/\hbar)^{\frac{1}{2}}(\mathbf{k}^2+\mu^2)^{\frac{1}{4}}\phi_j(\mathbf{x}_s)$.