tential has a depth of 67 Mev. One of the most obvious consequences of this change is that the spacing of shell-model energy states would be considerably increased. Other consequences of this change and their relevance to experiment will be discussed in a separate paper.

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Theory of Nuclear Models

R. J. EDEN* AND N. C. FRANCIS
Indiana University, Bloomington, Indiana†
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The framework for a unified theory of nuclear structure is described in which the wave functions for different nuclear models are obtained by transformations on the actual nuclear wave function. This formulation provides a basis for explaining the success of weak-coupling models of the nucleus and showing that they are not in conflict with the assumption that nucleons have very strong mutual interactions. The explanation lies in the fact that only in certain circumstances can the "particles" in a nuclear model be interpreted as nucleons.

We investigate the properties which transformation operators must have to change the nuclear wave function into a model wave function and consider how far these properties are satisfied in practice. Self-consistent equations are set up for a model having a product wave function in the particle variables, and it is shown that these equations can be solved in an approximation relevant to the problem of nuclear saturation.

I. INTRODUCTION

IN recent years a number of nuclear models have been developed which successfully describe many aspects of nuclear structure. The most striking successes have been obtained by the Mayer-Jensen[1] shell model and by Weisskopf's[2] cloudy crystal-ball model, and it is clear that for low energies there must be a close correspondence between these models and the actual nucleus. On the other hand, these models are based on an assumption of weak interaction between the particles they describe and this assumption appears to be in direct contradiction with the strong nucleon interactions which are observed in scattering experiments. We shall show in this paper that this apparent contradiction is not a real one but is explained essentially by the fact that the "particles" in these nuclear models are not nucleons—that is to say they cannot in all circumstances be interpreted as nucleons.

Our program is firstly to set up and explain a formalism in which the wave functions of various models can be transformed by "model operators" into the real nuclear wave function. Next we consider how actual nuclear models fit into the framework of our theory, and finally we consider the problem of using these model operators in practical applications of the theory.

Since this theory shows that the particles, in the shell model, for example, are not nucleons, it is necessary to show that shell-model results can still be explained with this new interpretation. We do not attempt to derive the explicit assumptions of the shell model, although our formalism leads to self-consistent equations which if solved should lead to the well potential of the model. Our primary concern is to show how the following aspects of the shell model are consistent with our method: the particles in the model obey the exclusion principle as though they are neutrons and protons, energy levels are predicted with sufficient accuracy to indicate the order of filling single-particle states, angular momentum and parity are accurately predicted as though the model was the real nucleus, and selection rules are well predicted. This is not by any means a complete list but should serve to indicate the problems of interpretation which have to be considered. Also, we must consider the relation of our theory to the failures of the model such as the failure to predict transition rates with any accuracy.

Since our methods are quite general and apply to any nuclear model, a complete presentation of this theory would require detailed consideration of very many aspects of the relation of nuclear models to experiment. We have in this paper attempted to select sufficient of these aspects of the theory to indicate the power of the method, and hope in future papers to examine other aspects. In particular, we have limited ourselves to consideration of low-energy nuclear models although we believe that the methods are also applicable to high-energy problems. In addition to the shell model, we

* Smithson Research Fellow of the Royal Society, on leave of absence from Clare College, Cambridge, England.
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[1] Maria Goeppert Mayer, Phys. Rev. **75**, 1969 (1949); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949).
[2] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).

consider the single-particle model and the related cloudy crystal-ball model. For the latter, we have to show how it is that the scattering of particles in the model has the same cross sections as elastic scattering of neutrons on the real nucleus.

We take as our starting point a representation in which the nuclear wave function is $\Psi(A)=\Psi(x_1,\cdots,x_A)$ where $x_1, \cdots, x_A$ are the coordinates of nucleons. The corresponding Schrödinger equation will have a Hamiltonian H which contains a potential energy $V(x_1,\cdots,x_A)$ which may include many-body potentials (it seems probable, however, that these can be neglected as a first approximation). It is known from nucleon-nucleon scattering data that there are very strong forces between two nucleons, and it seems unlikely that these forces will be much different when the nucleons are found in a nucleus. For strong nuclear forces the potential V is a rapidly varying function and the wave function $\Psi(A)$ will contain strong correlation effects; in particular, for short-distance repulsion between nucleons, $\Psi(A)$ will be small wherever $|x_i-x_j|$ is small. It is clear that the wave function $\Psi(A)$ will be very complicated and will not approximate (except in a very restricted sense) to the product wave functions used in nuclear models such as the independent-particle model of the shell model.

If the successes of these models are to be explained within the framework of present day quantum theory, it must be possible to make a transformation which will take the wave function $\Phi(A)$ for the model into the much more complicated wave function $\Psi(A)$ for the real nucleus. There will be different transformations corresponding to different nuclear models. We call the transformation operators "model operators" and much of this paper is devoted to considering their properties and uses. In general, observables will be represented in a model by different operators from those used for the actual nucleus, and it will be seen that the usefulness of a particular model depends on the behavior of observables under transformation by the appropriate model operator.

Our method is outlined as follows: (1) We consider the conditions on the transformed Hamiltonian $H^m=M^\dagger HM$ which are desirable in a useful model; (2) we deduce conditions on the model operator M; (3) we examine whether in general these conditions can be satisfied by a choice of M; (4) we consider how far these conditions enable us to regard the model as the real nucleus; (5) we consider whether these conditions are satisfied by existing nuclear models; (6) if the conditions are only satisfied approximately, we consider possible corrections which may allow for the resulting error; (7) we discuss the physical reasons why actual nuclear models satisfy the conditions mentioned above; (8) we develop the mathematical theory resulting from two particular forms of model operator relevant to the single-particle model and to the problem of nuclear saturation. We have not attempted to achieve completeness in the above program but have endeavored to carry it far enough to indicate the power and usefulness of the method.

The conditions (1) on H^m which are desirable cannot all be satisfied by any one model. They include the following: (*a*) H^m should be such that the transformed wave function $\Phi(A)$ is manageably simple, for example, a product wave function of some kind such as in the single-particle model or in the shell model. (*b*) H^m should be invariant for rotations and reflections of the coordinate axes and for rotations in isotopic spin space; when these conditions are satisfied, it is possible to define a new transformation which leaves the angular momentum, parity, and charge operators unchanged—then for these operators the model will appear to be the real nucleus. (*c*) H^m should give the same energy spectrum as the original Hamiltonian; this requires $\Psi(A)=M\Phi(A)$, where M is a unitary operator. (*d*) H^m should be the Hamiltonian for a system of particles which satisfy the exclusion principle in the same way as neutrons and protons; this condition sets certain symmetry requirements on M. (*e*) For a single-particle model of low-energy scattering the transformation operator must be chosen to commute with the elastic scattering part of the collision matrix. (*f*) for a single-particle model of a nucleus it may be desirable that $\Phi(A)=\psi(A)\Psi_0(A-1)$, where $\Psi_0(A-1)$ is the ground state of the $(A-1)$th nucleus. This list is far from a complete one but should be sufficient to indicate the conditions on H^m and M which we consider.

The basic point we wish to make about point (4) in our method arises because it is never in practice possible to satisfy all the above conditions simultaneously. It follows that no single nuclear model can be considered as the real nucleus; that is to say, the particles in a model cannot always be interpreted as nucleons without leading to incorrect results. Thus, for the shell model a "particle" in the model does not have the binding energy of a nucleon. For a single-particle model, we can choose the transformation so that the particle has the binding energy of the last nucleon added, but one can see that such a model cannot in general be expected to give the correct magnetic moment if the particle is interpreted as a nucleon in a definite state of angular momentum.

The nuclear models considered in point (5) have, of course, always been regarded as approximate, so our investigation is partly to the same degree of approximation. However, we are able also to show that some of these approximations are forced upon a particular model because one or more of its properties is not compatible with some of the desirable properties mentioned above. Some of these facts are well known, for example the use of a product wave function in the shell model precludes collective motion of a certain type, but the fact that limitations may be due to conflicting requirements does not appear to have been recognized before. We consider in step (6) some of the

corrections which have to be made because of the inherent limitations of a model, with particular reference to magnetic moments and deviations from the Schmidt lines.

The interpretation we given in step (7) is based essentially on the operation of the exclusion principle at low energies. This situation has been considered by Weisskopf[3] also with a view to justifying the shell model. His main point is that at low energies all nucleon states are occupied, so that because of the exclusion principle any one nucleon can only change its state by a transition in which high-energy states are involved; these transitions can be expected to occur with small probability, so a nucleon in a low-energy nuclear state will move with a long mean free path. One difficulty in interpreting this argument lies in giving a meaning to a "nucleon state" when the nucleon variables occur only in a strongly correlated wave function. In order to define a nucleon state, it is necessary to have some quantity like a single-nucleon wave function and this does not seem possible for a nucleon in a nucleus. We consider the problem from the viewpoint of transforming the shell-model wave function to the actual nucleus. It should be noted that the exclusion principle holds in the model because it holds in the actual nucleus. The arguments of Weisskopf can now be applied in considering the changes in observables under the transformation. Excited states of the model now occur as intermediate states, and it appears that for some operators the large energy differences in going to these excited states have a major simplifying effect, and in some cases make the operator commute with the transformation (approximately). However, it does not follow that there is always close resemblance between the actual nuclear wave function and the wave function for the model; in fact, consideration of the strong nucleon interactions shows that there cannot always be such a resemblance.

In the above sense, it seems that the arguments of Weisskopf provide some justification that a transformation exists which connects the real nucleus with a model consisting of particles moving in weak interaction and satisfying the other conditions of the shell model.

These physical ideas can be put in mathematical form to give an explicit model operator derived from two-body forces in a simplified problem connected with nuclear saturation. For this problem, the model operator is equivalent in a certain approximation to the transformation operator used by Brueckner, Levinson, and Mahmound[4] in their derivation of a nuclear radius in close agreement with observed values.

[3] V. F. Weisskopf, Helv. Phys. Acta **23**, 187 (1950).

[4] Brueckner, Levinson, and Mahmound, Phys. Rev. **95**, 217 (1954). See also K. A. Brueckner, Phys. Rev. **96**, 508 (1954), who notes that the "independent particle" of their coherent model does not refer to "independent nucleon" motion. It has also been noted by C. A. Levinson that the coherent model wave function is in a different space from the actual nuclear wave function.

In Sec. II of this paper, we illustrate our method by making a formal derivation of the single-particle model of a nucleus. In Sec. III, we consider the relation of this model to the elastic scattering of nucleons on a nucleus. In Sec. IV, we consider the general conditions on model operators which are related to the wave function of the model giving correctly eigenvalues of parity, angular momentum, and isotopic spin. In Sec. V, we deduce from conditions on model operators that the corresponding models will give selection rules correctly but will not in general give transition rates. In Sec. VI, we consider first the setting up of an "ideal" shell model and then examine how far it corresponds to the actual shell model. In Sec. VII, we consider in detail the development of a single-particle model having the same energy in one state as the actual nucleus; the energy condition is sufficient to determine the depth of the well in which the particle moves. In Sec. VIII, we derive a formula for the energy of a nucleus modified by extra boundary conditions; this is only approximately the formula used in reference 4, and our method appears to be of rather more general validity. Finally, in Sec. IX, we briefly state our conclusions and indicate further work which is proceeding.

II. FORMAL DERIVATION OF THE SINGLE-PARTICLE MODEL

We take as our starting point the many-body Schrödinger equation for A nucleons in a representation in which the kinetic energy is the sum of A single-nucleon kinetic energies and the potential energy is a function of the coordinates of the A nucleons:

$$(E-H_A)\Psi(x_1\cdots x_A)=0, \tag{1}$$

$$H_A=\sum_{i=1}^{A} T_i+V_A. \tag{2}$$

The potential V_A will depend on the mutual separations (x_i-x_j) of the nucleons and may include many-body potentials. It appears from scattering data that the two-nucleon parts of V are rapidly varying and non-monotonic functions of (x_i-x_j), and it is unlikely that the strong forces are compensated (i.e. smoothed out) by many-body interactions. It follows that in the representation defined by (1) and (2) the wave function $\Psi(A)[\equiv\Psi(x_1\cdots x_A)]$ contains strong correlations so that it is small when two nucleons are in a region of strong repulsion. It is, therefore, not possible to approximate to $\Psi(A)$ by product wave functions since these in general have considerable overlap between the positions of nucleons.

As an illustration of our method, we will consider a transformation which satisfies more rigid conditions than are necessary in practice. We will suppose the nucleons are distinguishable and assume that there

exists a unitary "model" operator M such that

$$\Psi(A)=M\Phi(A)$$
$$=\int dx_1'\cdots\int dx_A'(x_1\cdots x_A|M|x_1'\cdots x_A')\Phi(x_1'\cdots x_A'), \tag{3}$$

where

$$\Phi(A)=\psi(x_A)\Psi_0(A-1). \tag{4}$$

Then

$$(E-H_A{}^m)\Phi(A)=0, \quad H_A{}^m=M^{-1}H_AM. \tag{5}$$

In practice, it is often convenient to use non-unitary transformations. We have chosen M to be unitary here in order to obtain a model which has the same energy spectrum as the real nucleus. The wave function $\Psi_0(A-1)$ in (4) is defined to be the ground-state wave function of the $(A-1)$th nucleus with Hamiltonian H_{A-1}:

$$(E_{A-1}{}^0-H_{A-1})\Psi_0(A-1)=0. \tag{6}$$

From (4), (5), and (6),

$$\{E-E_{A-1}{}^0-H_A{}^m+H_{A-1}\}\Psi_0(A-1)\psi(x_A)=0. \tag{7}$$

Taking a scalar product with $\Psi_0(A-1)$ on the left, we get

$$\{E'-T_A'-W_c\}\psi(x_A)=0, \tag{8}$$

where $E'=E-E_{A-1}{}^0$ and

$$W_c=(\Psi_0(A-1),\{H_A{}^m-H_{A-1}-T_A'\}\Psi_0(A-1)). \tag{9}$$

Thus W_c is an operator depending only on the Ath particle's coordinates (or other variables associated with this particle). T_A' is the kinetic-energy operator of the Ath particle. Equation (8) is therefore a one-particle equation having eigenvalues of energy equal to the difference between the energy of different states of the Ath nucleus and the ground-state energy of the $(A-1)$th nucleus.

In practice, as a result of the very complicated energy spectrum of the nucleus, the potential W_c in (8) would be very complicated. It can be simplified if we relax the condition that M is unitary. Instead we replace M by an operator MP (to be called the new M), where P is a projection operator which selects only states corresponding to certain energy eigenvalues of the Ath nucleus. If this selection is made so that the new W is simple, we will have derived a model in which a one-particle equation predicts certain nuclear excited states of the nucleus. It is clear from the form of the transformation (3) that the particle in this model must not be identified with a nucleon. Such an identification would imply that other observables, which have not yet been considered, transform in a very special way under the transformation (3).

One freedom which may help to simplify W is worth noting here. The kinetic energy operator T_A' in (8) need not have the same mass as a nucleon, hence if there is a part of W which is proportional to $(\mathbf{k}_A)^2$ it can be incorporated in the nonrelativistic T_A' which is given by $(\mathbf{k}_A)^2/2m$ by introducing a modified mass m^* for the particle.[5]

It will become clear later that other operators will not take a simple form in a model as restrictive as indicated by (4). More generally, a transformation may be of use where a Φ is obtained which can be expanded as a sum of products of the eigenfunctions of Eqs. (6) and (10),

$$(E^r-T_A'-W_\alpha)\psi_r(x_A)=0. \tag{10}$$

$$\Phi(A)=\sum_{r,s}a_{r,s}\Psi_r(A-1)\psi_s(x_A). \tag{11}$$

In Eq. (10), W_α is assumed to be some potential for a single particle; it will approximate to the W_c in (8) if (4) is required to be the leading term in the expansion (11).

Another form of the single-particle equation which will be of use is obtained when a transformation M is such that the single-particle potential operator W has matrix elements:

$$W_{rs}=(\Psi_r(A-1),W\Psi_s(A-1)), \tag{12}$$

where W is the operator occurring in (9). Then the single-particle wave function will satisfy coupled equations:

$$(E_r-T_A'-W_{rr})a_{rr}\psi_r(x_A)=\sum_{t,s\neq r}\sum W_{rs}a_{st}\psi_t(x_A). \tag{13}$$

If the nondiagonal terms of W are small, this can be solved with zero-order equations:

$$(E_r-T_A'-W_{rr})\psi_r(x_A)=0, \tag{14}$$

and the coefficients are determined by perturbation methods.

It will be seen that in the approximation (14) the state of particle A depends on the state of the $(A-1)$ particles. The final approximation we wish to mention here is one in which the Ath particle satisfies equation (10), while the $(A-1)$th wave function $\Psi'(A-1)$ satisfies

$$(E-H_{A-1})a_{rr}\Psi_r'(A-1)=\sum_{r\neq s}\sum_t W_{rs}'a_{ts}\Psi_t'(A-1), \tag{15}$$

where

$$W_{rs}'=(\psi_r,\{H^m-H_{A-1}-T_A'-W_M\}\psi_s). \tag{16}$$

Again, for W_{rs} small when $r\neq s$, perturbation methods can be applied. In lowest approximation, the state of the $(A-1)$ particles depends on the state of the Ath particle. It will be seen later that the situation in the shell model resembles this approximation.

We have now considered formal transformations to the kind of approximate single-particle models which may be useful. The single particle may not have the same mass, and possibly the kinetic-energy operator T_A' will not be the operator for a Dirac particle, unless certain conditions are satisfied by the operator M; we will consider later whether these can be satisfied.

[5] J. A. Wheeler (private communication to K. A. Brueckner).

If we label the wave functions for the real nucleus and for the model in a suitable way, the model operator M will be formally equal to

$$M=\sum_r \Psi_r(A)\Phi_r^*(A). \tag{17}$$

If we are able to find a suitable operator W, the corresponding transformation will be obtained by solving the integral equation

$$M=1+[E-H^m+i\eta]^{-1}(V^{(A)}-W)M, \tag{18}$$

where $V^{(A)}$ is the part of the potential V which depends on the Ath nucleon. Equation (18) has boundary conditions appropriate to the scattering problem. (A different boundary condition is required for bound-state transformations.) The bound-state equation will be considered in Sec. VII where we make a more detailed investigation of the single-particle model.

It will be seen from (17) that since the wave function $\Psi(A)$ contains strong correlations, these correlations will be implicit in M. It is therefore not possible to use perturbation methods in solving Eq. (18) for M, and in general an approximate form of W may lead to a bad approximation to M. For these reasons, in considering explicitly the relation between models and the real nucleus—we will consider not the operator M but will investigate the expectation values of transformed operators $\langle O^m\rangle$. It will appear that in useful problems these operators are given by a rapidly convergent series although M may converge only very slowly.

The two most important extensions of the one-particle transformation are obtained by transforming $\Psi(A-1)$. When further transformations lead to single-particle equations, all of the form (10), one eventually obtains a complete product wave function analogous to the shell-model wave function. The other important transformation is to $\psi(x_{A-1})\Psi_0(A-2)$, when a model is obtained for two (or more) particles outside a core consisting of the ground state of the nucleus. The first of these transformations will be investigated in Secs. V, VI, and VIII.

III. ELASTIC SCATTERING

We have shown in the previous section that a transformation can be chosen to give some of the nuclear-energy levels by means of a single-particle model provided no restrictions are made on the complexity of the single-particle potential. In this section, we consider whether it is possible in general to choose a transformation which gives a single-particle model having the same elastic scattering matrix elements as a neutron on a nucleus.

The Schrödinger equation (4) can be written

$$(E-T_A-H_{A-1})\Psi_0(A-1)\psi(x_A)=W\Psi_0(A-1)\psi(x_A), \tag{19}$$

where

$$W=H_A{}^m-H_{A-1}-T_A. \tag{20}$$

We prove first that the elastic scattering matrix given by the potential W has an expectation value for the state $\Psi_0(A-1)$ which is equal to the elastic scattering matrix given for the one-particle problem by the potential W_c defined in (9).

Equation (19) has a solution corresponding to an initial plane wave state λ for the Ath particle, which can be written

$$\Psi_0(A-1)\psi(x_A)=\Omega\Psi_0(A-1)\lambda_A, \tag{21}$$

where

$$\Omega=1+(E-H_{A-1}-T_A+i\eta)^{-1}W\Omega. \tag{22}$$

Writing the scattering matrix $T=W\Omega$, we have

$$\begin{aligned}(\lambda_A'\Psi_0(A-1),\ T\Psi_0(A-1)\lambda_A)&=(\lambda_A'\Psi_0(A-1),\ W\Omega\Psi_0(A-1)\lambda_A)\\&=(\lambda_A'\Psi_0(A-1),\ W\Psi_0(A-1)\psi(x_A))\\&=(\lambda_A',\ W_c\psi(x_A)).\end{aligned} \tag{23}$$

Now $\psi(x_A)$ satisfies

$$(E_A-T_A)\psi(x_A)=W_c\psi(x_A), \tag{24}$$

which has a scattering solution for an initial plane wave state λ_A,

$$\psi(x_A)=\Omega_c\lambda_A, \tag{25}$$

where

$$\Omega_c=1+(E_A-T_A+i\eta)^{-1}W_c\Omega_c. \tag{26}$$

Upon writing

$$T_c=(\Psi_0(A-1),\ T\Psi_0(A-1)), \tag{27}$$

it follows from (23) and (25) that

$$T_c=W_c\Omega_c. \tag{28}$$

This completes the proof that the single-particle model gives the same elastic scattering as the many-particle model.

We next consider whether in general it is possible to choose the model operator M so that this single-particle scattering is the same as the elastic scattering of a neutron on a nucleus. We note first that M is by no means completely specified by the requirement that $\Phi(A)$ has the form given in (4). It is possible to vary it by any further transformation which acts only on $\psi(x_A)$. This should give enough freedom so that in principle a model operator M can be chosen so that it commutes with the elastic-scattering part of the collision matrix. It has already been noted that the purpose of the transformation M is to introduce correlations so that the model wave function $\Phi(A)$ goes in the real wave function $\Psi(A)$. Since the real wave function has no correlations between the Ath nucleon and the $(A-1)$ nucleons in the scattering limit, it will be possible to choose M so that it acts like a unit matrix in this limit. For this choice, M will not affect the boundary conditions of the problem and the model will give correctly the observed elastic scattering.

From the form of the wave function $\Phi(A)$ in (4), it is clear that the model only has end states for scattering which describes elastic scattering. The potential W_c cannot therefore give rise to excited states of the $(A-1)$ nucleus. It follows that W_c is a complex potential so the transformation operator M cannot be unitary. However, the lesser condition can be imposed on M, that it preserves the energy levels of single-particle excited states of the Ath nucleus. It appears that this condition is satisfied by the optical model of Watson, so we deduce that M can be chosen in practice so that W_c is equal to V_c, the optical model potential.

If we are not interested in resonances in the single-particle scattering, it will be sufficient to choose M only to preserve the lowest one-particle energy levels which do not lie in the continuous spectrum. It appears that these conditions on M are satisfied by the cloudy crystal-ball model of Weisskopf.[2] The approximate equality of the well depth in this model and the Mayer well depth indicates that single-particle energy levels are at least approximately preserved.

IV. PARITY, ANGULAR MOMENTUM, AND ISOTOPIC SPIN

In the previous two sections, we have considered conditions on the model operator M which are related to the correct prediction of energy levels and elastic scattering. In this section, we investigate the possibility of choosing M so that the parity, angular momentum, and isotopic spin operators take a simple form after they have been transformed.

In the representation used in our basic equation (1), the Hamiltonian H is invariant under reflections and rotations of the coordinate axes and under rotations in isotopic spin space if charge independence of nuclear forces is assumed. We will consider first the parity and angular momentum which, because of the invariance of H, can be used to label the energy eigenstates of (1). We denote a particular eigenfunction by $\Psi(E,P,j,m)$ where the eigenvalues of H, P, $\mathbf{J}^2$, J_z are E, P', $j(j+1)$, m.

If the transformation M leads to operators H^m, $P^m(\mathbf{J}^2)^m$, $J_z{}^m$, they will have the same eigenvalues E, P', $j(j+1)$, m, where

$$\Psi(E,P',j,m)=M\Phi(E,P',j,m). \tag{29}$$

We will now investigate the assumption that M can be chosen so that the Hamiltonian H^m of the model is invariant under rotations and reflections of the coordinate axes. From this assumption, it follows that H^m, P, $\mathbf{J}^2$, J_z form a commuting set so this set of operators can be used for labeling the wave functions of the model. Let these wave functions be $\Phi_1(E,P',j_1,m_1)$, where $j_1(j_1+1)$ and m_1 are the eigenvalues of $\mathbf{J}^2$ and $\mathbf{J}_z$. For a given value of energy, these wave functions can be expanded in terms of the wave functions $\Phi(E,P',j,m)$. If the degeneracy is the same, and this will in general be required for completeness of the two sets, we can deduce $(2j+1)=(2j_1+1)$. Hence, $j=j_1$ and the operators $\mathbf{J}^2$ and $(\mathbf{J}^2)^m$ have the same eigenvalues.

Now let

$$\Phi(E,P',j,m)=\sum_{m_1=-j}^{j} a_{mm_1}\Phi_1(E,P',j,m_1). \tag{30}$$

Then

$$\begin{aligned}\mathbf{J}^2\Phi(E,P',j,m)&=\sum a_{mm_1}\mathbf{J}^2\Phi_1(E,P',j,m_1)\\&=j(j+1)\Phi(E,P,j,m),\end{aligned} \tag{31}$$

since

$$\mathbf{J}^2\Phi_1(E,P',j,m_1)=j(j+1)\Phi_1(E,P',j,m_1). \tag{32}$$

Therefore $\Phi(E,P,j,m)$ is a common eigenfunction of both $\mathbf{J}^2$ and $(\mathbf{J}^2)^m$. This is true for every such wave function, and since these form a complete set,

$$\mathbf{J}^2=(\mathbf{J}^2)^m, \quad [\mathbf{J}^2,M]=0. \tag{33}$$

It will next be shown that M can be chosen so that J_z commutes with it. Let us assume that for an initial choice of M, $\Phi_1(E,P',j,m)$ is not the same as $\Phi(E,P',j,m)$. Define a unitary operator U whose components for fixed E, P', j are

$$U(E,P',j)=\sum_{m=-j}^{j}\Phi(E,P',j,m)\Phi_1{}^*(E,P',j,m). \tag{34}$$

Then U commutes with H, P, J^2, and

$$U\Phi_1(E,P',j,m)=\Phi(E,P',j,m). \tag{35}$$

Hence,

$$\begin{aligned}U^{-1}(J_z)^mU\Phi_1(E,P',j,m)&=U^{-1}(J_z)^m\Phi(E,P',j,m)\\&=m\Phi_1(E,P',j,m).\end{aligned} \tag{36}$$

But from the definition of $\Phi_1(E,P',j,m)$,

$$J_z\Phi_1(E,P',j,m)=m\Phi_1(E,P',j,m). \tag{37}$$

Since (36) and (37) hold for every eigenfunction in this complete set, it follows that

$$J_z=U^{-1}(J_z)^mU. \tag{38}$$

Hence, if we redefine M to be MU, for the new operator

$$[J_z,M]=0. \tag{39}$$

We now have

$$\Psi(E,P',j,m)=M\Phi(E,P',j,m). \tag{40}$$

Let $J_\pm=J_x\pm iJ_y$; then

$$J_\pm\Psi(E,P',j,m)=b_\pm(j,m)\Psi(E,P',j,\,m\pm1), \tag{41}$$

$$J_\pm\Phi(E,P',j,m)=b_\pm(j,m)\Phi(E,P',j,\,m\pm1). \tag{42}$$

Hence,

$$(J_\pm M-MJ_\pm)\Phi(E,P',j,m)=0. \tag{43}$$

Since this is true for each state in the complete set,

$$[J_\pm,M]=[J_x,M]=[J_y,M]=0. \tag{44}$$

From (39) and (44), we see that, provided that there

exists an M such that H^m is a scalar with respect to rotation, it is possible to choose a new M which is itself a scalar. The implications of this result will be considered after we have discussed parity and isotopic spin.

If the parity operator P as well as P^m commutes with H^m and there is no degeneracy of eigenstates with respect to parity,

$$P^m\Phi(E,P',j,m)=P'\Phi(E,P',j,m) \tag{45}$$

and

$$P\Phi(E,P',j,m)=P''\Phi(E,P',j,m), \tag{46}$$

for all eigenfunctions Φ in the transformed system. Hence,

$$[P^m,P]=0. \tag{47}$$

Using the fact that $P^2=1$, we get

$$[M^{-1},P]_+[M,P]_-=0. \tag{48}$$

Hence, either P always commutes with M or it always anticommutes with M. There is no loss of generality therefore in assuming the Φ state always has the same parity as the corresponding Ψ state.

The condition for the model to have the same charge as the real nucleus is much less stringent than the requirement of isotopic spin invariance. Provided it is possible to choose the model operator M so that it is symmetrical between neutrons and also symmetrical between protons, the total charge will be unaffected by the transformation. It appears that this can be achieved in practice for most useful models.

A much more valuable result can be achieved if complete charge independence of nuclear forces is assumed (i.e., neglect of Coulomb forces and neglect of neutron-proton mass difference). This assumption is approximately valid for light nuclei but not for heavy nuclei.

We consider a transformation for which H^m is invariant under rotations in isotopic spin space. Then (assuming similar invariance of H), the isotopic spin operators I^2 and I_3 can be considered in a manner exactly analogous to the angular momentum. It is then found that M can be chosen to be invariant under rotations in isotopic spin space:

$$[M,I_1]=[M,I_2]=[M,I_3]=0, \tag{49}$$

and also the corresponding wave functions Ψ and Φ of the real nucleus and the model have the same values for isotopic spin.

Up to now, we have considered the transformation properties of parity, angular momentum, and isotopic spin when the states Φ describing the model form a complete set. Since this is never satisfied by any of the nuclear models at present in use, it is useful to consider the situation when the states are not complete. We assume that the equation

$$\Psi(E,P,j,m)=M\Phi(E,P,j,m) \tag{50}$$

is satisfied for all Φ states but that it defines only a selection of the real nuclear states Ψ. In this case we can assume that M is unitary in the subspace based on the set of states given by the model. Provided that each state for which (49) is satisfied has the same degeneracy in the transformed system as in the real nucleus, the arguments already given about invariance go through except that Eqs. (33), (39), (44), and (49) hold only in the subspace defined by the model. It follows that departures from the invariance of M will be evident only as a result of transitions through states not in this subspace. If the model has been chosen so that these transitions have small probability, the concept of partial invariance of M will still be a useful one.

We will consider next how the results of this section can be used to justify the equivalence between the selection rules for a nuclear model and those for the real nucleus.

V. SELECTION RULES

The total Hamiltonian of the nuclear system and the radiation field can be written

$$H_{\text{total}}=H+H_{\text{int}}+H_{\text{rad}}. \tag{51}$$

Up to now we have considered only the part H which is the nuclear force Hamiltonian and primarily responsible for determining nuclear structure. The interaction H_{int} between the nuclear field and the radiation field has been assumed small so that to a good approximation H and H_{rad} can be treated independently. The model operator M is determined in this approximation by H only. Our discussion of the previous section has shown that when the transformed Hamiltonian H^m has certain invariance properties so does the operator M.

The interaction H_{int} will contain certain tensor operators A_i, B_{ij}, C_{ijk}, $\cdots$, which cause radiative transitions in the nucleus. Since M is a scalar with respect to rotations, the transformed operators $A_i{}^m$, $B_{ij}{}^m$, $C_{ijk}{}^m$, $\cdots$, will have the same tensor character as the untransformed operators. From the tensor character of one of these operators, it can be deduced that it will never cause transitions between states differing by certain values of angular momentum. Let us suppose that it is deduced that

$$(\Phi(E',P',j',m'),\ B_{ij}\Phi(E,P,j,m))=0. \tag{52}$$

Since this condition depends only on rotation properties of B_{ij} and the change in angular momentum specified by the matrix elements, it follows that when (52) is satisfied,

$$(\Phi(E',P',j',m'),\ M^{-1}B_{ij}M\Phi(E,P,j,m))=0. \tag{53}$$

Hence, using the results of Sec. IV,

$$(\Psi(E',P',j',m'),\ B_{ij}\Psi(E,P,j,m))=0. \tag{54}$$

This result shows that when a particular selection rule is obtained for a model forbidding transitions between

states of certain angular momenta, the same selection rule will be satisfied by the actual nucleus.

It will be noticed that in (54) we have assumed that M is unitary so that the energies E', E are the same as in (53). This assumption is not essential to the result provided that the invariance conditions on H^m are satisfied and the model gives the correct degeneracy of corresponding levels.

It is important to note that transition rates will not in general be given correctly by the model since nonzero matrix elements will not in general be equal to the matrix elements of the transformed operators:

$$(\Phi(E',P',j',m'),\ B_{ij}\Phi(E,P,j,m)) \neq (\Psi(E',P',j',m'),\ B_{ij}\Psi(E,P,j,m)). \tag{55}$$

Selection rules associated with changes in parity and isotopic spin can be shown to be the same for the model and the real nucleus in an analogous discussion to the above. It must be emphasized that the validity of the selection rules for isotopic spin depends on the assumption of complete charge independence. Thus, even an ideal model which satisfies all our other conditions cannot be expected to give correct isotopic spin selection rules except for light nuclei. When charge independence is not approximately true, it will not be possible in general to set up a model of the nucleus in which the transformed isotopic spin variables I^m resemble the untransformed variables I. It may be possible by considering explicit forms for M to estimate the difference $I_m{}^m - I$ and hence give a more precise meaning to isotopic spin of heavy nuclei.

When the states of a nuclear model form only a subspace of the complete set, the invariance of M holds only within the subspace. Then the selection rules predicted by the model will no longer be exact. However, they can only be violated by a process involving a transition with an intermediate state not in the subspace. If the coupling is small between states given by the model and other states, we can deduce that the selection rules will be satisfied approximately.

VI. THE SHELL MODEL

We consider first the setting up of an "ideal" shell model with a model operator M which satisfies as many desired conditions as possible; we will then consider how far this "ideal" model corresponds to the Mayer-Jensen shell model, and finally we will discuss the reasons why certain conditions on M are satisfied in practice.

The transformation corresponding to (3) is now required to give a complete product wave function:

$$\Psi(A) = M\Phi(A), \tag{56}$$

$$\Phi(A) = \prod_{i=1}^{A} \psi_i(x_i), \tag{57}$$

where each single-particle wave function satisfies the same Schrödinger equation:

$$(E_i - T_i - W^s)\psi_i(x_i) = 0, \tag{58}$$

$$H^m = M^\dagger H M = \sum_{i=1}^{A} \{T_i + W_s(x_i)\}. \tag{59}$$

If M is unitary, the energies E_i will satisfy

$$E = \sum_{i=1}^{A} E_i, \tag{60}$$

where E is an eigenvalue of the original Schrödinger equation,

$$(E - \sum_{i=1}^{A} T_i - V)\Psi(A) = 0. \tag{61}$$

The Φ equation, when (60) holds, is

$$(E - \sum_{i=1}^{A} T_i - \sum_{i=1}^{A} W^s(x_i))\Phi(A) = 0. \tag{62}$$

Provided that $\Psi(A)$ and $\Phi(A)$ satisfy suitable boundary conditions, it follows from (61) and (62) that the $W_s(x_i)$ must satisfy

$$(\Phi(A),\ \{V - \sum_{i=1}^{A} W^s(x_i)\}\Psi(A)) = 0. \tag{63}$$

In general, this will only be a necessary condition when $\Phi(A)$ and $\Psi(A)$ are bound states and tend to zero at infinity. Thus, a unitary M requires that the matrix elements (63) are zero between all states not in the continuous spectrum.

When (63) is satisfied, (61) and (62) have the same energy eigenvalues; then there is a solution to the equation:

$$(E - \sum_{i=1}^{A} T_i - \sum_{i=1}^{A} W^s(x_i))M = (V - \sum_{i=1}^{A} W^s(x_i))M, \tag{64}$$

given formally by

$$M = 1 + (E - \sum_{i=1}^{A}(T_i) - \sum_{i=1}^{A} W^s(x_i))^{-1}('V - \sum_{i=1}^{A} {}'W^s(x_i))M, \tag{65}$$

provided that

$$(\Phi(A),\ (V - \textstyle\sum W^s)M\Phi(A)) = 0. \tag{66}$$

The primes to the left of V and a W^s in (65) indicate that when M acts on a particular $\Phi(A)$ this state is excluded to the left of these operators. This exclusion means that (65) automatically satisfies (64) for nondiagonal matrix elements of M on the left of (64). Equation (64) is satisfied for diagonal matrix elements because of the condition (66).

In principle, Eqs. (62), (65), and (66) provide a set of equations for W^s, $\Phi(A)$, and M which could be solved by a self-consistent method beginning with a trial potential W^s in (62). This would present a formidable problem unless a self-consistent solution exists for a simple form of W^s. The success of the Mayer-

Jensen shell model (although not for binding energies) suggests that in fact W^s may be simple to a good approximation, so the solution of this problem may not be beyond attainment. In Sec. VIII, we will define a similar but more restricted problem and show that this can be solved.

It is clear from (65) that M can be taken to be symmetrical between coordinates of protons and coordinates of neutrons, so the particles satisfying Eq. (58) will satisfy the exclusion principle. Hence, the wave function for the model will be the antisymmetrical sum of products of eigensolutions to (58). If charge independence is not assumed, there will be different potentials for positive and negative particles in the model. Hence, the ground state of the model is obtained by filling up the lowest one-particle energy levels exactly as though the particles were neutrons and protons.

The well W^s will depend on the number of nucleons, so it should be written $W^s(A)$. For $W^s(A+1)$ all the energies will change by a small amount, and there will be corresponding changes in the wave function. Hence the difference in total energy $[E(A+1)-E(A)]$ will be equal to the energy of the $(A+1)$th particle $E_{(A+1)}$ together with the small changes ΔE_i, $i=1, 2, \cdots, A$. This can be contrasted with the ideal single-particle model which had energy $E(A+1)-E(A)$.

We assume that the transformed Hamiltonian is invariant under rotation and reflection of the coordinate axes. Then the model operator M can be chosen so that the parity and angular momentum of the model can be calculated as though it were the real nucleus. This insures that when the single-particle levels are filled up in some particular way, the resulting parity and angular momentum of the model will be the same as for the corresponding states of the real nucleus. For the ground states of the model, successive filling of levels will lead to occupation of all angular momentum states at a given energy so one obtains the closed-shell phenomena. From the work of Sec. V, the model will predict selection rules but in general will not give transition rates.

We next consider how far the Mayer-Jensen shell model corresponds to the ideal model considered above. It leads to a product wave function like (57) with single-particle equations like (58). The potential W^s is simple but has a spin-orbit coupling term; our general considerations do not offer any evidence why W^s should have a particular form although in principle W^s can be derived from the self-consistent equations. The total energies in the shell model do not satisfy (60), although the different eigenvalues E_i are ordered so that shells fill up in the correct way. Not all energy levels (e.g., rotational energy levels) are described by the shell model. We deduce that the model operator is not unitary but that for the "noncollective" energy levels M only scales down the energies without affecting the order of levels. The transformed Hamiltonian is invariant under rotations and reflections in the coordinate axes and gives correct degeneracy, so the parity and angular momentum commute with M in the subspace defined by the model. Hence, the model predicts parity and angular momentum quantum numbers correctly to the approximation implied by this subspace; similarly selection rules will be valid to the same approximation.

It will be seen that the shell model corresponds to the approximation discussed in Sec. II given by Eqs. (10), (15), and (16). The well in which $(A+1)$ particles in the model move depends on the presence of the Ath particle, and more accurately even on the state of this particle. The filling of single-particle states in the model is insensitive to this interdependence, so for this, $W^s(A)$ can have its A dependence neglected. Other observables depend more sensitively on the well—for example, the magnetic moment. If it is known that the $(A-1)$th nucleus has zero magnetic moment, we do not know that $(A-1)$ particles in the Ath nucleus also have zero magnetic moment. Hence the Ath particle in the shell model cannot be regarded as moving in a nucleus with an inert care of $(A-1)$ particles. One can either attempt to represent the effect of this case in some collective way as for example in the theory of Bohr and Mottelson,[6] or one can transform to the single-particle model of Eq. (8) where the care is the $(A-1)$th nucleus. In the latter case, however it cannot be deduced that the potential W_c is invariant under rotations since W_c differs from W^s in the shell model.

It appears that in practice the transformation M to the single-particle picture with potential W_c does not commute with angular momentum except for certain light nuclei. This conclusion follows from the fact that the magnetic moment of nuclei does not lie on the Schmidt lines in general. Mathematically it means that the single-particle eigenfunction $\psi_r(x_A)$ does not correspond to an angular momentum quantum number. It can, of course, be expanded:

$$\psi_r(x_A)=\sum b_{rs}\psi_s'(x_s), \tag{67}$$

where $\psi_s'(x_s)$ are eigenstates for example of the shell-model Hamiltonian.

From our work up to this point it appears that there are two main errors involved in representing the shell model as the real nucleus when evaluating magnetic moments (or quadrupole moments). The first error arises if these operators do not commute with the model operator leading to the shell model. The second error arises from treating the shell-model potential as independent of the number (or state) of the individual particles.

Our final discussion of the shell model in this section is concerned with the reasons for the existence of a transformation M from a strongly interacting nucleus to a weakly interacting model. It has already been noted in the introduction that there is not much meaning to the term "state of a nucleon in a nucleus." Due to the strong interaction of nucleons Ψ cannot be expressed

[6] A. Bohr and B. Mottelson, Kgl. Danske Videnskab, Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

as a product, and only for product states has the concept of single-nucleon states any clear meaning. It does not therefore appear valid to use qualitative arguments based on the state of a nucleon and the exclusion principle.[3] Even if such arguments were valid and led for example to an expectation that nucleons had long mean-free paths,[2] it would be incorrect to assume that this implied weak interaction between nucleons. The form of Eq (1) is such that the energy will diverge if Ψ has the form implied by a weak interaction model. It seems impossible to interpret the shell-model wave function as an approximation to the real wave function.[4]

It has been one of the principle objects of this paper to show that such an interpretation is unnecessary. In Sec. VIII, we will construct a correlated wave function Ψ from a model wave function Φ. The model consists of particles satisfying the exclusion principle and occuping the lowest of a set of energy levels. In calculating operators in the model, it appears that they are simple largely because of the effect of the exclusion principle and the occupation of all neighboring states. This suggests that the validity of the shell model arises from the action of the exclusion principle in the model. This is of course related to the action of the exclusion principle in the real wave function Ψ but not in any simply described way.

VII. FURTHER CONSIDERATION OF THE SINGLE-PARTICLE MODEL

In this section we investigate a simplified single-particle model of the nucleus. The model will describe a single particle moving in a square well of depth V_0, and we choose the model operator M so that the energy of the particle is the binding energy of the Ath nucleon as defined in II, Eq. (8). We will see that this condition is sufficient to determine the well depth V_0.

The total nuclear wave function $\Psi(A)$ satisfies the Schrödinger equation:

$$(E-H_A)\Psi(A)=0. \tag{68}$$

We assume that H_A has the simplified form,

$$H_A=\sum_{i=1}^{A} T_i+\sum_{i<j} V_{ij}, \tag{69}$$

which corresponds to only two-body forces between nucleons. Define V and V_i by

$$V_i=V_{Ai}, \quad V=\sum_{i=1}^{A-1} V_i. \tag{70}$$

Then (68) can be written

$$(E_0-H_{A-1}-T_A-V_0)\Psi_0(A)=(V-V_0)\Psi_0(A), \tag{71}$$

where the suffix 0 serves to indicate we are considering a particular eigenstate. We require a model operator M_0 such that

$$\Psi_0(A)=M_0\Phi_0(A)=M_0\Psi_0(A-1)\psi_0(x_A), \tag{72}$$

where

$$(E_0-H_{A-1}-T_A-V_0)\Phi_0(A)=0. \tag{73}$$

Provided that $\Phi(A)$ and $\Psi(A)$ satisfy boundary conditions for a bound state of the A nucleons, the condition for (71) and (73) to have the same eigenvalue is

$$(\Phi_0(A),\ (V-V_0)\Psi_0(A))=0. \tag{74}$$

We take

$$M_0=1+\frac{1}{a}{}'(V-V_0)M_0, \tag{75}$$

where $a=(E_0-H_{A-1}-T_A-V_0)$ and the prime to the left of $(V-V_0)$ indicates that all matrix elements involving $\Phi_0(A)$ on the left are to be omitted. It is easily verified that (74) and (75) ensure that if $\Phi_0(A)$ satisfies Eq. (73), then $\Psi_0(A)$ satisfies (71) when it is defined by (72). From (72), (74), and (75) we deduce that since V_0 denotes a well of constant depth,

$$V_0=(\Phi_0(A),\ VM_0\Phi_0(A)). \tag{76}$$

From (75),

$$M_0=1+[E_0-H_{A-1}-T_A-V_0+{}'V_0-{}'V]^{-1}{}'(V-V_0) \tag{77}$$

$$=1+[E_0-H_{A-1}-T_A-{}^0V_0{}^0-{}'V]^{-1}{}'V, \tag{78}$$

provided that we only consider M_0 acting on the state $\Phi_0(A)$. In (78), ${}^0V_0{}^0$ is defined to be the matrix which has elements between $\Phi_0(A)$ states equal to V_0, and has all other matrix elements zero. Since in (78) it acts on $'V$, it has no effect in the equation; hence,

$$M_0=1+[E_0-H_{A-1}-T_A-{}'V]^{-1}{}'V=M \tag{79}$$

when acting on $\Phi_0(A)$ to the right, where

$$M=1+[E_0-H_{A-1}-T_A]^{-1}{}'VM. \tag{80}$$

Hence,

$$V_0=(\Phi_0(A),\ VM\Phi_0(A)). \tag{81}$$

We prove next that

$$M\Phi_0(A)=F\left(1+\frac{1}{e}t_c\right)\Phi_0(A), \tag{82}$$

where F, e, and t_c will be defined in the following discussion:

$$e=E_0-H_{A-1}-T_A-t_c. \tag{83}$$

Define ${}'t_i{}^\alpha$ to be a solution of the equation

$${}'t_i{}^\alpha={}'V_i+{}'V_{\frac{1}{e}}{}'t_i{}^\alpha, \tag{84}$$

where the prime indicates the ground state $\Phi_0(A)$ is excluded. Then t_c is defined as a diagonal matrix with respect to $\Psi'(A-1)$ having matrix elements

$$t_c=\sum_{i=1}^{A}(\Psi'(A-1),\ {}'t_i{}^\alpha\Psi'(A-1)), \tag{85}$$

$$C_i=(\Psi'(A-1),\ {}'t_i{}^\alpha\Psi'(A-1)). \tag{86}$$

C_i is defined to be diagonal in the nuclear states of the $(A-1)$ particles. Since $\Psi'(A-1)$ is antisymmetrical, C_i is independent of the subscript i and depends only on the energy of the state $\Psi'(A-1)$. Hence,

$$C_i=C=(\sum C_i)/A=t_c/A. \tag{87}$$

Define

$$'I_i='t_i^\alpha-C. \tag{88}$$

From its definition, $'I_i$ has no diagonal elements with respect to the state $\Psi'(A-1)$. F in (82) is defined by

$$F=1+\frac{1}{e}\sum_{i=1}^{A}{}'I_iF_i, \tag{89}$$

$$F_i=1+\frac{1}{e}\sum_{i\neq j}{}'I_jF_j=F-\frac{1}{e}{}'I_iF_i. \tag{90}$$

We now prove (82) by showing that the right hand side of (82) is a solution of Eq. (80) for M. Substituting from (82) in the right hand side of (80), we get

$$1+\frac{1}{b}{}'VF\left(1+\frac{1}{e}t_c\right), \tag{91}$$

where $b=E_0-H_{A-1}-T_A$. Upon using (89), (70), and then (88), Eq. (91) becomes

$$1+\left[\frac{1}{b}{}'V+\frac{1}{b}\sum_i{}'V_i\frac{1}{e}{}'t_i^\alpha F_i+\frac{1}{b}\sum_{i,j,i\neq j}{}'V_1\frac{1}{e}I_jF_j-\frac{1}{b}\sum_i{}'V_i\frac{1}{e}CF_i\right]\left(1+\frac{1}{e}t_c\right). \tag{92}$$

Using (84) to remove the $'V_i(1/e)\,'t_i^\alpha$ term, we get

$$1+\left[\frac{1}{b}\sum_i{}'t_i^\alpha F_i-\frac{1}{b}\sum_i{}'V_i\frac{1}{e}CF_i\right]\left(1+\frac{1}{e}t_c\right). \tag{93}$$

If we use (88), this gives

$$1+\left[\frac{1}{b}\sum_i{}'I_iF_i-\frac{1}{b}\sum{}'V_i\frac{1}{e}CF_i+\frac{1}{b}\sum_i CF_i\right]\left(1+\frac{1}{e}t_c\right). \tag{94}$$

Now from (87), (89), and (90),

$$\frac{1}{b}\sum_i CF_i=\frac{1}{b}\left(1-\frac{1}{A}\right)t_cF+\frac{1}{b}\frac{1}{A}t_c. \tag{95}$$

From the definitions of e, (83) and b, (91), we have

$$\frac{1}{b}=\left(1-\frac{1}{b}t_c\right)\frac{1}{e}. \tag{96}$$

Upon using (89), (95), and (96), Eq. (94) becomes

$$1+\left[\left(1-\frac{1}{b}t_c\right)(F-1)+\frac{1}{b}t_c-\frac{1}{b}\sum_i{}'V_i\frac{1}{e}CF_i-\frac{1}{A}\frac{1}{b}t_cF+\frac{1}{A}\frac{1}{b}t_c\right]\left(1+\frac{1}{e}t_c\right). \tag{97}$$

If we use (87), this becomes

$$1+F\left(1+\frac{1}{e}t_c\right)-\left(1-\frac{1}{b}t_c\right)\left(1+\frac{1}{e}t_c\right)+\frac{1}{A}\left[\frac{1}{b}t_c-\frac{1}{b}\sum{}'V_i\frac{1}{e}t_cF_i-\frac{1}{b}t_cF\right]\left(1+\frac{1}{e}t_c\right). \tag{98}$$

By (96), this becomes

$$F\left(1+\frac{1}{e}t_c\right)+\frac{1}{A}\left[\frac{1}{e}t_c-\left(\frac{1}{b}\sum{}'V_i\frac{1}{e}t_cF_i+\frac{1}{b}t_cF\right)\times\left(1+\frac{1}{e}t_c\right)\right]. \tag{99}$$

Apart from the term involving A^{-1}, this completes the proof of Eq. (82). Terms involving A^{-1} of similar type to those in (99) have been considered by Watson[7] and are expected to be small.[8] We shall neglect these terms here but will discuss similar terms in greater detail in the next section.

We require V_0, which is given by (76), (79), and (82) as

$$V_0=\left[\Phi_0(A),\,VF\left(1+\frac{1}{e}t_c\right)\Phi_0(A)\right]. \tag{100}$$

We consider next the quantity VF. Using (70) and (89), we obtain

$$VF=V+\sum_i V_i\frac{1}{e}I_iF_i+\sum_{i,\,j,\,i\neq j}V_i\frac{1}{e}I_jF_j \tag{101}$$

$$=V+\sum_i V_i\frac{1}{e}{}'t_i^\alpha F_i+\sum_{i,j,i\neq j}V_i\frac{1}{e}I_jF_j-\sum_i V_i\frac{1}{e}CF_i. \tag{102}$$

We define a quantity t_i^β by the relation,

$$t_i^\beta=V_i+V_i\frac{1}{e}{}'t_i^\alpha. \tag{103}$$

Upon using (87), (90), and (103), Eq. (102) becomes

$$VF=\sum_i t_i^\beta F_i-\frac{1}{A}\sum_i V_i\frac{1}{e}t_cF_i. \tag{104}$$

We now obtain an equation for t_i^β which involves only scattering amplitudes. Define t_i to satisfy

$$t_i=V_i+V_i\frac{1}{e}t_i. \tag{105}$$

[7] K. M. Watson, Phys. Rev. **89**, 575 (1953).

[8] The neglect of these terms will not be valid at certain energies corresponding to resonances in single-particle scattering. For these energies the transformation leads to a series which at best would be slowly convergent and may diverge. If these terms are neglected, an "average" potential analagous to that of Weisskopf (reference 2) is obtained.

Then

$$V_i=t_i\left[1+\frac{1}{e}t_i\right]^{-1}, \tag{106}$$

$$'V_i='t_i\left[1+\frac{1}{e}t_i\right]^{-1}. \tag{107}$$

Also from (84),

$$'t_i{}^{\alpha}='V_i\left[1-\frac{1}{e}'V_i\right]^{-1}. \tag{108}$$

Substituting (106) and (108) into (103) and then using (107), we obtain

$$t_i{}^{\beta}=t_i\left[1+\frac{1}{e}{}^0t_i\right]^{-1}, \tag{109}$$

where

$${}^0t_i=t_i-'t_i. \tag{110}$$

Neglecting the A^{-1} term in (104), the expression (100) for V_0 becomes, from (104) and (109),

$$V_0=\sum_{i=1}^{A}\left(\Phi_0(A),\, t_i\left[1+\frac{1}{e}{}^0t_i\right]F_i\left[1+\frac{1}{e}t_c\right]\Phi_0(A)\right). \tag{111}$$

The detailed consideration of this expression can be carried out by the methods given by Francis and Watson.[9] It is sufficient to illustrate the method to make the approximation $F_i=1$. The term $(1/e)\,{}^0t_i$ can be neglected. Then to lowest order, we get

$$V_0=\sum_{i=1}^{A}(\Phi_0(A),\, t_i\Phi_0(A)). \tag{112}$$

If the expression (111) is considered in more detail, it is found that V_0 is modified in a way which corresponds to dispersion in the nuclear medium.

VIII. APPLICATION TO NUCLEAR SATURATION

In this section we consider the evaluation of the energy of the nuclear wave function Ψ when it is subjected to additional boundary conditions, in the form of a potential well. The basis of the method has been described in Sec. VI where it was shown that a model could be chosen having the same energy as a particular state of the real nucleus.

We consider a model having wave function Φ_0 which is a complete product of A single-particle wave functions and satisfies a Schrödinger equation

$$(E_0-T-V_0)\Phi_0=0. \tag{113}$$

The corresponding equation for the modified nuclear wave function Ψ_0 is

$$(E_0-T-V_0)\Psi_0=(V-V_0)\Psi_0. \tag{114}$$

We assume that Φ_0 describes a system of particles moving in a square-well potential with depth V_0 (measured from zero energy), infinite walls, and radius r. This imposes a corresponding boundary condition on Ψ_0 which makes it differ from the actual nuclear wave function.

The well depth V_0 is determined by the condition that Eqs. (113) and (114) have the same energy eigenvalue. It is assumed that E_0 is the ground-state energy.

$$V_0=(\Phi_0,V\Psi_0)=(\Phi_0,VM_0\Phi_0), \tag{115}$$

where

$$\Psi_0=M_0\Phi_0, \tag{116}$$

$$M_0=1+(E_0-T-V_0)^{-1}\,'(V-V_0)M_0, \tag{117}$$

where the prime to the left of $(V-V_0)$ denotes omission of the ground state. It follows from (115) and (117) that if Φ_0 satisfies (113), then Ψ_0 given by (116) satisfies (114). From (115) and (117), we deduce that

$$V_0=(\Phi_0,\, V[1+(a-\,'V)^{-1}\,'V]\Phi_0) \tag{118}$$

$$=(\Phi_0,VM\Phi_0), \tag{119}$$

where

$$a=E_0-T-{}^0V_0{}^0, \tag{120}$$

$$M=1+(a-\,'V)^{-1}\,'V=1+a^{-1}\,'VM. \tag{121}$$

Since a always appears with $'V$ on the right, the term ${}^0V_0{}^0$ in a always gives zero, so we redefine

$$a=E_0-T. \tag{122}$$

Define

$$'t_{ij}{}^{\alpha}=\,'V_{ij}+\,'V_{ij}\frac{1}{e}\,'t_{ij}{}^{\alpha}, \tag{123}$$

where

$$e=E_0-T-\,'t_c'=a-\,'t_c', \tag{124}$$

$$'t_c'=\sum\,'C_{ij}', \tag{125}$$

$$'C_{ij}'=\,'t_{ij}{}^{\alpha}{}'. \tag{126}$$

The quantity $'C_{ij}'$ is a diagonal matrix element between excited states. Let

$$'I_{ij}{}^{\alpha}=\,'t_{ij}{}^{\alpha}-\,'C_{ij}'; \tag{127}$$

this is the off-diagonal part of $'t_{ij}{}^{\alpha}$. Let

$$F=1+\frac{1}{e}\sum_{i<j}\,'I_{ij}{}^{\alpha}F_{ij}, \tag{128}$$

$$F_{ij}=1+\frac{1}{e}\sum_{i\neq l,\,j\neq m}\,'I_{lm}{}^{\alpha}F_{lm}, \tag{129}$$

$$=F-\frac{1}{e}\,'I_{ij}{}^{\alpha}F_{ij}. \tag{130}$$

We note that

$$\sum_{i<j}F_{ij}=\tfrac{1}{2}A(A-1)F-F+1. \tag{131}$$

We now prove that M in (119) can be replaced by F

[9] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).

with neglect of small terms. Consider $(1/a)\,'VF$,

$$\frac{1}{a}\,'VF=\frac{1}{a}\,'V+\frac{1}{a}\sum{}'V_{ij}\frac{1}{e}\,'I_{ij}{}^{\alpha}F_{ij}+\frac{1}{a}\sum_{i\neq l,j\neq m}{}'V_{ij}\frac{1}{e}\,'I_{lm}{}^{\alpha}F_{lm} \quad (132)$$

$$=\frac{1}{a}\sum{}'t_{ij}{}^{\alpha}F_{ij}-\frac{1}{a}\sum{}'V_{ij}\frac{1}{e}\,'V_{ij}\,'F_{ij} \quad (133)$$

$$=\frac{1}{a}\sum{}'I_{ij}{}^{\alpha}F_{ij}+\frac{1}{a}\sum{}'C_{ij}\,'F_{ij}-\frac{1}{a}\sum{}'V_{ij}\frac{1}{e}\,'C_{ij}\,'F_{ij} \quad (134)$$

$$=\frac{1}{e}\sum{}'I_{ij}{}^{\alpha}F_{ij}-\frac{1}{a}\,'t_c\,'\frac{1}{e}\sum{}'I_{ij}{}^{\alpha}F_{ij}+\frac{1}{a}\sum{}'C_{ij}\,'F_{ij}-\frac{1}{a}\sum{}'V_{ij}\frac{1}{e}\,'C_{ij}\,'F_{ij} \quad (135)$$

$$=(F-1)-\left(\frac{1}{a}\sum{}'C_{ij}\,'\frac{1}{e}\,'I_{ij}{}^{\alpha}F_{ij}+\frac{1}{a}\sum{}'V_{ij}\frac{1}{e}\,'C_{ij}\,'F_{ij}\right)+\frac{1}{a}\,'t_c\,'. \quad (136)$$

The last term in (136) gives zero when it acts on the the ground state Φ_0; the middle term is a correction term which will give a small contribution to the energy. Hence,

$$F\Phi_0\doteqdot M\Phi_0. \quad (137)$$

We next consider

$$V_0=(\Phi_0,VF\Phi_0), \quad (138)$$

where

$$VF=V+\sum_{i<j}V_{ij}\frac{1}{e}\,'t_{ij}{}^{\alpha}F_{ij}=\sum_{i<j,i\neq l,j\neq m}V_{ij}\frac{1}{e}\,'I_{lm}{}^{\alpha}F_{lm}-\sum_{i<j}V_{ij}\frac{1}{e}\,'C_{ij}\,'F_{ij}. \quad (139)$$

Let

$$t_{ij}{}^{\beta}=V_{ij}+V_{ij}\frac{1}{e}\,'t_{ij}{}^{\alpha}. \quad (140)$$

Then, neglecting the last term in (141), we get

$$V_0=(\Phi_0,\sum t_{ij}{}^{\beta}F_{ij}\Phi_0). \quad (141)$$

$t_{ij}{}^{\beta}$ can be obtained in terms of other scattering amplitudes by using (125), (140) and defining

$$t_{ij}=V_{ij}+V_{ij}\frac{1}{e}t_{ij}. \quad (142)$$

From (123),

$$'t_{ij}{}^{\alpha}=\,'V_{ij}\left(1-\frac{1}{e}V_{ij}\right)^{-1} \quad (143)$$

Hence, by using (140) and (143),

$$t_{ij}{}^{\beta}\left(1-\frac{1}{e}\,'V_{ij}\right)=V_{ij}. \quad (144)$$

Solving (142) for V_{ij}, and hence obtaining $'V_{ij}$, we get after some simplification

$$t_{ij}{}^{\beta}=t_{ij}\left(1+\frac{1}{e}{}^0t_{ij}\right)^{-1}, \quad (145)$$

where

$${}^0t_{ij}=t_{ij}-\,'t_{ij}. \quad (146)$$

Hence, from (138) and (145),

$$V_0=\sum_{i<j}\left(\Phi_0,\,t_{ij}\left[1+\frac{1}{e}{}^0t_{ij}{}^0\right]^{-1}F_{ij}\Phi_0\right). \quad (147)$$

To lowest order this gives

$$V_0=\sum_{i<j}^{A}(\Phi_0,t_{ij}\Phi_0), \quad (148)$$

where Φ_0 is the antisymmetrized wave function for the ground state of the model.

The potential V_0 given by (148) is identical with that used by Brueckner, Levinson, and Mahmoud in their discussion of nuclear saturation.[10] It should be noted, however, that our assumptions about the relation between the model and the nuclear wave functions differ in some important respects from the assumptions of these authors. Probably the most important difference lies in our use of the consistency condition (63) and (115) to determine the potential well of the model and hence the energy of the system. This method not only gives a more rigorous derivation of the energy in the present approximation, but it can also be applied to the more general problem of obtaining a shell-model potential.

If we label the states of the model by the single particle momenta k, the wave function Φ can be written:

$$\Phi=(A\,!)^{-\frac{1}{2}}\begin{vmatrix}|k_1)_1 & |k_1)_2 & \cdots & |k_1)_A\\ |k_2)_1 & |k_2)_2 & \cdots & \\ \vdots & & & \\ |k_A)_1 & \cdots & & |k_A)_A\end{vmatrix}. \quad (149)$$

The chief drawback to the above method lies in the fact that the potential V has been split up in an unsymmetrical way so that the intermediate states which arise in calculation will not be antisymmetrized. This means that in considering the magnitude of the cor-

[10] Further details of the nuclear saturation problem have been considered by K. A. Brueckner, Phys. Rev. **96**, 508 (1954), and forthcoming papers.

rection terms in the series for V_0, one cannot restrict intermediate states to those above the Fermi gas. It seems probable therefore that the series will not converge as rapidly as it would if use could be made of antisymmetric intermediate states. If one uses a formulation with antisymmetrized intermediate states, new difficulties occur, however. As an illustration of these, we consider a formulation in which V is split into symmetric parts, each of which can only cause transitions to antisymmetric states provided that the initial state is antisymmetric. Let

$$V=\lambda v, \tag{150}$$

where $\lambda=A(A-1)/2$. Define

$$t=v+v\frac{1}{e}{}'t, \tag{151}$$

$$e=E_0-T-t_c, \tag{152}$$

$$t_c=\lambda\,'C'=\lambda\,'t', \tag{153}$$

$$I=\,'t-\,'C', \tag{154}$$

$$F=1+\frac{1}{e}\lambda IF. \tag{155}$$

Neglecting terms of order A^{-1}, we get

$$F\Phi_0=M\Phi_0, \tag{156}$$

$$V_0\doteqdot(\Phi_0,\lambda tF\Phi_0) \tag{157}$$

$$=\lambda(\Phi_0,t\Phi_0)+\lambda^2\left(\Phi_0,\frac{1}{e}t\text{–}F\Phi_0\right)+\cdots. \tag{158}$$

Although the intermediate states can now all be treated as antisymmetrical and the discussion of convergence is thereby simplified, it can be seen that there is now a difficulty at another point. This new difficulty lies in the solution to Eq. (151) for the new matrix t. It can be argued[11] that the two equations (151) and (105) have nearly the same solution if one is only interested in the coherent scattering part of t, provided that the interaction potential v is highly singular. The second term in (158) is not negligible (it is about one-quarter of the leading term at most) but does not appear in the work of Brueckner and Levinson,[11] due to their use of different equations for t and F. The other correction terms which we have neglected appear to be small; they are to be discussed in detail in a forthcoming paper by Brueckner and Levinson.[11]

[11] These questions are considered in an accompanying paper by Brueckner and Levinson, Phys. Rev. **97**, 1344 (1955), who are also developing a more detailed treatment of the use of antisymmetrization in the problem of nuclear saturation.

IX. CONCLUSION

The considerations of the present paper indicate that it is profitable to examine the theory of nuclear structure from a new viewpoint in which nuclear models are no longer to be interpreted as giving approximations to the actual nuclear wave function. The actual nuclear wave function can be obtained by acting on the wave function of the model with a suitable model operator. It appears that the success of actual nuclear models such as the shell model is due to a special choice of model operator which commutes with certain observables. It has been shown that a model operator can be used explicitly to define nuclear energy in a problem connected with nuclear saturation.

We have examined some of the simpler properties of model operators and investigated conditions which they must satisfy to ensure the "reality" of models with respect to certain observables such as angular momentum and parity. It is hoped in further work to consider in greater detail the special properties of model operators which are related to observables which are not accurately predicted by existing models. Preliminary work indicates that this study may give a clearer indication of the kind of correction terms which are needed in evaluating the magnetic moment for example.

The derivation of a formula for the energy of an "equivalent Fermi gas," although complicated, indicates that a derivation of the Mayer potential for the shell model may be possible by the method of this paper. Further work is proceeding on these aspects of the problem.

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