Magnetic Moments of Conjugate Nuclei

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Many experimentally-observed magnetic moments of conjugate (semi-mirror) nuclei deviate approximately the same amount from the Schmidt limits. It is shown that the theoretical magnetic moments provided by a simple form of the collective model are such that the difference between the deviations of a pair of conjugate nuclei is small. Those experimentally-observed exceptions to this rule where the difference between the deviations is quite large (such as $I^{127}-Mo^{97}$) can be explained by modifications of the collective model to include the effects of (a) shell model structure and (b) configuration admixing. In fact, the relative importance of the various dynamical features of the nucleus can be assessed, qualitatively at least, from the size and sign of the magnetic moment deviation difference of a pair of conjugate nuclei.

INTRODUCTION

THE symmetry properties, angular momentum and parity, of the ground states of most odd-even nuclei are correctly predicted by the independent-particle shell model of Mayer[1] and Haxel, Jensen, and Suess.[2] But this model seems less successful in determining the dynamical features of the nucleus as exemplified by the deviations of the experimental magnetic moments of odd-even nuclei from the predicted Schmidt values. The magnetic moment deviations indicate that either an incomplete magnetic moment operator is being used or that the independent-particle shell-model wave functions are a rather poor description of the complex nuclei.

Attempts to explain the magnetic moment deviations of the heavier nuclei by means of additional magnetic moments arising from meson effects have not been too successful. A detailed investigation by Ross[3] on meson exchange moments, normalized to explain the H^3 and He^3 magnetic moment anomalies, indicates that the experimentally observed magnetic moment deviations cannot be adequately accounted for on the basis of meson effects.

Recent attempts to explain the observed magnetic moment deviations have centered about the collective model approach of Bohr[4,5] and Mottelson and/or the modification of the Mayer-Jensen shell model due to configuration mixing. Both methods have resulted in a substantial improvement for $I=L+1/2$ nuclei. The pure collective model (without configuration mixing) contradicts observations by predicting no deviation for $I=1/2$ nuclei and by predicting deviations outside of the Schmidt line for many $I=L-1/2$ nuclei. The configuration mixing approach in an extreme form as suggested by Volkov[6] predicts deviations for $I=L-1/2$ nuclei too far *inside* the appropriate Schmidt line. The more fundamental configuration mixing calculations of Blin-Stoyle[7,8] gives excellent agreement for all $I=1/2$ nuclei ($I=L+1/2$ and $I=L-1/2$) as well as an unambiguous explanation for the anomalous closed shell plus one nucleus, Bi^{209}. However, it is difficult to believe that the many observed large quadrupole moments can be explained in terms of only mixed configurations which involve only a few particles.

Undoubtedly, complex nuclei must be described both in terms of collective phenomena and mixed configurations. In fact, Bohr and Mottelson considered the mixing of the configuration of the last odd particle due to interaction with the surface oscillations. However, if possible, it would be desirable to try to determine the relative importance to the magnetic moment of the collective motion of the nucleons as opposed to the admixing of different configurations.

CONJUGATE NUCLEI

Schawlow and Townes[9] observed that conjugate (semi-mirror) nuclei, i.e., an odd-even nuclear pair (one nucleus having Z odd protons while the other has Z odd neutrons), have nearly the same magnetic moment deviation from their respective Schmidt lines. This result can be explained by assuming that (a) the odd-particle wave functions are primarily responsible for the magnetic moment deviation, and (b) the odd-particle wave functions are the same in both nuclei.

In this connection, it is convenient to define the

[1] M. Goeppert-Mayer, Phys. Rev. **78**, 16 (1950).
[2] Haxel, Jensen, and Suess, Z. Physik **128**, 295 (1950).
[3] Marc Ross, Phys. Rev. **88**, 939 (1952). Additional references concerning meson effects and magnetic moments are to be found in this reference.
[4] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952).
[5] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).
[6] A. B. Volkov, Phys. Rev. **94**, 1664 (1954). Additional references concerning configuration mixing are to be found in this reference.
[7] R. J. Blin-Stoyle, Proc. Phys. Soc. (London) **A66**, 11 (1953).
[8] R. J. Blin-Stoyle and M. A. Perke, Proc. Phys. Soc. (London) **A67**, 885 (1954).
[9] A. L. Schawlow and C. H. Townes, Phys. Rev. **82**, 268 (1951).

magnetic moment deviations in the following manner:

$$\Delta^P=\frac{\mu_S{}^P-\mu^P}{\mu_P-\frac{1}{2}},\quad \Delta^N=\frac{\mu_S{}^N-\mu^N}{\mu_N}, \tag{1}$$

where μ_P and μ_N are the intrinsic magnetic moments of the proton and neutron, $\mu_S{}^P$ and $\mu_S{}^N$ are the respective Schmidt values of the magnetic moment, and μ^P and μ^N are the corresponding calculated or experimental magnetic moments. If the even particles are coupled to zero angular momentum, then regardless of the complexity of the odd-particle wave function,

$$\Delta^P=\Delta^N \quad\text{or}\quad \Delta^P-\Delta^N=0, \tag{2}$$

provided the odd-particle wave functions are the same for the nuclear pair (charge symmetry).[6]

The experimental deviations for sixteen cases of conjugate nuclei are listed in Table I.[10] The last column lists the quantity $\Delta^P-\Delta^N$ which is a measure of the validity of Eq. (2). It can be seen that with the exception of four cases $\Delta^P-\Delta^N$ is either quite small or else small compared to Δ^P and Δ^N themselves. Of these four cases, two ($Au^{197}-Ba^{135}$ and $Pr^{141}-Pd^{105}$) are poorly measured. In the worst case ($I^{127}-Mo^{97}$), I^{127} with a spin of 5/2 is in a region of very close competition with nuclei having a spin of 7/2 (the same competition is not evident among the corresponding odd-neutron nuclei). The competition between the $g_{7/2}$ and $d_{5/2}$ levels probably accounts for the extremely large magnetic moment deviation of I^{127} and the resulting large value for $\Delta^P-\Delta^N$. In the final case which is poor, the deviation of Nb^{93} appears to be too small when compared to other $g_{9/2}$ nuclei. The interpretation of this case in terms of shell-model structure is discussed later.

CONJUGATE NUCLEI AND THE COLLECTIVE MODEL

It has been suggested[6] that the mirror property of the magnetic deviations of the conjugate nuclei represents an argument against the collective model of the nucleus. The number of even particles in the nuclei of the conjugate pair are generally quite different because of the neutron excess. For this reason it would seem likely that the core contribution should be quite different for the two nuclei, leading to rather large values of $\Delta^P-\Delta^N$. However, it is found that the collective model of Bohr and Mottelson without configuration mixing also predicts small values (not zero as in the configuration case) for $\Delta^P-\Delta^N$.

The small values predicted for $\Delta^P-\Delta^N$ are a consequence of ignoring any possible shell structure in the core, e.g., by assuming a particular coupling (strong or weak) to be valid for both nuclei of the pair. The data for the magnetic moments of conjugate nuclei show for instance that the strong-coupling limit of the collective model without configuration mixing cannot be valid for several pairs. While this conclusion is not new, the approach may lead to some clarification as to the limitations which must be placed on the various forms of the collective model.

The magnetic moment operator for the odd-proton nucleus may be written as

$$\boldsymbol{\mu}^P=\mu_0[\mathbf{I}+(\mu_P-\tfrac{1}{2})\boldsymbol{\sigma}_P'+\textstyle\sum_P(\mu_P-\tfrac{1}{2})\boldsymbol{\sigma}_P+\sum_N(-\mathbf{j}_N+\mu_N\boldsymbol{\sigma}_N)], \tag{3}$$

where primes indicate the last odd particle and the summations extend over the core particles. μ_0 is the nuclear Bohr magneton, $\boldsymbol{\sigma}$ is the Pauli spin operator, and $\mathbf{I}$ is the total angular momentum of the nucleus.

The corresponding magnetic moment operator for the odd-neutron nucleus is

$$\boldsymbol{\mu}^N=\mu_0\{\mu_N\boldsymbol{\sigma}_N'+\textstyle\sum_P[\mathbf{j}_P+(\mu_P-\tfrac{1}{2})\boldsymbol{\sigma}_P]+\sum_N\mu_N\boldsymbol{\sigma}_N\}. \tag{4}$$

The wave function for the odd-proton nucleus is written as

$$\psi_I{}^M(P)=\sum_{\gamma,I,j}\alpha_{\gamma,\lambda,j}(A_P)\sum_{\mu}\langle j,\lambda,M-\mu,\mu|j,\lambda,I,M\rangle \times C_\lambda{}^\mu(A_P)\phi_j{}^{M-\mu}(P). \tag{5}$$

$C_\lambda{}^\mu(A_P)$ is the angular part of the core function of the odd-proton nucleus having angular momentum λ, a z component of angular momentum μ, and A_P particles in the core. $\langle j,\lambda,M-\mu,\mu|j,\lambda,I,M\rangle$ is the appropriate Clebsch-Gordon coefficient as defined in Condon and Shortley.[11] $\phi_j{}^m$ is the wave function of the odd particle having total angular momentum j. The summation over j thus includes the possibility of configuration admixing due to the interaction of the odd particle with the surface. γ includes all other required quantum numbers to specify the wave function such as the occupation numbers for the surfons (quantized surface vibrations), as well as all summations necessary to give the required symmetry properties of the wave function. $\alpha_{\gamma,\lambda,j}$ includes the remainder of the core wave function as well as the normalized probability amplitudes for the expansion. A similar wave function holds for the odd-neutron nucleus.

In most calculations involving low-lying states or ground states only the values $\lambda=0$, 2 ($\lambda=1$ represents a dipole vibration at much higher energy) are considered important. In the perturbation calculation of Milford[12] the values $\lambda=0$, 2, 4 are used since it can be shown that the odd particle interacts with the odd λ in the second order. In the following work it is assumed that λ assumes only even values.

With the use of these wave functions and the magnetic moment operators given by Eq. (3) and Eq. (4),

[10] All data are from P. F. A. Klinkenberg, Revs. Modern Phys. 24, 63 (1952) unless otherwise noted.

[11] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1951), Chap. III.

[12] F. J. Milford, Phys. Rev. 93, 1297 (1954).

it is found that

$$\Delta^P-\Delta^N=\sum_{\gamma,\lambda,j,\mu}|\langle j,\lambda,I-\mu,\mu|j,\lambda,I,I\rangle|^2$$
$$\times\Bigg\{\alpha_{\gamma,\lambda,j}{}^2(A_N)\Bigg[\langle\phi_j{}^{I-\mu}|\sigma^z|\phi_j{}^{I-\mu}\rangle$$
$$+\Bigg\langle C_\lambda{}^\mu(A_N)\Bigg|\frac{\sum_P j_P{}^z}{\mu_N}+\sum_N\sigma_N{}^z$$
$$+\frac{\mu_P-\frac{1}{2}}{\mu_N}\sum_P\sigma_P{}^z\Bigg|C_\lambda{}^\mu(A_N)\Bigg\rangle\Bigg]-\alpha_{\gamma,\lambda,j}{}^2(A_P)$$
$$\times\Bigg[\langle\phi_j{}^{I-\mu}|\sigma^z|\phi_j{}^{I-\mu}\rangle+\Bigg\langle C_\lambda{}^\mu(A_P)\Bigg|\frac{-\sum_N j_N{}^z}{\mu_P-\frac{1}{2}}$$
$$+\sum_P\sigma_P{}^z+\frac{\mu_N}{\mu_P-\frac{1}{2}}\sum_N\sigma_N{}^z\Bigg|C_\lambda{}^\mu(A_P)\Bigg\rangle\Bigg]\Bigg\}$$
$$\sum_{\gamma,\lambda,j',j'',\mu}\langle j',\lambda,I-\mu,\mu|j',\lambda,I,I\rangle$$
$$\times\langle j'',\lambda,I,I|j'',\lambda,I-\mu,\mu\rangle$$
$$\times[\alpha_{\gamma,\lambda,j'}(A_N)\alpha_{\gamma,\lambda,j''}(A_N)-\alpha_{\gamma,\lambda,j'}(A_P)\alpha_{\gamma,\lambda,j''}(A_P)]$$
$$\times\langle\phi_{j''}{}^{I-\mu}|\sigma^z|\phi_{j'}{}^{I-\mu}\rangle.\quad(6)$$

No cross terms occur between core states since all values of λ are even.

It is now necessary to simplify Eq. (6) in a manner which is consistent with the usual assumptions of the various forms of the collective model. An especially simple case is the Bohr-Mottelson strong-coupling model without configuration mixing. In this case there is no admixture of different single particle j states. The only value of j is assumed to be that prescribed by the shell model, i.e., $j=I$. Furthermore, the core angular momentum is restricted to $\lambda=2$. With these conditions Eq. (6) becomes

$$\Delta^P-\Delta^N=\sum_\mu|\langle I,2,I-\mu,\mu|I,2,I,I\rangle|^2$$
$$\times\Bigg\{\Bigg\langle C_\lambda{}^\mu(A_N)\Bigg|\frac{\sum_P j_P{}^z}{\mu_N}+\sum_N\sigma_N{}^z$$
$$+\frac{\mu_P-\frac{1}{2}}{\mu_N}\sum_P\sigma_P{}^z\Bigg|C_\lambda{}^\mu(A_N)\Bigg\rangle$$
$$-\Bigg\langle C_\lambda{}^\mu(A_P)\Bigg|\frac{-\sum_N j_N{}^z}{\mu_P-\frac{1}{2}}+\sum_P\sigma_P{}^z$$
$$+\frac{\mu_N}{\mu_P-\frac{1}{2}}\sum_N\sigma_N{}^z\Bigg|C_\lambda{}^\mu(A_P)\Bigg\rangle\Bigg\}.\quad(7)$$

However, a more general simplification of Eq. (6) is desirable for the purposes of further study. The different numbers of even particles in the nuclei of the conjugate pair can affect $\Delta^P-\Delta^N$ in two ways. First, if the core shares in the total angular momentum of the nucleus, then the respective share of the even particles in the core angular momentum would be expected to be different for the two nuclei of the conjugate pair. Thus regardless of specific nuclear forces and the coupling of the core to the single odd particle, the neutron excess alone could possibly lead to large values of $\Delta^P-\Delta^N$.

The second effect of the unequal number of particles is related to the coupling of the single odd particle to

TABLE I. The magnetic moments and magnetic moment deviations of conjugate nuclei in order of ascending total angular momentum and mass.

Conjugate nuclei	Z	N	Single-particle state l_j	Magnetic moment	Δ^P or Δ^N	$\Delta^P-\Delta^N$
H^3	1	2	$s_{1/2}$	2.979	−0.081	+0.034
He^3	2	1	$s_{1/2}$	−2.127	−0.115	
P^{31}	15	16	$s_{1/2}$	+1.312	0.724	+0.014
Si^{29}	14	15	$s_{1/2}$	−0.555	0.710	
N^{15}	7	8	$p_{1/2}$	−0.283	−0.008	−0.041
C^{13}	6	7	$p_{1/2}$	+0.702	+0.033	
B^{11}	5	6	$p_{3/2}$	2.689	0.482	+0.098
Be^9	4	5	$p_{3/2}$	−1.177	0.384	
Cu^{63}	29	34	$p_{3/2}$	2.226	0.683	−0.058
Cu^{65}	29	36	$p_{3/2}$	2.385	0.614	−0.127
Cr^{53}[a]	24	29	$p_{3/2}$	−0.474	0.741	
Cl^{35}	17	18	$d_{3/2}$	2.226	−0.304	−0.036
Cl^{37}	17	20	$d_{3/2}$	2.385	−0.244	+0.024
S^{33}	16	17	$d_{3/2}$	+0.644	−0.268	
Au^{197}[b]	79	118	$d_{3/2}$	0.16	−0.02	+0.14
Ba^{135}	56	79	$d_{3/2}$	+0.835	−0.163	
Al^{27}	13	14	$d_{5/2}$	+3.641	0.502	−0.051
Mg^{25}	12	13	$d_{5/2}$	−0.855	0.553	
I^{127}	53	74	$d_{5/2}$	+2.809	0.865	0.353
Mo^{97}	42	53	$d_{5/2}$	−0.914	0.512	
Pr^{141}[c]	59	82	$d_{5/2}$	+3.58	0.53	−0.2
Pd^{105}	46	59	$d_{5/2}$	−0.6	0.68	
Rb^{85}	37	48	$f_{5/2}$	1.353	−0.214	+0.052
Zn^{67}[b]	30	37	$f_{5/2}$	+0.876	−0.256	
Mn^{55}	25	30	$(f_{7/2})^5$, $J=5/2$	3.468	0.291	0.004
Ti^{47}[d]	22	25	$(f_{7/2})^5$, $J=5/2$	−0.787	0.287	
V^{51}	23	28	$f_{7/2}$	5.148	0.281	−0.030
Ca^{43}[e]	20	23	$f_{7/2}$	−1.318	0.311	
Co^{59}	27	32	$f_{7/2}$	4.648	0.499	0.077
Ti^{49}[d]	22	27	$f_{7/2}$	−1.104	0.422	
Nb^{93}	41	52	$g_{9/2}$	6.166	0.273	−0.268
Ge^{73}[d]	32	41	$g_{9/2}$	−0.877	0.541	
In^{113}[b]	49	64	$g_{9/2}$	5.522	0.554	0.124
In^{115}[b]	49	66	$g_{9/2}$	5.534	0.545	0.115
Sr^{87}[f]	38	49	$g_{9/2}$	−1.089	0.430	

[a] F. Alder and K. Halbach, Helv. Phys. Acta **26**, 426 (1953).
[b] H. Walchli, Oak Ridge National Laboratory Report ORNL-1469, 1952 (unpublished).
[c] Hin Lew, Phys. Rev. **91**, 619 (1953).
[d] C. D. Jeffries, Phys. Rev. **92**, 1262 (1953).
[e] C. D. Jeffries, Phys. Rev. **90**, 1130 (1953).
[f] C. D. Jeffries and P. B. Sogo, Phys. Rev. **91**, 1286 (1953).

the nuclear core. The excitation energies of the nuclear core oscillations depend among other things on the nuclear radius and, therefore, on the number of particles in the core. Thus, even assuming the charge independence of nuclear forces, the coupling of the single odd particle to the surface oscillations would be different for the two nuclei of the conjugate pair due to the different polarizabilities of the nuclear cores. Formally, this means that in general, $\alpha_{\gamma,\lambda,j}(A_P) \neq \alpha_{\gamma,\lambda,j}(A_N)$ in Eq. (6) since $A_P \neq A_N$. However, the problem of calculating the correct excitation energies of the nuclear core oscillation is sufficiently complicated that most investigations involving calculations of magnetic moments assume an average excitation, i.e., an average polarizability of the nuclear core. These statements, of course, deal with a simple form of the collective model which ignores all shell structure.

In the light of the previous considerations, a general simplification of Eq. (6) which is consistent with the magnetic moment calculations of most investigators is to assume that

$$\alpha_{\gamma,\lambda,j}(A_P)=\alpha_{\gamma,\lambda,j}(A_N) \quad \text{for all } \gamma,\lambda,j. \tag{8}$$

Physically, Eq. (8) embodies the assumptions that (a) the nuclear forces are charge-independent and (b) the properties of the core are independent of the number of particles constituting it.

In the absence of shell effects, the two assumptions leading to Eq. (8) are at best approximate. As the number of protons in the core increases, the effect of Coulomb repulsions will tend to destroy the charge independence of the nuclear forces (as exemplified in the different ordering of the shell model states for protons and neutrons). However, the properties of the core tend to become more uniform with increasing numbers of core particles, which justifies to some extent the use of Eq. (8) as a first approximation. In a later section, Eq. (8) is modified to include the influence of shell structure, lack of charge symmetry, and the different polarizabilities of the nuclear cores.

Besides the simplification of Eq. (6) entailed by the assumption of Eq. (8), it is necessary to make some further assumptions to allow the evaluation of the matrix elements of Eq. (6). In treating the interaction of the odd particle with the core in terms of surface oscillations, the identities of the particles of the core are ignored. Thus it seems reasonable to assume that the angular momentum of the core is equally distributed among all the particles participating in the oscillation. For the liquid drop model the number of particles participating in the surface vibration can be shown[13] to be of the order of $(3/\lambda)A$, where A is the number of particles in the core and λ is the angular momentum of the oscillation. Since we shall assume that on the average $\lambda=2$, all the particles are assumed to participate in the oscillation. The work of Bohr and Mottelson indicates that this is probably an overestimate of the number of particles participating in the vibration.

If every particle has an equal share of the angular momentum, then

$$\begin{aligned}\langle C_\lambda^\mu(A)|\sigma^z|C_\lambda^\mu(A)\rangle &= \frac{1}{A}\langle C_\lambda^\mu(A)|\sum_P \sigma_P{}^z+\sum_N \sigma_N{}^z|C_\lambda^\mu(A)\rangle \\ &= \frac{1}{A}\langle C_\lambda^\mu(A)|S^z|C_\lambda^\mu(A)\rangle \\ \langle C_\lambda^\mu(A)|j^z|C_\lambda^\mu(A)\rangle &= \frac{1}{A}\langle C_\lambda^\mu(A)|\sum_P j_P{}^z+\sum_N j_N{}^z|C_\lambda^\mu(A)\rangle \\ &= \frac{1}{A}\langle C_\lambda^\mu(A)|J^z|C_\lambda^\mu(A)\rangle=\frac{\mu}{A},\end{aligned} \tag{9}$$

where A is the number of particles in the core, S^z is the z component of the total spin (in the spin-orbit sense) of the core, and J^z is the z component of the total angular momentum of the core (this is, of course, equivalent to λ^z).

The core wave functions can be expanded in an LS representation

$$C_\lambda^\mu(A)=\sum_{L,S}\beta_{LS}(A)\sum_{m_S}\langle L,S,\mu-m_S,m_S|L,S,\lambda,\mu\rangle \times \mathcal{L}_L{}^{\mu-m_S}\mathcal{S}_S{}^{m_S}, \tag{10}$$

where $\mathcal{L}_L{}^{m_L}$ and $\mathcal{S}_S{}^{m_S}$ are the orbital and spin wave functions, respectively. $\beta_{LS}(A)$ are the required probability amplitudes.

Implicit in the applications of the collective model (weak or strong coupled) is the assumption discussed in connection with Eq. (8), that the properties of the surface for sufficiently heavy nuclei are at most slightly dependent on the number of particles A_P or A_N of the core. This assumption leads to the simplification Eq. (8) and to the related simplifying assumption

$$\beta_{LS}(A_N)=\beta_{LS}(A_P)=\beta_{LS}. \tag{11}$$

Furthermore, the various forms of the collective model assume that the magnetic moment of the core is due to a gyromagnetic ratio

$$g_c=Z/A \tag{12}$$

associated with the angular momentum λ of the core. This is equivalent to the assumption that

$$S=0, \quad L=\lambda. \tag{13}$$

This assumption or the more general assumption that $S<L$ and small is probably justified if the nuclear forces are predominantly of the Wigner or Majorana

[13] J. M. Blatt and V. F. Weiskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 302.

(space exchange) type.[14] The energetically most favored state of given λ would be that state with maximum space symmetry, which justifies the assumption Eq. (13). With these various assumptions, Eq. (7) becomes

$$\Delta^P-\Delta^N=\frac{3}{I+1}\left(0.436\frac{N_P}{A_P}-0.523\frac{Z_N}{A_N}\right), \quad (14)$$

while Eq. (6) becomes

$$\Delta^P-\Delta^N=\tfrac{1}{2}\sum_{\gamma,\lambda,j,L,S}\alpha_{\gamma,\lambda,j}{}^2\beta_{LS}{}^2 \times\left[\frac{I(I+1)+\lambda(\lambda+1)-j(j+1)}{I+1}\right] \times\left\{\left(0.436\frac{N_P}{A_P}-0.523\frac{Z_N}{A_N}\right) +\left[\frac{\lambda(\lambda+1)+S(S+1)-L(L+1)}{2\lambda(\lambda+1)}\right] \times\left[\frac{(N_N-1.195Z_N)}{A_N}-\frac{(Z_P-0.835N_P)}{A_P}\right]\right\}, \quad (15)$$

where the assumption Eq. (13) has not been used in this case. In these equations N_P, Z_P are the number of neutrons and the number of protons in the core of the odd-proton nucleus while N_N, Z_N are the same for the odd-neutron nucleus.

When calculating magnetic moments in the strong coupling limit of the simple collective model, Bohr and Mottelson make the approximation

$$g_c=Z/A\sim 0.45. \quad (16)$$

With this value, Eq. (14) becomes

$$\Delta^P-\Delta^N=0.012/(I+1), \quad (17)$$

TABLE II. The values of $\Delta^P-\Delta^N$ for the conjugate pairs $I^{127}-Mo^{97}$ and $Au^{197}-Ba^{135}$ for different values of S, L, and λ.

		$\Delta^P-\Delta^N$		
		$I^{127}-Mo^{97}$		$Au^{197}-Ba^{135}$
S	L	$\lambda=2$	$\lambda=4$	$\lambda=2$
0	2	0.023	0.095	0.052
1	3	−0.004	0.179	−0.024
1	2	0.037	0.227	0.090
1	1	0.064	0.262	0.166
2	4	0.032	0.143	−0.100
2	3	0.023	0.213	0.052
2	2	0.064	0.262	0.166
2	1	0.089	0.297	0.241
2	0	0.107	0.311	0.278
4	6	−0.087	0.070	−0.251
4	2	0.160	0.382	0.430
6	8	−0.165	0.011	−0.403
6	4	0.215	0.448	0.580

[14] R. G. Sachs, *Nuclear Theory* (Addison-Wesley Press, Cambridge, 1954), p. 190.

which is sufficiently small compared to the average magnitude of Δ^P or Δ^N to constitute a quasi-mirror theorem for the magnetic moments of conjugate nuclei as calculated by the assumptions of the Bohr-Mottelson strong-coupling collective model.

A result similar to Eq. (17) can be obtained by a direct calculation using the magnetic moment given by Bohr and Mottelson [their Eq. (IV-6)]. It is found that if the strength of the coupling of the odd particle to the surface oscillations is the same for both nuclei of the conjugate pair, then

$$-0.044<\Delta^P-\Delta^N<0.071 \quad (18)$$

in the range of experimental interest regardless of the coupling strength. In the strong-coupling limit, the same result holds regardless of the specific individual coupling strengths for the two nuclei.

An examination of the nineteen entries in the last column of Table I shows that ten cases, or more than half, fall outside the limits set by Eq. (18). Even more interesting is a consideration of those cases involving nuclei with more than 65 nucleons since the collective model should be most valid for heavier nuclei. Of the seven cases involving heavier nuclei, only one (an $I=L-1/2$ conjugate pair) satisfies the criterion of Eq. (18). Thus the mirror property of conjugate nuclei is most apparent for lighter nuclei where the conditions leading to Eq. (2) are more likely to hold.

CONJUGATE NUCLEI AND NUCLEAR MODELS

The various expressions obtained for the quantity $\Delta^P-\Delta^N$ are of considerable interest since any systematic error arising from the assumption made would probably tend to cancel. Thus a study of $\Delta^P-\Delta^N$ is a more sensitive means of investigating some aspects of the different nuclear models than a similar study of Δ^P or Δ^N alone. However, the very complicated nature of the nucleus as well as the extensive nature of the assumptions made to obtain Eq. (14) and Eq. (15) allow conclusions of only a qualitative nature. In this spirit the effect of modifying the various assumptions leading to the above equations will be studied.

Equation (15) is used to determine the effect of replacing $g_c=0.45$ by the appropriate values of Z_P, N_P, A_P and Z_N, N_N, A_N, as well as for different values of S, L, and λ for the core. Only $j=I$ is considered in the following discussion. For a comparatively light conjugate pair such as $V^{51}-Ca^{43}$, it is found that theoretically $|\Delta^P-\Delta^N|\leqslant 0.030$ for all values of S, L up to $S=6$ and $L=8$. Thus the observed small value of $\Delta^P-\Delta^N=-0.030$ is even compatible with Eq. (15) for the values of $S\neq 0$ and $S>L$. This observation is generally true for the lighter ($A<65$) conjugate nuclei because of the relatively small neutron excess and the consequent cancellations of terms in Eq. (15).

The application of Eq. (15) to a medium-heavy conjugate pair such as $I^{127}-Mo^{97}$ and a heavy conjugate pair such as $Au^{197}-Ba^{135}$ is given in Table II. The importance of the increasing neutron excess is shown in the general increase in magnitude of $\Delta^P-\Delta^N$ for the same λ, S, and L. The core angular momentum $\lambda=4$ is not allowed for the $Au^{197}-Ba^{135}$ conjugate pair with the assumption that $j=I=3/2$. The application of Eq. (15) with $\lambda=4$ is given for the $I^{127}-Mo^{97}$ conjugate pair, and it is seen that there is a significant increase for given S and L of $\Delta^P-\Delta^N$ as λ increases. This may be of considerable importance for some conjugate pairs, since Milford[12] has shown that $\lambda=4$ occurs with significant probability in the straightforward perturbation calculation with the collective-model Hamiltonian. However, it is interesting to note that the values obtained for $\Delta^P-\Delta^N$ from Eq. (15) for $I^{127}-Mo^{97}$ are almost all smaller than the observed value of $\Delta^P-\Delta^N=0.353$. The results of Table II give further confirmation, for this conjugate pair, of the importance of configuration mixing mentioned earlier, i.e., the occurrence of $j\neq I$ and the breakdown of the assumption of Eq. (8).

For the conjugate pair $Au^{197}-Ba^{135}$ it is seen that the experimental value of $\Delta^P-\Delta^N=0.14$ can be obtained by many values of $0\leqslant S\leqslant 6$, $0\leqslant L\leqslant 8$ and $\lambda=2$. The majority of the calculated values of $\Delta^P-\Delta^N$ are positive, and the magnitude increases with increasing S, L, and λ. This general property leads to the expectation that $\Delta^P-\Delta^N$ if the collective model is valid. For the heavier conjugate nuclei ($A>65$ for both nuclei), it is found that $\Delta^P-\Delta^N>0$ for five out of seven cases. It is interesting to note that the remaining two cases which have $\Delta^P-\Delta^N<0$ have even neutron shells which are either closed or two more than closed.

It is now desirable to investigate qualitatively the possible effect of shell structure on the values of $\Delta^P-\Delta^N$. The occurrence of shell structure manifests itself by the increased stability (binding energy) associated with certain given (magic) numbers of either protons or neutrons in the nucleus. The collective model interprets this increased stability in terms of an increase of the surface tension of the core with a subsequent increase in the energy necessary to excite core oscillations. Simple perturbation considerations relate the probability amplitudes $\alpha_{\gamma,\lambda,j}$ to this excitation energy by the proportionality $\alpha_{\gamma,\lambda,j}\sim 1/(E_\lambda-E_0)$, where E_λ is the excitation energy of the λ core oscillation and E_0 is the energy of the zero-order state, in our case the shell-model wave function. In considering the influence of shell effects on the value of $\Delta^P-\Delta^N$, this perturbation approach will be assumed.

Equation (6) is simplified by assuming that only $j=I$ and $\lambda=0$, 2 occur, that $S=0$, and that Eq. (9) is valid. Furthermore, it is assumed as a consequence perturbation theory that

$$\alpha_{\gamma,2,I}(A)=\frac{C_I}{E_2(A)-E_0(A)}, \tag{19}$$

TABLE III. The values of $\Delta^P-\Delta^N$ taking into account the effects of shell structure on the collective model.

Conjugate nuclei	l_I	Z	N	$\Delta E(A)$ (Mev)	Exp. $\Delta^P-\Delta^N$	Theoretical $\Delta^P-\Delta^N$
Au^{197}	$d_{3/2}$	79	118	0.35	0.143	−0.227
Ba^{135}	$d_{3/2}$	56	79	0.79		
I^{127}	$d_{5/2}$	53	74	0.64	0.353	0.190
Mo^{97}	$d_{5/2}$	42	53	0.77		
Pr^{141}	$d_{5/2}$	59	82	1.6	−0.2	−0.472
Pd^{105}	$d_{5/2}$	46	59	0.56		
Rb^{85}	$f_{5/2}$	37	48	0.85	0.052	0.021
Zn^{67}	$f_{5/2}$	30	37	1.04		
Nb^{93}	$g_{9/2}$	41	52	0.93	−0.268	−0.099
Ge^{73}	$g_{9/2}$	32	41	0.68		
In^{113}	$g_{9/2}$	49	64	0.55	0.124	0.224
In^{115}	$g_{9/2}$	49	66	0.55	0.115	0.227
Sr^{87}	$g_{9/2}$	38	49	1.9		

where the constant C_I is taken to be independent of both A and the energy of excitation $\Delta E(A)$. Generally C_I will vary with these quantities, but this variation does not significantly alter the following considerations. With these assumptions, Eq. (6) becomes

$$\Delta^P-\Delta^N=\frac{3|C_I|^2}{I+1}\left\{\frac{1}{[\Delta E(A_P)]^2}\left[\frac{1}{\mu_P-\frac{1}{2}}\frac{N_P}{A_P}+\frac{1}{I}\right]+\frac{1}{[\Delta E(A_N)]^2}\left[\frac{1}{\mu_N}\frac{Z_N}{A_N}-\frac{1}{I}\right]\right\} \tag{20a}$$

for $I=L+1/2$ nuclei, and

$$\Delta^P-\Delta^N=\frac{3|C_I|^2}{I+1}\left\{\frac{1}{[\Delta E(A_P)]^2}\left[\frac{1}{\mu_P-\frac{1}{2}}\frac{N_P}{A_P}-\frac{2}{2I+1}\right]+\frac{1}{[\Delta E(A_N)]^2}\left[\frac{1}{\mu_N}\frac{Z_N}{A_N}+\frac{2}{2I+1}\right]\right\} \tag{20b}$$

for $I=L-1/2$ nuclei.

The above equations now include a contribution to $\Delta^P-\Delta^N$ from the single odd particle states which previously had vanished because of the assumption of Eq. (8). Equations (20) are applied to those heavier conjugate nuclei mentioned previously, i.e., conjugate nuclei both of whose members have $A>65$. The results of this application are given in Table III. The energy of excitation of the core $\Delta E(A_P)$ or $\Delta E(A_N)$ is assumed to be the experimentally-observed $\lambda=2$ first excited state for the appropriate even-even nucleus given by Scharff-Goldhaber.[15] For convenience, the coefficient C_I is chosen to make $\alpha_{\gamma,2,I}{}^2(A)=1$ for that nucleus of the conjugate pair having the lowest excitation energy. This last choice would lead to $|\Delta^P-\Delta^N|$ (theoretical) $\geqslant|\Delta^P-\Delta^N|$ (experimental), with the signs of both

[15] Gertrude Scharff-Goldhaber, Phys. Rev. 90, 587 (1953).

TABLE IV. The values of $\Delta^P-\Delta^N$ when the single particle states are different for the nuclei of the conjugate pair.

$I=$	3/2	5/2	7/2	9/2
Conjugate pairs with $I=L+\frac{1}{2}$	0.34	1.01	1.27	1.41
Conjugate pairs with $I=L-\frac{1}{2}$	0.62	−0.92	−1.27	−1.43

being the same if the assumptions leading to Eq. (20) are valid.

It is seen in Table III that with the exception of the $Au^{197}-Ba^{135}$ conjugate pair, $\Delta^P-\Delta^N$(theoretical) agrees in sign with $\Delta^P-\Delta^N$(experimental), with shell effects leading to the negative signs of the $Pr^{141}-Pd^{105}$ and $Nb^{93}-Ge^{73}$ conjugate pairs. These shell effects appear in the increased value of $\Delta E(A)$ for that nucleus at or near a closed shell. All those conjugate pairs which either have different signs (one case) or which do not satisfy the condition $|\Delta^P-\Delta^N|$(theoretical) $\geqslant|\Delta^P-\Delta^N|$(experimental) are either (a) $I=L-1/2$ conjugate nuclei, or (b) conjugate nuclei in which one nucleus is in a region of very close competition between several single particle states. Nuclei with $I=L-1/2$ are not adequately explained by the simple collective model,[5,12] with the theoretical deviations tending in the opposite direction with increasing I compared to those observed. Therefore, it is not surprising that the conjugate pairs with $I=L-1/2$ do not display optimum behavior in Table III.

In all other cases where $|\Delta^P-\Delta^N|$(theoretical) $<|\Delta^P-\Delta^N|$(experimental), and in the case of $Au^{197}-Ba^{135}$, there are different degrees of close competition between single-particle states of different j for the odd-proton nuclei and odd-neutron nuclei. This competition can be seen by examining the observed ground state spins of nuclei in the region of the conjugate nuclei. For a nucleus in a region of close competition between different single particle states, it is no longer likely that $\alpha_{\gamma,2,j\neq I}$ is negligible, and it is, therefore, necessary to consider the case of configuration mixing. The term configuration mixing is used in the sense that different single-particle states (single-particle configurations) are admixed in the general odd-proton wave function, Eq. (5), or the odd-neutron equivalent.

In order to obtain an indication of the importance of configuration mixing, two simple but extreme cases are considered: for conjugate pairs with $I=L+1/2$,

$$\begin{aligned}\alpha_{\gamma,2,j=I}{}^2(A_N)&=1, \quad \alpha_{\gamma,\lambda,j\neq I}(A_N)=0,\\ \alpha_{\gamma,2,j=I+1}{}^2(A_P)&=1, \quad \alpha_{\gamma,\lambda,j\neq I+1}(A_P)=0,\end{aligned} \tag{21a}$$

while for conjugate pairs with $I=L-1/2$,

$$\begin{aligned}\alpha_{\gamma,2,j=I}{}^2(A_N)&=1, \quad \alpha_{\gamma,\lambda,j\neq I}(A_N)=0,\\ \alpha_{\gamma,2,j=I-1}{}^2(A_P)&=1, \quad \alpha_{\gamma,\lambda,j\neq I-1}(A_P)=0.\end{aligned} \tag{21b}$$

In either case the parities of the odd-particle states must be the same. These cases correspond to the Bohr-Mottelson strong-coupling collective model, where the final angular momentum I is obtained by coupling a different single-particle state to the surface oscillations for each of the two nuclei of the conjugate pair.

The values obtained for $\Delta^P-\Delta^N$ by using the conditions Eqs. (21) in Eq. (6) subject to the assumption Eqs. (9) are given in Table IV. It is immediately seen that the calculated values of $\Delta^P-\Delta^N$ are, for the appropriate I, larger than any experimentally observed $\Delta^P-\Delta^N$. The particular cases considered are appropriate for the $I^{127}-Mo^{97}$ $(I=L+1/2=5/2)$ conjugate pair and the $Au^{197}-Ba^{135}$ $(I=L-1/2=3/2)$ conjugate pair listed in Table III. I^{127} is in a region of very close competition between $d_{5/2}$ and $g_{7/2}$ single particle states, while Au^{197} is in a region of close competition between $d_{3/2}$ and $s_{1/2}$ single particle states. It is significant that the appropriate values of $\Delta^P-\Delta^N$ in Table IV have the correct sign to account for the observed values of $\Delta^P-\Delta^N$ for these two conjugate pairs. A more general calculation shows that the value of $\Delta^P-\Delta^N$ depends linearly, to a first approximation, on the probability (not probability amplitude) of occurrence of the admixed state. Therefore, if shell effects are not considered, the large $I^{127}-Mo^{97}$ value of $\Delta^P-\Delta^N$ can be explained by about a 30 percent probability for the $g_{7/2}$ state occurring in the I^{127} wave function. Similarly, about a 25 percent probability for the $s_{1/2}$ state in the Au^{197} wave function will account for the $Au^{197}-Ba^{135}$ experimental value of $\Delta^P-\Delta^N$.

CONCLUSIONS

The calculated magnetic moment deviations of conjugate nuclei are exactly equal (mirror property) according to the nuclear model of Volkov[6] and are nearly equal (quasi-mirror property) according to the strong-coupling Bohr-Mottelson[5] model. An analysis of the observed magnetic moment deviations shows that neither property (mirror or quasi mirror) holds for those heavier conjugate nuclei which should most closely approximate collective-model nuclei. However, it has been shown that the experimental results can be explained in terms of a simple form of the collective model, subject to modifications which incorporate shell structure in the core and/or the configuration mixing of different single-particle states.