

Two-Body Forces and Nuclear Saturation. III. Details of the Structure of the Nucleus

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(Received November 22, 1954)

The characteristics of particle motion in the nuclear medium have been examined in detail. The origin of the strong dependence of the potential energy on the nucleon momentum is discussed and an equivalent formulation is exhibited in which a uniform and constant potential is assumed but the nucleon moves with markedly reduced mass. The determination of the potential is shown to lead to a self-consistency problem which is to some extent similar to that appearing in the Hartree method of self-consistent fields. Solution of this problem shows that the self-consistency requirements impose severe restrictions on the solution and have a strong stabilizing influence in the saturation problem. The volume energy of the nucleus has been evaluated by using two-body potentials closely equivalent to a combination of central and tensor forces which agree with the low-energy parameters and predict scattering correctly up to 90 Mev. The result agrees closely with the observed values of energy and density.

The collective character of the nucleon potential is described and shown to manifest itself markedly in the excitation energy of single-particle levels where a considerable fraction of the energy is taken up in small adiabatic shifts of the states of the remaining nucleons.

The origin of the surface and symmetry energy is discussed; an evaluation of these effects gives a result in good agreement with empirically derived values. The stability of the nucleus against distortion arising from the polarizing effects of the tensor force is also examined and it is found that the nucleus shows marked stability against such distortion.

I. INTRODUCTION

IN previous papers on the problem of nuclear saturation¹ (to be referred to as NS I and NS II), an approximation method was developed and applied to the determination of certain properties of the volume and surface energies of nuclear matter in the presence of central and tensor forces. It is the purpose of this paper to discuss a variety of additional structural details which considerably increase the completeness of this nuclear model. In Sec. II we shall consider various aspects of the motion of nuclear particles in nuclear matter and the problem of deriving the potential energy felt by such a particle. In this connection we shall develop some concepts of the dispersion law relating energy and wave number in nuclear matter and of the self-consistency requirements in the formulation of the problem. In Sec. III we shall discuss some of the collective aspects of the nuclear states which are implicit in the work of Sec. II; in Sec. IV we shall discuss the problems of the surface and symmetry energies of the nucleus and the stability of the nucleus against distortion arising from polarizing action of the tensor force. Finally in Sec. V we summarize our results and make some concluding remarks.

In this paper we shall not discuss the basic assumptions of the method as developed in NS I; a detailed discussion of these together with a physical interpretation of various aspects of the model will form the content of another paper now in preparation.

* Supported in part by a grant from the National Science Foundation. The work was also done in part while the author was a visiting Associate Physicist at Brookhaven National Laboratory.

¹ Brueckner, Levinson, and Mahmoud, *Phys. Rev.* **95**, 217 (1954); K. A. Brueckner, *Phys. Rev.* **96**, 908 (1954).

II. DISPERSION LAW IN NUCLEAR MATTER; SELF-CONSISTENCY PROBLEMS

A. Origin of Dispersion Effect and of Self-Consistency Requirement

In NS I and NS II it was shown that the potential energy of a particle moving in a plane wave state in uniform nuclear matter was in general a function of its momentum, so that the energy of a particle of momentum k was related to k by

$$E(k) = k^2/2M + V(k), \quad (1)$$

with $V(k)$ not a constant. This expression is a statement of a dispersion law relating E and k . In many typical applications such as those encountered in the development of the shell model, it is assumed that the potential is independent of the particle state. For example in the case of a square well the presence of the potential merely serves to define a boundary and a new zero point of energy without otherwise affecting the particle motion. It is, however, not in general possible to assume that the potential is independent of the state. We shall discuss qualitatively the origin of the dispersion effect stated in Eq. (1) before proceeding to an examination of its further implications.

The dependence of the potential energy on k arises from the origin of this energy in the interaction of the particle with its neighbors in the nuclear medium. In the model which we have developed, the potential energy is a manifestation of the effect of collisions on the wave function; since this effect drops off rapidly as the relative energy in collision increases (and the scattering decreases), one would expect to find a decrease in the potential strength as the momentum of the particle under consideration rises. The direction of the effect is of course intimately related to the satu-

rating character of the two-body interaction considered, since the amplitude for forward scattering arises from this interaction characteristically becomes small (or eventually changes sign) at high energy. Thus we expect a lessening of the potential strength for a rapidly moving particle.

Another interesting way of describing these effects has been suggested to the author by Wheeler.² He observes that near zero momentum the potential energy $V(k)$ can be expanded as

$$V(k) = V(0) + bk^2 + \dots, \quad (2)$$

where in some cases only these first two terms in the expansion need be considered. We expect the leading term $V(0)$ to be negative, while the quadratic correction term will in general be positive. Thus we can now write the dispersion law of Eq. (1) as

$$\begin{aligned} E(k) &= k^2/2M + V(0) + bk^2 \\ &= k^2/2M^* + V(0), \end{aligned} \quad (3)$$

where M^* as defined by this equation is related to the normal mass by

$$M^* = M/(1 + 2bM) \quad (4)$$

and consequently is less than the normal mass. Thus we are led to an equivalent description in which the nuclear potential is constant but the "nucleon" mass is modified. This effect can be expected to manifest itself in other ways to which we shall return later.

We have so far not discussed a consequence of this modification in nuclear matter of the propagation characteristics of a nucleon which, as we shall see, leads to a self-consistency requirement on the formulation of the problem. As developed in NS I, the potential energy of a nucleon could be expressed in terms of a summation over nuclear states of the two-body forward scattering amplitude *evaluated in the nuclear medium*. We shall express this requirement formally in the next section; we remark here only that already an obvious difficulty arises in that to evaluate the scattering *in the medium* it is necessary to know the dispersion law for propagation before carrying out this evaluation. Thus the dispersion law determined is a function of the dispersion law assumed and a self-consistency requirement arises. As we shall see, this presents a rather formidable mathematical problem which we shall discuss in the following sections.

B. Formal Statement of Self-Consistency Problem

The formal statement equivalent to the requirement that we determine the scattering amplitude "in the nuclear medium" is the following³: let v_{12} be the inter-

action between nucleons 1 and 2. We further define the two quantities t_{12} and V_C by the equations:

$$\begin{aligned} t_{12} &= v_{12} + v_{12}[E_1 + E_2 - H_0(1) - H_0(2) \\ &\quad - V_C(1) - V_C(2)]^{-1}t_{12}, \end{aligned} \quad (5)$$

$$V_C(i) = \sum_{i \neq j} t_{Cij}, \quad (6)$$

where t_C is the part of the scattering operator t which is diagonal (or coherent) with respect to the nuclear states. We define the singularity in the energy denominator of Eq. (5) by taking the principal value⁴; this corresponds to the stationary character of the problem. The operator t defined this way is often called the reaction matrix. The potential $V_C(i)$ is the potential energy of the i th particle; the modification from the t defined for free-particle scattering arises from the presence of the potential energy V_C in the propagation function.⁵ The complication of the problem arises from the appearance of V_C in the defining equation for t which must be solved to obtain V_C ; the mathematical problem presented is the solution of a nonlinear integral equation for V_C .

The presence of the modified propagator in this problem represents the expected and quite reasonable many-body effect on the interaction of two nucleons in dense nuclear matter. We note that it does not arise from a change in the two-body potential v_{12} ; this effect has been previously discussed in NS I and shown to be small.

This problem has some of the aspects of the self-consistent field problems encountered in atomic physics. In our case we know the wave functions (degenerate Fermi gas) but the interaction is not known, while in the Hartree method the interaction is specified but the wave function is unknown. The self-consistency requirement of our problem is to specify such an average potential, solve for the interactions and finally from the interactions determine the average potential. Thus the solution of this problem bears a certain resemblance to the atomic case but is fortunately somewhat easier to obtain.

In solving the pair of Eqs. (5) and (6) the simplest approximation method is that previously used in NS I and II. To exhibit this, we assume that the transformation to the center-of-mass system and relative coordinates is possible [as it is if $V_C(k_1)$ depends at most on a constant and a term quadratic in k_1] and write in momentum space, where k is the relative

⁴ B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

⁵ The exclusion principle also acts to modify the scattering operators; it can be shown, however, that the most important class of intermediate states included in the construction of t which would appear to violate the exclusion principle are exactly cancelled when the expectation value in the many particle state is evaluated. In addition, most of the contribution to the integral over intermediate states comes from high excited states which are unaffected by the exclusion principle.

² J. A. Wheeler (private communication).

³ See N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).

momentum,⁶

$$t_C(k) = (\mathbf{k}|t|\mathbf{k}) = (\mathbf{k}|v|\mathbf{k}) + (2\pi)^{-3} \int d\mathbf{k}' (\mathbf{k}|v|\mathbf{k}') \times [E(k) - k'^2/M - V_C(k')]^{-1} (\mathbf{k}'|t|\mathbf{k}). \quad (7)$$

The pole in the denominator appears at the energy-conserving point,

$$E(k) = k^2/M + V_C(k), \quad (8)$$

which allows us to write

$$t_C(k) = (\mathbf{k}|v|\mathbf{k}) + (2\pi)^{-3} \int d\mathbf{k}' (\mathbf{k}|v|\mathbf{k}') [k^2/M - k'^2/M + V_C(k) - V_C(k')]^{-1} (\mathbf{k}'|t|\mathbf{k}). \quad (9)$$

Then, if $V_C(k)$ is independent of k or if the variation in $V_C(k) - V_C(k')$ is slow relative to the variation in $(k^2 - k'^2)/M$, then the terms in V_C can be dropped and we have approximately:

$$t_C(k) = (\mathbf{k}|v|\mathbf{k}) + M(2\pi)^{-3} \times \int d\mathbf{k}' (\mathbf{k}|v|\mathbf{k}') (k^2 - k'^2)^{-1} (\mathbf{k}'|t|\mathbf{k}). \quad (10)$$

In this case, which is now exactly analogous to the free scattering at momentum k , the effect of the potential has been merely to replace the (negative) energy E by k^2/M which is the kinetic energy in the nuclear medium. This approximation was used in NS I and II since it was assumed that an appreciable correction would not arise from this approximation to the properties of the medium; we shall now proceed, however, to develop a more rigorous method for solving Eqs. (5) and (6).

The defining equation for t_{12} of Eq. (5) can be rewritten

$$t_{12} = v_{12}\Omega, \quad (11)$$

where

$$\Omega = 1 + (E - H_0 - V_C)^{-1} v_{12}\Omega. \quad (12)$$

Thus the construction of Ω is sufficient to determine t . This equation, however, is analogous to the usual scattering wave equation (except for the modified boundary condition); hence we know that the t can be expressed in terms of the phase shifts which we obtain by solving the wave equation

$$(E - H_0 - V_C)\Omega = v_{12}\Omega. \quad (13)$$

Knowing V_C in momentum space allows us to make the transformation to coordinate space in a straight forward manner.

⁶ In this and the rest of the paper we shall assume that the diagonal part of t is given only by the forward scattering. The inclusion of the exchange scattering has been discussed in NS I and shown there to give a trivial change to the formalism.

C. Approximation Methods for Solving the Self-Consistent Problem

We consider first the simple case in which only constant and quadratic terms in V_C are used. As we shall show, this is a quite good approximation over the range of momenta which enter in the ground state but breaks down somewhat in evaluating t since a considerably larger range of momenta enters (in virtual states) in solving the wave equation for t . Making this approximation, we can write the problem:

$$(E - 2V_0 + 2b\nabla^2 - H_0)\Omega = v\Omega. \quad (14)$$

Since the energy E is

$$E(k) = 2V_0 + 2bk^2 + k^2/M, \quad (15)$$

this is equivalent to

$$(k^2 + \nabla^2)\Omega = M^*v\Omega, \quad (16)$$

with M^* defined by Eq. (4). The effect of the dispersion of the medium therefore (as was remarked earlier) is to introduce a mass modification. The solution of Eq. (16) for the phase shifts now follows standard methods; the usual scattering amplitude $a_k(\theta)$ is finally related to $t_C(k)$ by

$$t_C(k) = -\frac{4\pi}{M^*v} a_k(0). \quad (17)$$

It is interesting to note that t is independent of M^* in Born approximation since in such a case $a_k(0)$ is proportional to M^* .

The self-consistency requirement on $V_C(k)$ thus is reduced to a single-parameter problem, since it is now sufficient to find a value M^* (or b) in the dispersion law which will predict scattering amplitudes and hence a $V_C(k)$ which depends on k in the same way (i.e., is self-consistent). This procedure will be described in more detail in the following section.

Before proceeding, we make one further remark about a slightly more general choice of $V_C(k)$. Let

$$V_C(k) = V_0 + bk^2 + ck^4, \quad (18)$$

which is a somewhat better representation of the k dependence of the potential for k values rather larger than the Fermi momentum, although the term in k^4 is negligible for small values of k . According to Eq. (13) we now can write the Schrödinger equation:

$$[E - k_1^2/2M - k_2^2/2M - 2V_0 - b(k_1^2 + k_2^2) - c(k_1^4 + k_2^4)]\Omega(1,2) = v_{12}\Omega(1,2), \quad (19)$$

where \mathbf{k}_1 and \mathbf{k}_2 are to be interpreted as operators on the coordinates of particles 1 and 2. Transforming to the center-of-momentum system and relative coordinates by the relations

$$\mathbf{k}_1 = \mathbf{k} + \frac{1}{2}\mathbf{P}, \quad \mathbf{k}_2 = -\mathbf{k} + \frac{1}{2}\mathbf{P}, \quad (20)$$

we are led to the transformed equation:

$$[E - (1/M + 2b)(k^2 + \frac{1}{4}P^2) - 2V_0 - c(2k^4 + 3k^2P^2 + \frac{1}{8}P^4)]\Omega = v\Omega. \quad (21)$$

Next, introducing the relation between energy, P , and k_0 (the asymptotic value of k),

$$E = (1/M + 2b)(k_0^2 + \frac{1}{4}P^2) + 2V_0 + c(2k_0^4 + 3k_0^2P^2 + \frac{1}{8}P^4), \quad (22)$$

we find

$$[(1/M + 2b)(k_0^2 - k^2) - 2c(k^4 - k_0^4 + \frac{3}{2}k^2P^2 - \frac{3}{2}k_0^2P^2)]\Omega = v\Omega. \quad (23)$$

This is of standard form (with modified mass) except for the term proportional to c . This we approximate in the following way; as we have remarked, the terms in c are unimportant except for large k values. Thus we expect only the term in k^4 to be appreciable for low values of k_0 and P . Treating this term as a perturbation, we make the replacement

$$ck^4\Omega \cong M^*ck^2[(1/M + 2b)k_0^2 - v]\Omega \approx -M^*ck^2v\Omega. \quad (24)$$

Next, introducing a new function

$$\phi = (1 - 2cM^{*2}v)\Omega, \quad (25)$$

Eq. (23) reduces to

$$(k_0^2 + \nabla^2)\phi = \frac{M^{*2}v}{1 - 2cM^{*2}v}\phi, \quad (26)$$

if we again drop some small terms in ck_0^2 , with $M^* = M/(1 + 2bM)$ as before [Eq. (4)]. The asymptotic forms of Ω and ϕ are the same, and for nonsingular potentials the boundary condition at the origin is unchanged; thus a solution for the phase shifts using Eq. (26) is equivalent to a solution of the original problem.

Another way of expressing the result of Eq. (26) is that the "effective mass" M^{**} defined by

$$M^{**} = M^*/(1 - 2cM^{*2}v) \quad (27)$$

differs to some extent from the asymptotic value M^* . The direction of this correction depends on the size of c ; as we shall see later the departure from the simple quadratic law for $V_C(k)$ given in Eq. (2) corresponds to a small negative c . Thus the "effective mass" used in solving Eq. (26) is somewhat larger than the asymptotic value or equivalently, the effective potential

$$v^* = v/(1 - 2cM^{*2}v) \quad (28)$$

is somewhat stronger. The effect is not very large; we shall, however, make at least qualitative use of this result in the next section.

These considerations have shown how the self-consistent problem may be reduced to a two-parameter problem, the self-consistent determination of the constants b and c in the expansion of the potential

$$V(k) = V_0 + bk^2 + ck^4. \quad (29)$$

The problem is however hardly a trivial one in that it is necessary to obtain a complete solution for all phase

shifts over a large range of energies for each choice of the parameters b and c . Thus we must assume in practice a "zeroth order" form for $V_C(k)$ which we call

$$V_C^{(0)}(k) = V_0^{(0)} + b_0k^2 + c_0k^4, \quad (30)$$

and for this choice construct the scattering amplitudes and finally a first iterate $V_C^{(1)}$ which will be a function of b_0 and c_0 , i.e.,

$$V_C^{(1)}(k) = V_C^{(1)}(k, b_0, c_0). \quad (31)$$

Then, expanding the first iterate,

$$V_C^{(1)}(k, b_0, c_0) = V_0^{(1)} + b_1k^2 + c_1k^4, \quad (32)$$

we expect the new constants b_1 and c_1 to be functions of b_0 , c_0 , i.e.,

$$b_1 = b_1(b_0, c_0), \quad c_1 = c_1(b_0, c_0). \quad (33)$$

The self-consistency requirement is finally

$$b_1 = b_0, \quad c_1 = c_0. \quad (34)$$

D. A Self-Consistent Solution

The method of the last section can in principle be applied to any two-body interaction; it is of course in the general case (in which tensor forces are included) extremely tedious since a considerable range of values of the parameters (particularly of the most important parameter b) must be studied to map out the dependence of V_C . We shall, however, in part avoid this difficulty by using simple potentials approximately equivalent to the ones considered in NS I and NS II. In these earlier works, a potential derived from meson theory was used which had a rather complicated exchange dependence and which contained a large tensor force. Since, however, the many-body potential energy derived depended primarily on the low-energy scattering characteristics of the potentials, it is possible to replace (at least at low energies) the original potentials used by any choice agreeing with the low-energy parameters. An important restriction on the potentials, however, is that the phase shifts at high energy show the rapid drop resulting from the presence of repulsive cores in the potentials. A further requirement is that the triplet even scattering approximate to the tensor scattering at high energies, which is rather smaller than the scattering from a central potential because of the first-order averaging to zero of the tensor force scattering.

Potentials which approximately satisfy these conditions and which allow relatively simple evaluation of the self-consistency problem are square wells of approx-

TABLE I. Equivalent square well parameters. The ranges are given in units of 1.40×10^{-13} cm.

	Singlet	Triplet
Range	1.53	0.93
Core radius	0.40	0.30
Depth (Mev)	40.6	148

appropriate range and strength, with a region of central repulsion. The parameters of these potentials are given in Table I; the s -wave scattering predicted by them is given in Fig. 1 in comparison with the s -wave scattering of NS II. The remaining partial waves have been omitted for the following reasons. (Since they can be treated in Born approximation and thus are only weakly affected by the modifications of the effective mass the following remarks based on the studies of NS II are valid.) (1) The investigations of the tensor scattering show that the effect of the higher waves is very small, considerable cancellation occurring among the D -waves of different J and m_J . (2) The D -wave singlet scattering is appreciable but is approximately cancelled by the scattering arising from the odd-state singlet repulsive potentials. (3) The magnitude of the energy is almost entirely determined by the interaction at low energy and thus is very well approximated by the s -wave scattering alone. These remarks have all been quantita-

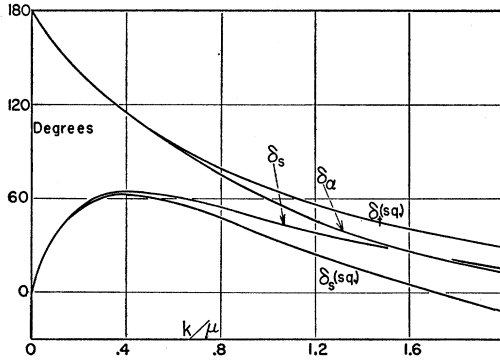


FIG. 1. S -wave scattering from square wells with parameters given in Table I, compared with S -wave scattering calculated for central and tensor wells of NS I and NS II. In the latter case for the tensor force, the δ_α eigen phase shift is given.

tively verified by a comparison with the calculation in NS II of the energy as a function of density predicted by the equivalent square wells.

Thus we can proceed to an evaluation of the self-consistency problem using the simple equivalent square wells. We notice first that the equivalent Schrödinger equation is

$$\begin{aligned}
 (k_0^2 + \nabla^2)\phi &= M^{**}v\phi \\
 &= -\frac{M^*V_0}{1+2cM^{**2}V_0}\phi, \quad r_c < r < R, \quad (35)
 \end{aligned}$$

where r_c is the radius of repulsion and R is the range of the well. This equation determines phase shifts as a function of

$$M^{**} = M^*/(1+2cM^{**2}V_0), \quad (36)$$

and as a function of energy. The phase shifts predicted for the singlet and triplet states are shown as a function of the mass parameter in Fig. 2. The potential energy per particle is related to these phase shifts (see NS I

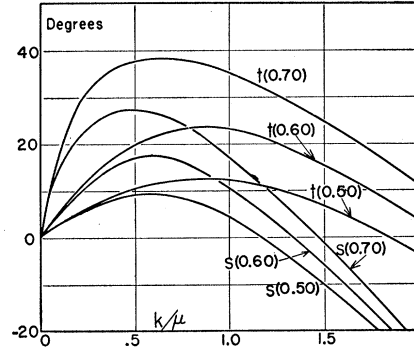


FIG. 2. Singlet (s) and triplet (t) phase shifts from equivalent square wells calculated as a function of mass parameter. The values of M^{**}/M are indicated.

and NS II) by the equations (for s states only):

$$\begin{aligned}
 V_C(k) &= -\frac{6}{\pi M^*} \int P(k, k') [\tan \delta_s(k', M^{**}) \\
 &\quad + \tan \delta_t(k', M^{**})] k' dk', \quad (37)
 \end{aligned}$$

where

$$\begin{aligned}
 P(k, k') &= 2, \quad 0 < k' < \frac{1}{2}(k_F - k) \\
 &= [\frac{1}{4}k_F^2 - (\frac{1}{2}k - k')^2]/kk', \\
 &\quad \frac{1}{2}|k_F - k| < k' < \frac{1}{2}(k_F + k). \quad (38)
 \end{aligned}$$

Let us first consider the simple case for which $c=0$ and $M^*=M^{**}$ [see Eq. (36)]. In this case the self-consistency requirement is rather easy to satisfy since we have an additional parameter at our disposal, namely the adiabatic density parameter η . This parameter appears through the dependence of k_F on the density through the relation

$$k_F = 1.52\mu/\eta. \quad (39)$$

We thus proceed in the following way: for a given value of M_0^* , we compute

$$V_C(k, M_0^*, \eta) = V_0 + b(M_0^*, \eta)k^2, \quad (40)$$

and adjust the density parameter η so that the value of M^* , predicted by the relation

$$M_1^* = M/[1 + 2b(M_0^*, \eta)M], \quad (41)$$

agrees with M_0^* . This requirement gives M^* as a function of the density (see Fig. 3) and thus also the potential energy V_C as a function of k and the density. We can then proceed to the mean potential energy per particle by evaluating the average

$$V_{Av} = \frac{1}{2} \int_0^{k_F} k^2 V_C(k) dk / \int_0^{k_F} k^2 dk, \quad (42)$$

the factor of $\frac{1}{2}$ compensating for the double summation over all particle coordinates. Finally, the mean energy per particle as a function of density is given by

$$E_{Av} = T_F + V_{Av}. \quad (43)$$

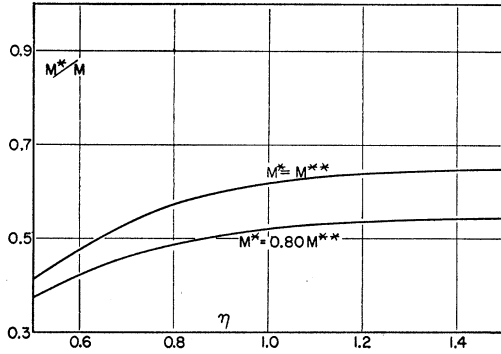


FIG. 3. Self-consistent values of mass parameter as a function of density parameter η [defined in Eq. (39)]. The two cases $M^* = M^{**}$ and $M^* = 0.80M^{**}$ are given.

The self-consistency requirements having already been satisfied for each value of η , it is sufficient to determine the value of η for which the energy is a minimum, thus determining the equilibrium density and binding energy. This relation between E and η is given in Fig. 4, the minimum occurs at a density of $\eta = 1.30$ or $R = 1.82 \times 10^{-13} A^{1/3}$ and a binding energy of about 4.5 Mev per particle.

This rather unsatisfactory solution to the problem is not yet fully consistent since the k^4 dependence of the potential is not negligible. The function $V_C(k)$ as determined for the approximately self-consistent parameters $M^*/M = 0.70$ and $\eta = 0.9$ is given in Fig. 5 as a function of k up to a value of k considerably larger than the Fermi momentum k_F . A good approximation to this curve is:

$$V_C(k) = (-119 + 57.3k^2/k_F^2 - 7.18k^4/k_F^4) \text{ Mev}, \quad (44)$$

so that although the quadratic approximation is valid for $k < k_F$, the term in k^4 is not negligible for values of k/k_F appreciably larger than one. The sign of this term in V_C is negative; inserting this value in Eq. (36) for M^{**} , we find

$$M^* \simeq 0.80M^{**}, \quad (45)$$

taking an average for the singlet and triplet states.

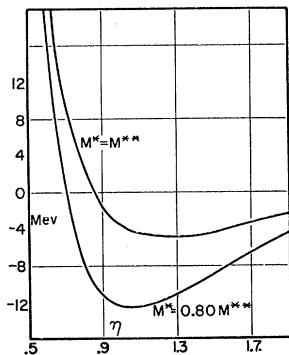


FIG. 4. Energy as a function of density for self-consistent potentials with $M^* = M^{**}$ and with $M^* = 0.80M^{**}$.

This relation is of course not very well determined since it is a higher order correction to the dominant quadratic effect in the dispersion law. In addition its precise value depends rather sensitively on the high-momentum behavior of the scattering which is not given very reliably by these approximation methods. Nevertheless, it is probably correct to assume that the effective mass in the high-momentum region is somewhat greater than the effective mass at low-momentum values.

By using the ratio between M^{**} and M^* of Eq. (45), it is again possible to carry out a self-consistent evaluation of the effective mass M^* and consequently of the potential energy. These results are given in Figs. 3 and 4. For this case the binding energy per particle is now about 12.8 Mev at a density $\eta = 1.00$ or $R = 1.40 \times 10^{-13} A^{1/3}$. At this value of the density the effective low-momentum mass M^* is $0.54M$, and a check on the self-consistency at high momentum values shows that the ratio $M^* = 0.8M^{**}$ is still approximately satisfied.

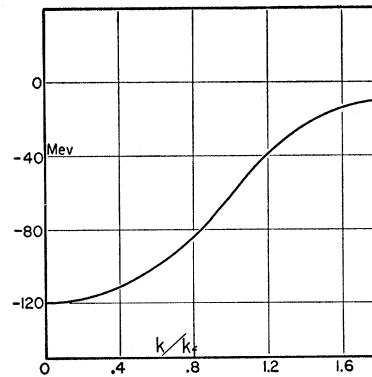


FIG. 5. Potential energy as a function of k for $M^* = 0.70M$ and $\eta = 0.90$.

Thus we have an approximately self-consistent solution to the two-parameter problem.

E. Discussion

Some rather striking features of these results are readily seen. The most evident effect of the self-consistent inclusion of the dispersion effects is the reduction in the mean potential energy and a simultaneous change in the position of the density minimum of the total energy curve (see Fig. 6). This modification arises from the impossibility of finding a self-consistent solution to the problem when the potential energy is too large or equivalently, when the "effective mass" is close to the nucleon mass. Thus a strong stabilizing influence on the nuclear density arises from the self consistency requirement on the solution. The effect is intimately associated with the many-body character of the problem, arising as it does from the modification of the propagation characteristics of a nucleon in nuclear matter. As such it gives an interesting insight into the

interplay of effects which lead to the equilibrium configuration of the nucleus.

III. COLLECTIVE ASPECTS OF THE MOTION

As we have shown in the previous section, the motion of nucleons through nuclear matter is determined by a potential which varies rapidly with the nucleon momentum; it is also true in our treatment that the potential energy arises from two-body interactions. In this section we would like to discuss the significance of these results and how they may be related to the single-particle and collective aspects of the nuclear structure.

Let us consider first the relation of the energies of the nucleons to such quantities as mean binding energy and the binding energy of the last nucleon. The potential energy of a nucleus of momentum k we call $V_c(k)$ as

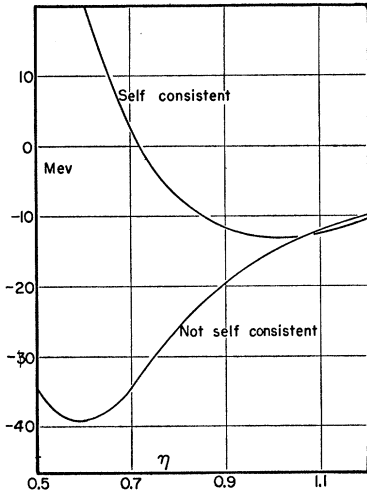


FIG. 6. Comparison of self-consistent energy E for $M^* = 0.80M^{**}$ with the result obtained in NS II with no self-consistency requirement imposed. For the latter case $M^{**} = M$ and M^* is in the range 0.2 to 0.4.

before. This energy cannot, however, be unambiguously assigned to one nucleon, since it is the result of two-body interactions and thus is shared between the nucleon under consideration and the rest of the nucleus. Thus, in an obvious way, the potential energy is already a manifestation of a collective characteristic. The mean energy of the system we get by averaging (the kinetic energy and the potential energy) over the momentum distribution, i.e.,

$$E_{Av} = T_{Av} + V_{Av} = \int k^2 dk [k^2/2M + \frac{1}{2}V_c(k)] / \int k^2 dk, \quad (46)$$

as has been observed earlier.

We now proceed to the relation between the binding energy of a nucleon and the kinetic and potential

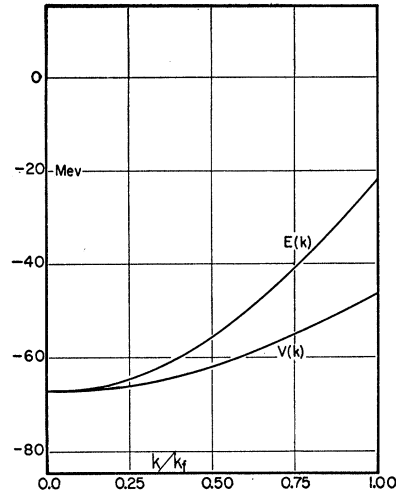


FIG. 7. Total energy and potential energy as a function of momentum at equilibrium density.

energies. At first we might expect to obtain this by simply evaluating $E(k)$; this, however, is obviously incorrect since we are not taking into account the "shared" character of the potential energy. A further complication arises from the possibility that the nuclear system will readjust its density to bring the energy to a minimum, which must also be taken into account in determining the binding energy. In spite of the complicated character of these questions, we can easily obtain an answer by considering the physical character of the binding energy, which is the difference between the energy of a nucleus of A nucleons in the ground state and the energy of a nucleus of $(A-1)$ nucleons in the ground state plus a removed nucleon. This difference is simply the mean binding energy as given in Eq. (46).

This simple result actually originates in a rather complicated interplay of effects which can be somewhat unphysically separated in the following way: we suppose that we formally consider the energy of removal of a nucleon as arising first from removal with the remaining nuclear configuration held fixed and second the energy of rearrangement of the remaining nucleus into its lowest state. The first of these is clearly $E(k) = T(k) + V_c(k)$; the rearrangement energy then makes up the difference between this value and the mean binding energy. The latter is a manifestation of the collective origin of the energy; in our example (see Fig. 7) the values of $E(k_F)$ is -22 Mev, of E_{Av} is -12.8 Mev; thus the rearrangement energy is 9.2 Mev. The origin of the rearrangement energy is fairly obvious; it arises in the adiabatic shift of all of the nuclear energy levels as one of the particles contributing to the binding is removed; it also arises from the reduction of the nuclear equilibrium volume as the particle is removed.

We can also express this result for the more general case of nuclear excitation. For simplicity let us consider a particle outside a core (for example a closed shell

+one nucleon). In this case the excitation of the outside particle to an excited state is also reflected in a change in the potential energy of each of the core particles, so that the energy of each level in the closed system shifts slightly as a consequence of the excitation of the outside particle. The excitation energy thus must be supplied not only to excite nonadiabatically the last nucleon but also to shift adiabatically the states of the core. Because of the related origin of these two effects in the two-body nature of the binding, they are of the same order of magnitude. We therefore see that in no sense are the "single-particle" excitation energies of the shell model single-particle in origin; they are rather of collective character and measure the sum of the changes in energy which take place in every state of the system upon excitation of any particle.

It might be remarked that the collective effect described here is rather different from the collective modes of excitation discussed by Bohr and Mottelson.⁷ The effect we describe is present even in excitations which their theory would describe as of pure single-particle origin, since we find that a true single-particle picture has no zeroth order validity.

IV. SURFACE AND SYMMETRY ENERGIES: STABILITY AGAINST DISTORTION

A. Surface Energy

In this section we shall make a determination of the nuclear surface energy, i.e., that part of the energy of the nucleus which is associated with surface effects and hence has a different dependence on the total number of nucleons than does the volume energy. In this development we shall follow closely the procedure used in NS II with modifications, however, which lead to more reasonable results than those earlier obtained.

In NS II an approximate result of Hill and Wheeler⁸ was used to represent the finite volume effect on the distribution of momentum states in the nucleus and was shown to lead to a surface energy. The result of Hill and Wheeler is that the effect of the finite volume is to modify the density of momentum states from the result valid for an infinite medium,

$$N_{\infty}(k)d\mathbf{k} = v d\mathbf{k} / (2\pi)^3, \quad (47)$$

to a result valid for a spherical nucleus of radius R ,

$$\begin{aligned} N_R(k)d\mathbf{k} &= v d\mathbf{k} [1 - 3\pi / (4kR)] / (2\pi)^3 \\ &\equiv v d\mathbf{k} (1 - k_0/k) / (2\pi)^3, \end{aligned} \quad (48)$$

with $k_0 = 3\pi/4R$. In this expression v is the volume on the surface of which the nuclear wave function is required to vanish. It is convenient to change the normalization of this density distribution in the following way: we notice first that the total number of

states with momentum less than k_F is

$$n_R(k_F) = \int_{k < k_F} N_R(k) d\mathbf{k} = v \frac{4\pi k_F^3}{3(2\pi)^3} (1 - \frac{3}{2}k_0/k_F). \quad (49)$$

Defining a modified volume v_M by the relation

$$v_M = v(1 - \frac{3}{2}k_0/k_F), \quad (50)$$

we can write

$$n_R(k_F) = v_M (4/3)\pi k_F^3 / (2\pi)^3. \quad (51)$$

This modified volume thus is an equivalent volume in which the number of states with momenta less than k_F has the same form as in the case of an infinite medium. Let us consider the physical implication of this change in normalization volume. First we observe that the change in volume is equivalent to a change in radius:

$$\begin{aligned} R_M &= R(1 - \frac{3}{2}k_0/k_F)^{1/3} \\ &\simeq R - \frac{3}{8}\pi\lambda_F, \end{aligned} \quad (52)$$

or that the effective radius R_M is less than the radius R of the original normalization volume by an amount

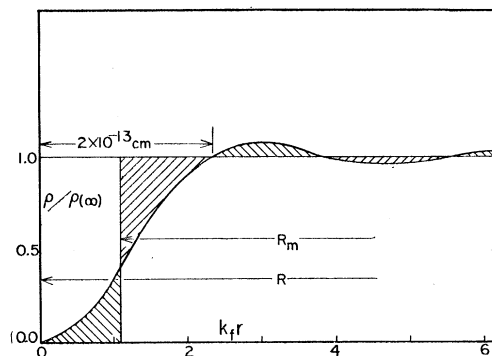


Fig. 8. Plot of the density of particles in a Fermi gas with the wave functions required to vanish on a plane boundary at $r=0$. The mean radius is indicated, as is also the scale in absolute units, showing that the nuclear edge in this case has a depth of the order of 2×10^{-13} cm.

very nearly equal to the wavelength λ_F of the particles of maximum momentum. This, however, is a result previously pointed out by Swiatecki⁹ and shown in Fig. 8. Physically the requirement that the wave function vanish on the surface of the volume has the effect of compressing the nuclear matter into a region of smaller mean radius, the compression being given by precisely the change from R to R_M . Thus the modified volume v_M represents more accurately the physical volume in which the nucleons are on the average confined. Using this result, we find for the relations between atomic number A and the Fermi momentum (with 2 neutrons and 2 protons per state):

$$A = 4v_M (4/3)\pi k_F^3 / (2\pi)^3, \quad (53)$$

⁷ A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab. Mat.-fys. Medd. **27**, No. 16 (1953).

⁸ D. Hill and J. Wheeler, Phys. Rev. **89**, 1102 (1953).

⁹ W. J. Swiatecki, Proc. Phys. Soc. (London) **A64**, 226 (1951).

and the average kinetic energy per particle:

$$T_F = \int \frac{k^2}{2M} (1 - k_0/k) d\mathbf{k} / \int (1 - k_0/k) d\mathbf{k}$$

$$= \frac{3}{5} \frac{k_F^2}{2M} \left(1 - \frac{5}{4} \frac{k_0}{k_F}\right) / \left(1 - \frac{3}{2} \frac{k_0}{k_F}\right). \quad (54)$$

To first order in k_0 (or to first order in $A^{-1/3}$), there is thus an increase in the kinetic energy per particle over that for an infinite medium:

$$T_F(R) - T_F(\infty) = -\frac{3}{5} \frac{k_F^2}{2M} \times \frac{1}{4} \frac{k_0}{k_F}$$

$$= (5.69/\eta^2) A^{-1/3} \text{ Mev}. \quad (55)$$

We consider next the mean potential energy per particle. In this case we need evaluate a sum over states of the form:

$$V_{Av} = \frac{1}{A} \sum_i \sum_j t_C(k_{ij}), \quad (56)$$

where t_C is the transition operator for forward scattering. Previously we have used the relation between $t_C(k)$ and the scattering amplitude $a_k(\theta)$

$$t_C(k) = \frac{4\pi}{M^*v} a_k(0), \quad (57)$$

where v is again the normalization volume for the nuclear states. The same effect, however, should be included here as we have just discussed, namely, the effective volume is more accurately represented by the modified (and reduced) volume v_M , the reason being precisely the same as that already discussed. Going over in Eq. (56) from the double sum to an integration and using the modified relation between t_C and a_k , we find

$$V_{Av}(R) = \frac{\frac{3}{8\pi^2 M^*} \int d\mathbf{k}_1 d\mathbf{k}_2 \left[a_s \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) + a_t \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) \right] \left(1 - \frac{k_0}{k_1}\right) \left(1 - \frac{k_0}{k_2}\right)}{\int d\mathbf{k}_1 \left(1 - \frac{3}{2} \frac{k_0}{k_F}\right)^2}. \quad (58)$$

This expression is identical with that used previously in NS II except for one normalization factor $(1 - \frac{3}{2} k_0/k_F)^{-1}$. This change is the result of the change in the normalization volume, which was not taken into account previously in a fully consistent way.

We again can separate this result into a term independent of R (or of k_0) and a correction term linear in

k_0 . The result is

$$V_{Av}(R) - V_{Av}(\infty) = \frac{9k_0}{32\pi^3 M^*} k_F^{-3} \int d\mathbf{k}_1 d\mathbf{k}_2$$

$$\times \left[a_s \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) + a_t \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) \right] \left(\frac{2}{k_1} - \frac{3}{k_F} \right). \quad (59)$$

In evaluating this correction we make use of the self-consistent results of Sec. II for M^* and the corresponding results for the scattering amplitudes. The result is a repulsive correction to the potential energy, of magnitude

$$V_{Av}(R) - V_{Av}(\infty) = \left(\frac{2.99}{\eta^2} + \frac{2.73}{\eta^4} \right) A^{-1/3} \text{ Mev}. \quad (60)$$

Combining these results with the kinetic energy effect of Eq. (55), we find

$$E = E_\infty + E_\sigma, \quad (61)$$

where the surface energy is

$$E_\sigma = \left(\frac{8.68}{\eta^2} + \frac{2.73}{\eta^4} \right) A^{-1/3} \text{ Mev}. \quad (62)$$

At the equilibrium density of $\eta = 1.00$, $E_\sigma = 11.4$ Mev $A^{-1/3}$ which is approximately equal to the usually accepted value.¹⁰ It is interesting to note that the surface energy is almost exactly equally divided between effects on the potential and kinetic energies.

This result is considerably less than that previously obtained in NS II; the improvement is a consequence of the more nearly correct treatment of the effective nuclear volume. The great sensitivity of the surface energy to the precise details of the treatment shows that the results obtained are not completely quantitative; it is clear, however, that this model gives results which are quite compatible with the empirically derived surface energy.

B. Symmetry Energy

The symmetry energy of the nucleus arises from a departure of the nuclear state from the most symmetrical one, i.e., that with equal neutron and proton numbers. Observed displacements of the neutron-to-proton ratio from unity are the consequence of the Coulomb interaction which leads to a neutron excess. We wish to make a quantitative study of these effects. In this discussion we shall omit the surface effects since these have a different origin than the symmetry energy and can be omitted (in first approximations) in a discussion of the latter.

We shall assume the following nuclear model on which to base our considerations. We let the maximum Fermi momentum for the neutrons be k_N and for the protons k_P . We also introduce the numbers character-

¹⁰ E. Feenberg, Revs. Modern Phys. **19**, 239 (1947).

izing the neutron excess:

$$N_N - N_P = I, \quad N_N + N_P = A, \quad (63)$$

or

$$N_N = \frac{1}{2}A(1+\epsilon), \quad N_P = \frac{1}{2}A(1-\epsilon), \quad (64)$$

where $\epsilon = I/A$. We further introduce the Fermi momentum k_F defined as before with $\epsilon = 0$. As a function of ϵ , the maximum neutron and proton momenta are

$$k_N = k_F(1+\epsilon)^{\frac{1}{3}}, \quad k_P = k_F(1-\epsilon)^{\frac{1}{3}}, \quad (65)$$

and the mean kinetic energy is

$$T_\epsilon = \frac{3}{5}(k_F^2/2M)^{\frac{1}{2}}[(1+\epsilon)^{5/3} + (1-\epsilon)^{5/3}]. \quad (66)$$

The linear term in ϵ vanishes; consideration of the quadratic term is sufficient for our purposes. Thus we find

$$T_\epsilon - T_F = (5/9)T_F\epsilon^2, \quad (67)$$

which is a repulsive term opposing departure of the neutron-proton ratio from unity.

We treat the potential energy similarly. First we define an integral

$$J(\epsilon) = (4\pi)^{-2} \left\{ \int_{k_N} d\mathbf{k}_1 \int_{k_N} d\mathbf{k}_2 a_{NN} + \int_{k_P} d\mathbf{k}_1 \int_{k_P} d\mathbf{k}_2 a_{PP} + \int_{k_P} d\mathbf{k}_1 \int_{k_N} d\mathbf{k}_2 a_{NP} \right\}, \quad (68)$$

where the three terms represent contributions from the neutron-neutron, proton-proton, and neutron-proton interactions, respectively. Restricting ourselves to consideration of the s -wave interaction alone, we can express this integral as a function of integrals over the amplitudes for scattering in the singlet and triplet spin states, i.e.,

$$J(\epsilon) = (4\pi)^{-2} \left\{ \int_{k_N} d\mathbf{k}_1 \int_{k_N} d\mathbf{k}_2 a_s \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) + \int_{k_P} d\mathbf{k}_1 \int_{k_P} d\mathbf{k}_2 a_s \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) + \int_{k_N} d\mathbf{k}_1 \int_{k_P} d\mathbf{k}_2 \left[a_s \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) + 3a_t \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) \right] \right\}. \quad (67')$$

Let us denote the result of carrying out the angular integrals over \mathbf{k}_1 and \mathbf{k}_2 by

$$(4\pi)^{-2} \int d\Omega_1 \int d\Omega_2 a \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) = \bar{a}(k_1, k_2). \quad (68)$$

Then we can write

$$J(\epsilon) = \int_0^{k_N} dk_1 \int_0^{k_N} dk_2 k_1^2 k_2^2 \bar{a}_s(k_1, k_2) + \int_0^{k_P} dk_1 \int_0^{k_P} dk_2 k_1^2 k_2^2 \bar{a}_s(k_1, k_2) + \int_0^{k_N} dk_1 \int_0^{k_P} dk_2 k_1^2 k_2^2 [\bar{a}_s(k_1, k_2) + 3\bar{a}_t(k_1, k_2)]. \quad (69)$$

We now carry out an expansion of the integral $J(\epsilon)$ about the point $\epsilon = 0$. The leading term $J(0)$ then corresponds to zero isotopic excess and therefore gives rise to the undisturbed potential energy. The linear term in ϵ is easily shown to vanish, if one makes use of the relations:

$$\partial k_N / \partial \epsilon = \frac{1}{3} k_F (1+\epsilon)^{-\frac{2}{3}}, \quad \partial k_P / \partial \epsilon = -\frac{1}{3} k_F (1-\epsilon)^{-\frac{2}{3}}. \quad (70)$$

Retaining only the quadratic term in ϵ , we find

$$J(\epsilon) - J(0) = \frac{1}{3} \epsilon^2 k_F^6 \left\{ \frac{1}{3} \bar{a}_s(k_F, k_F) - \bar{a}_t(k_F, k_F) + k_F^{-2} \int_0^{k_F} k^2 dk \frac{\partial}{\partial k_F} [\bar{a}_s(k_F, k) + \bar{a}_t(k_F, k)] \right\}. \quad (71)$$

The term depending on the difference in the scattering for singlet and triplet states represents the effect on the relative populations of the singlet and triplet spin states of increasing the isotopic excess. The second term is the result of the effect on the potential energy of the general shift to higher-momentum states. We finally can use the result expressed in Eq. (71) to find the effect on the potential energy, which is

$$[V(\epsilon) - V(0)]/V(0) = [J(\epsilon) - J(0)]/J(0),$$

or

$$[V(\epsilon) - V(0)]/V(0) = \frac{1}{9} \epsilon^2 k_F^6 \left\{ \frac{1}{3} \bar{a}_s(k_F, k_F) - \bar{a}_t(k_F, k_F) + k_F^{-2} \int_0^{k_F} k^2 dk \frac{\partial}{\partial k_F} [\bar{a}_s(k_F, k) + \bar{a}_t(k_F, k)] \right\} \times \left\{ \int_0^{k_F} k_1^2 dk_1 \int_0^{k_F} k_2^2 dk_2 [\bar{a}_s(k_1, k_2) + \bar{a}_t(k_1, k_2)] \right\}^{-1}. \quad (72)$$

The evaluation of this integral is most easily effected if we choose an explicit dependence of the sum $a_s(\frac{1}{2}\mathbf{k}_1 - \frac{1}{2}\mathbf{k}_2) + a_t(\frac{1}{2}\mathbf{k}_1 - \frac{1}{2}\mathbf{k}_2)$ on the two vectors $\mathbf{k}_1, \mathbf{k}_2$. A good representation of these amplitudes for the self consistent mass $M^* = 0.54M$ is

$$a_s + a_t = 2a |\mathbf{k}_1 - \mathbf{k}_2|^{-1} + b\mu^{-1} + \frac{1}{2}c |\mathbf{k}_1 - \mathbf{k}_2| \mu^{-2}, \quad (73)$$

where $a=0.467$, $b=1.258$, $c=-0.885$. Inserting this expansion into the integrals of Eq. (72), we find

$$\frac{V(\epsilon)-V(0)}{V(0)} = \frac{1}{9}\epsilon^2 \left(\frac{4a}{15} + \frac{bk_F}{9\mu} + \frac{ck_F^2}{9\mu^2} \right)^{-1} \left[\frac{k_F}{3} \bar{a}_s(k_F, k_F) - k_F \bar{a}_t(k_F, k_F) - \frac{2}{3}a + \frac{2}{15}c \frac{k_F^2}{\mu^2} \right]. \quad (74)$$

At equilibrium density, taking $\eta=1.00$ and inserting the values of the constants, we find

$$V(\epsilon)-V(0) = -0.62\epsilon^2 V(0) = 16.9\epsilon^2 \text{ Mev}. \quad (75)$$

Confirming this with the kinetic effect, we find for the symmetry energy:

$$E(\epsilon)-E(0) = 25.1\epsilon^2 \text{ Mev} = 25.1 \text{ Mev} \left(\frac{N-Z}{N+Z} \right)^2. \quad (76)$$

This is to be compared with the empirical value given by Feenberg¹⁰ of $18.1\epsilon^2$ Mev. Thus the agreement between the observed and calculated symmetry energy is rather good. The result we have obtained depends, however, rather sensitively on the precise character of the scattering near the Fermi momentum through the first two terms on the right side of Eq. (74). While these contribute about $\frac{1}{3}$ of the total symmetry energy, their precise values (more exactly their difference) are not so critical elsewhere. Thus the agreement we have found is probably as good as we can expect considering the uncertainties of our methods on details such as this.

C. Stability of the Nucleus against Distortion

In this section, we shall examine the stability of the nucleus against spin polarization and shape distortion. In general we can expect that a repulsive symmetry energy of the type discussed in Sec. IV will arise in connection with polarization of the spin and thus that the most stable configuration of the nucleus will be that of greatest symmetry. This effect can be compensated to some extent, however, by the action of the tensor force which, coupling together the spin polarization and shape distribution of the nucleus, can possibly lead to a stable nonspherical configuration with spin polarization. In this effect the tensor force is somewhat analogous to the Coulomb force which polarizes the isotopic spin of the system.

We proceed in a manner similar to the previous section. We let k_+ be the maximum momentum of states with spin parallel to an axis (to be taken the axis of distortion) \mathbf{n} , and k_- the maximum for states with antiparallel spin. We further let

$$N_+ - N_- = S, \quad N_+ + N_- = A, \quad (77)$$

or

$$N_+ = \frac{1}{2}A(1+s), \quad N_- = \frac{1}{2}A(1-s), \quad (78)$$

where s is a parameter analogous to the isotopic spin

polarization parameter ϵ of the last section. To allow for physical distortion of the nucleus, we introduce a distortion parameter ρ by the relation:

$$R = R_0[1 + \rho P_2(\cos\theta)]. \quad (79)$$

This distortion of shape is the simplest possible for the nucleus, a distortion proportional to $P_1(\cos\theta)$ corresponding merely to a displacement of the surface. The density of momentum states is, following the method of Hill and Wheeler,⁸

$$N(k)d\mathbf{k} = \frac{vd\mathbf{k}}{(2\pi)^3} \left\{ 1 - \frac{3\pi}{4R_0k} [1 + \rho P_2(\cos\theta)]^{-1} \right\}, \quad (80)$$

where θ is now the angle between \mathbf{k} and the axis of distortion \mathbf{n} . The determination of the kinetic energy gives to second order in ρ :

$$T = \frac{3}{5} \frac{1}{2M} \left\{ k_+^5 \left[1 - \frac{5}{4} \frac{k_0}{k_+} \left(1 + \frac{1}{5} \rho^2 \right) \right] + k_-^5 \left[1 - \frac{5}{4} \frac{k_0}{k_-} \left(1 + \frac{1}{5} \rho^2 \right) \right] \right\} \times \left\{ k_+^3 \left[1 - \frac{3}{2} \frac{k_0}{k_+} \left(1 + \frac{1}{5} \rho^2 \right) \right] + k_-^3 \left[1 - \frac{3}{2} \frac{k_0}{k_-} \left(1 + \frac{1}{5} \rho^2 \right) \right] \right\}^{-1}, \quad (81)$$

with k_0 defined as before in Eq. (48). Writing

$$k_+ = (1+s)^{\frac{1}{2}} k_F, \quad k_- = (1-s)^{\frac{1}{2}} k_F, \quad (82)$$

and expanding the kinetic energy about ρ and s equal to zero, we find

$$T = T_F \left[1 + \frac{1}{4} \frac{k_0}{k_F} + \frac{5}{9} s^2 + \frac{1}{20} \frac{k_0}{k_F} \rho^2 \right]. \quad (83)$$

In this result the second term is a surface correction, the third arises from spin polarization, and the fourth from shape distortion.

We consider next the potential energy. We define an integral related to the potential energy by the equation:

$$k(s, \rho) = (4\pi)^{-2} \sum_{\text{spins}} \int d\mathbf{k}_1 d\mathbf{k}_2 \left\{ 1 - \frac{k_0}{k_1} [1 + \rho P_2(1)]^{-1} \right\} \times \left\{ 1 - \frac{k_0}{k_2} [1 + \rho P_2(2)]^{-1} \right\} a \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right), \quad (84)$$

where a is the scattering amplitude. For the case of tensor scattering which differs for the three substates of spin 1, it is convenient to introduce projection operators for the states in which the spin is parallel, perpendicular, or antiparallel to the relative momentum vector

$\mathbf{k} = \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)$. Thus we write for the triplet states:

$$\begin{aligned} a_i(k) &= \frac{2}{3}t_1(k) \left(1 + \frac{1}{4}S_{12}\right) + \frac{1}{3}t_0 \left(1 - \frac{1}{2}S_{12}\right) \\ &= \frac{1}{3}(2t_1 + t_0) + \frac{1}{6}(t_1 - t_0)S_{12}, \end{aligned} \quad (85)$$

where

$$S_{12} = 3\boldsymbol{\sigma}_1 \cdot \mathbf{k} \boldsymbol{\sigma}_2 \cdot \mathbf{k} / k^2 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, \quad (86)$$

and t_1 (or t_0) is the amplitude for scattering with spin parallel or antiparallel (or perpendicular) to \mathbf{k} . The matrix elements of S_{12} for the three triplet substates are (relative to a direction of quantization at an angle θ with the relative momentum vector k):

$$\begin{aligned} \langle 1 | S_{12} | 1 \rangle &= (-1 | S_{12} | -1) = 3 \cos^2 \theta - 1, \\ \langle 0 | S_{12} | 0 \rangle &= -2(3 \cos^2 \theta - 1). \end{aligned} \quad (87)$$

By using these results and separating the integration over momenta and summation over spins into integrations over the regions of spin up (+) and spin down (-), Eq. (84) can be written

$$\begin{aligned} k(s, \rho) &= \frac{1}{3}(4\pi)^{-2} \left\{ \int_{k_+} d\mathbf{k}_1 \int_{k_+} d\mathbf{k}_2 N(k_1) N(k_2) \right. \\ &\quad \times [2t_1 + t_0 + (t_1 - t_0)P_2(\cos\theta)] \\ &\quad + \int_{k_-} d\mathbf{k}_1 \int_{k_-} d\mathbf{k}_2 N(k_1) N(k_2) \\ &\quad \times [2t_1 + t_0 + (t_1 - t_0)P_2(\cos\theta)] \\ &\quad + \int_{k_-} d\mathbf{k}_1 \int_{k_+} d\mathbf{k}_2 N(k_1) N(k_2) [2t_1 + t_0 + 9S \\ &\quad \left. - 2(t_1 - t_0)P_2(\cos\theta)] \right\}. \end{aligned} \quad (88)$$

In calculating this integral we shall keep only the first nonvanishing terms in ρ and s^2 ; in so doing we omit terms in ρ^2 which are of the same magnitude as those kept in the kinetic energy and which are also repulsive. As we shall see, only a rough determination of the ρ^2 terms is sufficient to show that the system is stable against distortion so that we shall for simplicity make use of only the quadratic kinetic effect (and thus underestimate the set quadratic term in ρ). In expanding the integral $k(s, \rho)$, we find that as in the case of the symmetry energy the terms linear in s vanish. The result of this development is

$$k(s, \rho) = k(0, 0) + s^2 G + s^2 \rho H, \quad (89)$$

where

$$\begin{aligned} k(0, 0) &= (4\pi)^{-2} \int_{k_F} d\mathbf{k}_1 \int_{k_F} d\mathbf{k}_2 \left(1 - \frac{k_0}{k_1}\right) \left(1 - \frac{k_0}{k_2}\right) \\ &\quad \times (2t_1 + t_0 + 3S), \end{aligned} \quad (90)$$

$$\begin{aligned} G &= 2 \frac{\partial^2 k(0, 0)}{\partial s^2} = \frac{k_F^6}{27} \left\{ 2\bar{t}_1(k_F, k_F) + \bar{t}_0(k_F, k_F) \right. \\ &\quad - 9\bar{S}(k_F, k_F) + 3k_F^{-2} \int_0^{k_F} k^2 dk \frac{\partial}{\partial k_F} [2\bar{t}_1(k_F, k) \\ &\quad \left. + \bar{t}_0(k_F, k) + 3\bar{S}(k_F, k)] \right\}, \end{aligned} \quad (91)$$

$$\begin{aligned} H &= 2 \frac{\partial^3 k(0, 0)}{\partial s^2 \partial \rho} = \frac{8}{9} k_0 k_F^5 \int \frac{d\Omega_1 d\Omega_2}{(4\pi)^2} P_2(\cos\theta) \\ &\quad \times P_2(\cos\theta_1) \left[t_1 \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) - t_0 \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{2} \right) \right] \end{aligned} \quad (92)$$

with in the last integral the vectors \mathbf{k}_1 and \mathbf{k}_2 equal to k_F in magnitude. The term $k(0, 0)$ is simply related to the potential energy without distortion, the repulsive "spin symmetry" energy arises from G , and the magnitude of H determines the amount of distortion. In the absence of tensor force, $t_1 = t_0$ and no distortion will occur.

Before evaluating these integrals let us examine the general features of the change in kinetic and potential energy with distortion. After combining Eqs. (83) and (89) to give the total energy, the result is (dropping the $\frac{1}{4}k_0/k_F$ term in the kinetic energy):

$$\begin{aligned} E &= T_0 \left[1 + \frac{5}{9}s^2 + \frac{1}{20} \frac{k_0}{k_F} \rho^2 \right] \\ &\quad + V_0 \left[1 + s^2 \frac{G}{k(0, 0)} + s^2 \rho \frac{H}{k(0, 0)} \right]. \end{aligned} \quad (93)$$

Minimizing the energy with respect to ρ , we find

$$\begin{aligned} E &= T_0 + V_0 + s^2 \left[-T_0 + V_0 \frac{G}{k(0, 0)} \right] \\ &\quad \times \left[1 - \frac{s^2 (V_0 H / k(0, 0))^2}{\frac{1}{5} \frac{k_0}{k_F} T_0 \left(\frac{G V_0}{k(0, 0)} + \frac{5}{9} T_0 \right)} \right]. \end{aligned} \quad (94)$$

The symmetry correction terms to the kinetic and potential energies are both repulsive and of the same order of magnitude; also T_0 and V_0 are nearly equal in magnitude. Thus let us write very roughly (for nuclei with $A \approx 200$ that $\frac{1}{5}k_0/k_F \approx 1/25$):

$$E - E_0 \approx |V_0| s^2 \left[1 - s^2 \frac{|V_0|}{T_0} \left(\frac{5H}{k(0, 0)} \right)^2 \right]. \quad (95)$$

For the energy of the nucleus to be lower for large distortions, it is necessary for the term proportional

to s^4 to dominate (with s^2 less than one). The ratio $H/k(0,0)$ is generally much less than unity since it involves a ratio of essentially a surface term to a volume term; this can be compensated for, however, by a sufficiently large ratio of V_0/T_0 . This cannot occur in our problem because of the strong stabilization of the nuclear density near a normal density for which the kinetic and potential energies are of the same magnitude. Thus we can expect rather generally for saturating forces that the presence of an admixture of tensor force will not lead to polarization and distortion. Consistent with these remarks, we content ourselves with approximate evaluation of these integrals since only a qualitative result is needed.

In $k(0,0)$ and G we neglect the tensor force splitting of the spin substates since this affects the result only weakly. The integrals then may be evaluated following the methods used in evaluating the symmetry energy. In the integral for H we make the approximation of replacing the difference t_1-t_0 by its angular average leaving the integration over the product of the Legendre functions alone. We also get an approximate value for the tensor force splitting by examining the results of NS II which show that $(t_1-t_0)k_F$ is approximately equal to 0.30. Again the precise value of this quantity is not needed since we need only a very rough measure of the energy. The results of these approximations are

$$V(s,\rho) - V(0,0) = 11.5 \text{ Mev } s^2 + 0.25 \text{ Mev } s^2 \rho. \quad (96)$$

Combining this with the kinetic energy and minimizing with respect to the distortion parameter ρ , we find

$$E - E_0 = 21.5 \text{ Mev } s^2 (1 - 0.0029s^2). \quad (97)$$

Thus, for possible values of s^2 (less than unity), the increment in the tensor force energy due to the distortion (the negative term in the bracket) is entirely negligible relative to the dominant repulsive energy associated with the spin polarization (the term in s^2). We therefore see that there is no tendency for the nucleus to depart from its spherical shape or to polarize its spin, the stability against such effects being marked.

V. SUMMARY OF RESULTS

In the preceding sections we have discussed some of the consequences of a treatment of the saturation and structure problems which is more general than the considerations of earlier papers (NS I and NS II). In considering the dispersive characteristics of the nuclear medium we have been led to a problem of self-consistency which we have formulated and approximately solved in Sec. II. The problems encountered there are mathematically difficult to solve and also depend on some features of the theory which are less certain than the central aspects of the formulation. In particular, the dispersive characteristics of the medium for very fast particles appear to be rather important; conse-

TABLE II. Summary of results for volume, surface, and symmetry energy. For comparison the results of Feenberg^a are also given. The energies are all in units of Mev.

	Radius	E_{volume}	E_{surface}	E_{symmetry}
Calculated	1.40×10^{-13}	12.8	11.4	25.2
Feenberg	1.46×10^{-13}	14.0	13.1	18.1

^a See reference 10.

quently a detailed knowledge of very high-energy scattering is necessary. Thus we can no longer claim that we have fully solved the problem originally formulated, although it is probable that the main features have been handled on the whole correctly. The quantitative results of this paper are, considering the approximations made, in very satisfactory and perhaps surprising agreement with the empirical values for density, binding energy, surface energy, and symmetry energy (see Table II). Since the theory has no free parameters this success is to some extent persuasive of the correctness of the main concepts involved.

The picture of the nucleus which we have developed is one in which the nucleons move to a good approximation in the independent-particle states determined by a uniform potential. The properties of the highly dispersive nuclear medium are deduced from the two-body interactions and are a rapidly varying function of the density of the nuclear matter. The equilibrium density at the energy minimum occurs as the result of a rather complex balance between the kinetic and potential energy.

The properties of the nucleus are collective to a remarkable extent as a result of the codetermination by the nucleons of the potential in which they move. A striking consequence of this is shown in the collective character of the excitation energies of "single-particle" excited states, this energy being shared almost equally by the excited particle and by the remainder of the nucleus. The unique features of the nuclear structure are also expressed clearly in an equivalent formulation of the problem in which the potential is assumed constant but the nucleon mass reduced by a factor of approximately two. Thus the "nucleons" can be considered to move in a conventional uniform and constant potential but with a very different "effective mass."

The results of this work can of course immediately be extended to the shell model. A reasonable procedure in this case would be to assume that the potential energy appropriate to a shell-model state is for some mean value of the momentum in the state. The results of this paper could then be immediately applied to the case of spherical quantization. A simpler procedure would be to introduce the equivalent problem in which the nuclear mass is reduced and the appropriate constant potential used. In our case, for example, the mass reduction is by a factor 0.54 and the equivalent po-

tential has a depth of 67 Mev. One of the most obvious consequences of this change is that the spacing of shell-model energy states would be considerably increased. Other consequences of this change and their relevance to experiment will be discussed in a separate paper.

The author would like to acknowledge a number of helpful discussions of topics related to the contents of this paper with Dr. R. J. Eden, Dr. N. Francis, and Dr. C. A. Levinson. He is also indebted to Professor John A. Wheeler whose interest has been greatly stimulating to this work.

Theory of Nuclear Models

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 (Received November 22, 1954)

The framework for a unified theory of nuclear structure is described in which the wave functions for different nuclear models are obtained by transformations on the actual nuclear wave function. This formulation provides a basis for explaining the success of weak-coupling models of the nucleus and showing that they are not in conflict with the assumption that nucleons have very strong mutual interactions. The explanation lies in the fact that only in certain circumstances can the "particles" in a nuclear model be interpreted as nucleons.

We investigate the properties which transformation operators must have to change the nuclear wave function into a model wave function and consider how far these properties are satisfied in practice. Self-consistent equations are set up for a model having a product wave function in the particle variables, and it is shown that these equations can be solved in an approximation relevant to the problem of nuclear saturation.

I. INTRODUCTION

IN recent years a number of nuclear models have been developed which successfully describe many aspects of nuclear structure. The most striking successes have been obtained by the Mayer-Jensen¹ shell model and by Weisskopf's² cloudy crystal-ball model, and it is clear that for low energies there must be a close correspondence between these models and the actual nucleus. On the other hand, these models are based on an assumption of weak interaction between the particles they describe and this assumption appears to be in direct contradiction with the strong nucleon interactions which are observed in scattering experiments. We shall show in this paper that this apparent contradiction is not a real one but is explained essentially by the fact that the "particles" in these nuclear models are not nucleons—that is to say they cannot in all circumstances be interpreted as nucleons.

Our program is firstly to set up and explain a formalism in which the wave functions of various models can be transformed by "model operators" into the real nuclear wave function. Next we consider how actual nuclear models fit into the framework of our theory, and finally we consider the problem of using these model operators in practical applications of the theory.

Since this theory shows that the particles, in the shell model, for example, are not nucleons, it is necessary to show that shell-model results can still be explained with this new interpretation. We do not attempt to derive the explicit assumptions of the shell model, although our formalism leads to self-consistent equations which if solved should lead to the well potential of the model. Our primary concern is to show how the following aspects of the shell model are consistent with our method: the particles in the model obey the exclusion principle as though they are neutrons and protons, energy levels are predicted with sufficient accuracy to indicate the order of filling single-particle states, angular momentum and parity are accurately predicted as though the model was the real nucleus, and selection rules are well predicted. This is not by any means a complete list but should serve to indicate the problems of interpretation which have to be considered. Also, we must consider the relation of our theory to the failures of the model such as the failure to predict transition rates with any accuracy.

Since our methods are quite general and apply to any nuclear model, a complete presentation of this theory would require detailed consideration of very many aspects of the relation of nuclear models to experiment. We have in this paper attempted to select sufficient of these aspects of the theory to indicate the power of the method, and hope in future papers to examine other aspects. In particular, we have limited ourselves to consideration of low-energy nuclear models although we believe that the methods are also applicable to high-energy problems. In addition to the shell model, we

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† This work was supported by grants from the National Science Foundation, the Office of Naval Research, and the U. S. Atomic Energy Commission.

¹ Maria Goeppert Mayer, *Phys. Rev.* **75**, 1969 (1949); Haxel, Jensen, and Suess, *Phys. Rev.* **75**, 1766 (1949).

² Feshbach, Porter, and Weisskopf, *Phys. Rev.* **96**, 448 (1954).