32 mb/Q or roughly a quarter of the total yield is due to post-resonance photons. The reason for the predominance of resonance reactions with 320-Mev x-rays is due not only to the large size of the cross section at those energies, but also to the shape of the bremsstrahlung spectrum.

Jones and Terwilliger[27] have recently determined neutron yields from copper irradiated by 320-Mev x-rays. Neutron yields can also be computed from our data by multiplying the mb/Q for each separate yield by the corresponding neutron multiplicity and adding the results. In carrying out this sum, the emission of composite particles (α particles, deuterons) was ignored. The resulting overestimate of the total neutron emission is, however, not very serious in view of the known relatively small yield of these heavier particles, the total neutron production yield comes out to be 180 mb/Q of which 94 mb/Q or roughly half is due to post resonance photons. Both these results are quite consistent with the more direct and precise measurements of Jones and Terwilliger.

For purposes of comparison, several other types of yields have been recorded in Table III. Some of these yields are due to fairly direct measurements but some are based on interpolations of data obtained for elements other than copper. For example, one of the listed meson production yields is based on the π^+ production yield in carbon,[32] the $A^{\frac{2}{3}}$ dependence of this yield,[33] on the observed ratio of π^- to π^+ production[33] and on estimates of the π^0 production rate.[34] A final estimate based on so many components is at best rather rough, but it was thought to be useful nevertheless to record a number of different types of yields in one place. If one believes all of the numbers in the table there are some disconcerting things about some of their relative sizes. For example, it is possible to estimate the yield for the production and recapture of mesons in a nucleus from the yield of those mesons that manage to get out, if one is willing to interpret the observed $A^{\frac{2}{3}}$ dependence of the meson-production cross sections in terms of a very short mean free path for mesons in nuclear matter. But such an estimate, together with a reasonable estimate for neutron multiplicity in meson-recapture events,[35] leads to an expected neutron production rate a few times larger than what is actually observed. In view of all the uncertainties involved in the determination in some of the yields quoted, it is hard to know how seriously to regard these discrepancies.

[32] J. Steinberger and A. S. Bishop, Phys. Rev. **86**, 171 (1952).
[33] R. M. Littauer and D. Walker, Phys. Rev. **82**, 746 (1951).
[34] Panofsky, Steinberger, and Steller, Phys. Rev. **86**, 180 (1952).
[35] V. Tongiorgi and D. A. Edwards, Phys. Rev. **88**, 145 (1952).

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Scattering of Fast Neutrons and Protons by Atomic Nuclei*

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The effects of the Pauli principle on the analysis of the scattering of fast neutrons and protons by atomic nuclei are considered. This modifies the usual multiple scattering treatment of such problems in three ways: (1) It is necessary to agree on a convention for deciding which are "scattered" and which are "nuclear" nucleons. (2) The two-body scatterings obtained from the impulse approximation must be properly antisymmetrized. (3) Exchange corrections occur because of the non-orthogonality of the plane wave states for scattered particles and the states for bound particles. The latter corrections seem to be negligible for energies sufficiently high that the multiple-scattering approach is expected to be useful anyway. The present analysis is also applicable to other types of multiple-scattering problems.

I. INTRODUCTION

In two previous publications[1,2] the theory of the scattering of fast particles by atomic nuclei was formulated as a multiple-scattering process. In the present work we wish to extend this to the scattering of fast neutrons and protons by atomic nuclei.[3] At first sight this might appear difficult, since the concept of a single particle passing through a medium and being scattered by particles of the medium does not lend itself conveniently to a description in which all the particles are treated as indistinguishable, as demanded by the Pauli principle.[4] Nevertheless, we shall be able to conclude that under such conditions that the multiple-scattering formulation is expected to be useful anyway, the Pauli principle adds no significant complication.

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[1] K. M. Watson, Phys. Rev. **39**, 575 (1953). This paper will henceforth be referred to as I.
[2] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953). This paper will henceforth be referred to as II.
[3] G. Takeda and K. Watson, Phys. Rev. **94**, 1087 (1954), have given an application of the conclusions in the present paper.
[4] We use the generalized Pauli principle by which neutrons and protons are two states of the *nucleon*. The wave functions describing such systems are to be antisymmetrized with respect to all nucleons.

To see this, let us first suppose that the incident particle is distinguishable from the nucleons in the nucleus and review the physical basis for the multiple-scattering model.[5] The first and most significant step is the introduction of the *impulse approximation*,[6] by which the many-body dynamics of the problem is related to that of the two-body problem. This approximation permits one to describe the encounter of the incident particle with a nucleon in the nucleus as a scattering between two free particles. The criterion for the success of this approximation is essentially that the energy of the incident particle be much larger than the binding potential of the struck nucleon. Actually, this condition may be somewhat stronger than is needed, since it is the change, ΔU, in the binding potential during the scattering process which is important. Thus for a fairly uniform nuclear potential the impulse approximation might be valid for rather low-energy scatterings.[7] Indeed, when the ratio

$$\Delta U/E \tag{1}$$

is small (where E is the energy of the incoming particle),[8] we can consider the impulse approximation to be valid. (A more careful discussion is given in reference 6.) Since ΔU may be of the order of 10–20 Mev, we expect to find the impulse approximation useful for E larger than 50–100 Mev.

The formulation of the complete process in terms of such two-body scatterings has been given in reference 1. This involves expressions for such nuclear properties as the nucleon density in the nucleus, which are assumed to be known. After the first scattering, we expect the nucleus to be excited and thus these nuclear properties to be changed. As described in I, however, if the incident particle is fast, it should outrun the "storm" which it has created and is expected to encounter at each subsequent scattering a medium which is essentially in its ground state. For incident nucleon energies above 50–100 Mev,[8] we should expect this condition to obtain and the impulse approximation also to be valid (until, of course, the nucleon energy has been sufficiently degraded through successive scatterings).

A further requirement that a scattering between the incident particle and one in the nucleus be essentially the same as the scattering between free particles is that the distance between scatterings (mean free path) be sufficiently great that the energy of the scattered particle between collisions is fairly well restricted by energy conservation. This condition can be specified by the relation

$$\Delta E \Delta t \simeq \hbar,$$

where ΔE is the uncertainty in the energy of the scattered particle at the second scattering and Δt is the time between the two collisions. If the mean free path is λ, then

$$\Delta t \simeq \lambda/v$$

(where v is the velocity of the particle). For most problems of interest,

$$\lambda \simeq 4a_0,$$

where $a_0 \simeq \hbar/\mu c$ is the mean separation between nucleons in the nucleus. Thus

$$\Delta E/E \simeq \tfrac{1}{4}(v/c)(\mu c^2/E), \tag{2}$$

where E is the energy of the scattered particle. For $E > 100$ Mev, the uncertainty in the energy is small.[9]

We turn now to the point of major concern to us in this work—i.e., the consequences of the Pauli principle when the bombarding particle is a neutron or a proton. Perhaps the most obvious point to require clarification is the decision as to which is the "scattered particle" and which are the "nuclear particles" after the first and subsequent scatterings. The distinction implied here is evidently one of convenience only and also one which must be chosen appropriately for the problem at hand. Nevertheless, it appears useful for many cases to define the "scattered nucleon" as the one having the *greatest kinetic energy*. Since we are considering bombarding energies much larger than nuclear kinetic energies, this distinction seems meaningful and will be adopted, unless specified otherwise.[10] An alternate distinction which will be useful in studying the prong distribution of nuclear stars induced by high-energy nucleons will be to define as "scattered nucleons" all those whose kinetic energy exceeds a certain lower limit, which is presumably larger than that for the "bound nucleons."[11] In any case, it seems that the most convenient "labeling" of the fast nucleons will be in terms of their kinetic energy.

The above consideration involved only the choice of a useful convention. There are also other consequences of the Pauli principle which require more elaborate and detailed development. That there is a limit for which the multiple-scattering approximation is valid is apparent—to see this we need only consider the example of a proton being scattered in diffuse hydrogen gas. Here the only significant consequence of the Pauli principle is the proper antisymmetrization of the two-body wave function for each proton-proton scattering.[12] When the target is a nucleus, it is also necessary to antisymmetrize the two-body nucleon-nucleon wave

[5] R. Serber, Phys. Rev. **72**, 1114 (1947).

[6] G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952); G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

[7] See references 6 and Brueckner, Levinson, and Ahmoud, Phys. Rev. **95**, 217 (1954).

[8] The energy E should here be considered to be the energy of the incident particle *inside* rather than *outside* the nucleus. That is, we should add to the original energy the depth of the nuclear potential well. (See reference I for a justification of this statement.)

[9] The rapidity with which the cross section varies with energy is of some importance here. However, most cross sections in the energy range of a few hundred Mev do not seem to vary very rapidly with energy.

[10] This choice is evidently the proper one for studying the elastic scattering of fast nucleons by means of the optical method.

[11] This is essentially the point of view adopted by Goldberger for his semiclassical model of high-energy nuclear reactions [M. L. Goldberger, Phys. Rev. **74**, 1269 (1948)].

[12] This we shall call the "primary exchange" effect.

function for each two-body scattering. When the energy is sufficiently high that the impulse approximation is valid, this should adequately describe the scattering event. The reason for this is that the condition of validity of the impulse approximation is essentially just that the two scattering particles interact only with each other and not with other nuclear particles during a scattering event.

There is one final aspect of the Pauli principle which might appear to make the multiple-scattering formulation difficult at low energies. This is a process of the sort by which the incoming nucleon "2" strikes the bound nucleon "1," causing nucleons "1" and "0" to recoil (carrying off the energy of "2") and leaving "2" in "0"'s position in the nucleus. Because of the Pauli principle which makes it impossible to distinguish between "0," "1," and "2," this type of process will interfere with that by which "0" strikes "1" and "0" and "1" recoil ("2" not being affected). This, we feel, will complicate the multiple-scattering analysis. However, the process described above can happen only because of the nonvanishing binding energy of the nucleus. When the kinetic energy of the colliding nucleon is large compared to that of a bound nucleon, this correction seems to be negligible (as will be described in more detail below).[13]

We may describe the conclusions of the last paragraph in another way. Exchange effects due to the Pauli principle for two (or more) particles are not important in general unless the wave functions of these particles overlap appreciably. Referring to the example above, we may then say that this process is not important unless a significant part of the *spectrum* of *kinetic energies* for particle "0" extends up to energies as high as that of the faster of the two nucleons. Instead of overlap of energies, we may think in terms of the overlap of wave functions in configuration space. Since particle "0" is just "any place in the nucleus," we can construct a wave packet for "2" which overlaps very little of "0's" wave function if the energy of "2" is high.

Our conclusion is thus that for energies sufficiently high that the impulse approximation is valid, the Pauli principle neither invalidates nor complicates the multiple-scattering description of high-energy nuclear reactions. This will be developed more precisely below, where we use the notation and techniques of references I and II.

II. FORMAL CONSEQUENCES OF THE PAULI PRINCIPLE

We shall use the notation of II. The nucleus is considered to have a complete set of eigenfunctions,

$$g_\nu(\xi_1,\xi_2,\cdots\xi_A),$$

of energy W_ν. Here ν is a set of variables used to denote the complete set of nuclear states ($\nu=0$ refers to the ground state) and the ξ's represent a complete set of nucleon variables for the A nucleons. The g's are supposed to be properly antisymmetrized as required by the Pauli principle. An incident nucleon in a plane wave state of momentum $\mathbf{p}$ and spin orientation ν has a wave function

$$\lambda_{\mathbf{p},\nu}(\xi_0).$$

The Hamiltonian describing the interaction of these $(A+1)$ nucleons is

$$H=K+V, \tag{3}$$

where K is the sum of the kinetic energies of the nucleons and

$$V=\sum_{\alpha>\beta}^{A} V_{\alpha\beta}. \tag{4}$$

$V_{\alpha\beta}$ is the interaction energy between the αth and βth nucleons.

Starting with the initial state,

$$g_0(\xi_1,\cdots,\xi_A)\lambda_{\mathbf{p},\nu}(\xi_0),$$

which describes nucleon "0" as incident on the nucleus in its lowest state, we suppose the Schrödinger equation to have a solution $\psi_a^{(0)}(\xi_0,\cdots,\xi_A)$:

$$H\psi_a^{(0)}=E\psi_a^{(0)}. \tag{5}$$

From $\psi_a^{(0)}$ we may construct an antisymmetrized[14] wave function:

$$\Psi_a=(A+1)^{-\frac{1}{2}}\Lambda_0\psi_a^{(0)}, \tag{6}$$

where

$$\Lambda_0=I-\sum_{\alpha=1}^{A} P_{\alpha 0}. \tag{7}$$

Here, $P_{\alpha 0}$ is the operator which interchanges ξ_0 and ξ_α and I is the identity permutation.

To evaluate the scattering cross section from Eq. (6), we write $\psi_a^{(0)}$ as

$$\psi_a^{(0)}=\psi_a^{(0)\mathrm{in}}+\psi_a^{(0)\mathrm{sc}}, \tag{8}$$

where the two terms contain, respectively, the incoming wave and outgoing scattered waves. The corresponding wave functions,

$$\psi_a^{(\alpha)}\equiv P_{\alpha 0}\psi_a^{(\alpha)}, \tag{9}$$

may be similarly decomposed. Letting

$$\mathbf{z}_0,\quad \mathbf{z}_1,\quad \cdots,\quad \mathbf{z}_A$$

be the coordinates of the nucleons and

$$S_0^\nu,\quad S_1^\nu,\quad \cdots,\quad S_A^\nu$$

[13] We shall henceforth use the term "target exchange" for this effect, which may also be described as a "non-orthogonality" correction.

[14] We recall that the g's are already antisymmetrized.

be their spin wave functions, we have

$$\lim_{z_0\to\infty}\psi_a{}^{(0)\mathrm{sc}}=-\sum_f\frac{\exp(i\mathbf{p}_f\cdot\mathbf{z}_0)}{z_0}\times S_0{}^{\nu f}g_f(\xi_1,\cdots,\xi_A)[(2\pi)^2MT_d],$$

$$\lim_{z_\alpha\to\infty}\psi_a{}^{(0)\mathrm{sc}}=-\sum_f\frac{\exp(i\mathbf{p}_f\cdot\mathbf{z}_\alpha)}{z_\alpha}\times S_\alpha{}^{\nu f}g_f(\xi_1,\cdots,\xi_A)[(2\pi)^2MT_e],\quad(10)$$

where M is the nucleon mass and T_d and T_e are the respective *direct* and *exchange* scattering amplitudes for scattering into a final momentum and spin state $(\mathbf{p}_f,\nu_f)$ with the nucleus in the state f.[15] Evidently,

$$(p^2/2M)+W_0=(p_f{}^2/2M)+W_f.\quad(11)$$

To calculate the cross section, we require the expectation value of the current operator ($\mathbf{p}_\rho$ is the momentum of the ρth nucleon):

$$\mathbf{J}(\mathbf{x})=\sum_{\rho=0}^{A}\mathrm{Re}\{\delta(\mathbf{x}-\mathbf{z}_\rho)(\mathbf{p}_\rho/M)\}\quad(12)$$

for large $|\mathbf{x}|$. This is

$$\langle\mathbf{J}\rangle_{\mathrm{sc}}=\frac{1}{A+1}(\Lambda_0\psi_a{}^{(0)\mathrm{sc}},\mathbf{J}(\mathbf{x})\Lambda_0\psi_a{}^{(0)\mathrm{sc}})=\frac{1}{A+1}\sum_{\rho=0}^{A}\left(\Lambda_0\psi_a{}^{(0)\mathrm{sc}},\delta(\mathbf{x}-\mathbf{z}_\rho)\frac{\mathbf{p}_\rho}{M}\Lambda_0\psi_a{}^{(0)\mathrm{sc}}\right).\quad(13)$$

Evaluation of Eq. (13) using Eqs. (10) is straightforward. We obtain

$$\langle J\rangle_{\mathrm{sc},f}=(1/x^2)(p_f/M)(2\pi)^4M^2\mathcal{S}|T_d-AT_e|^2,\quad(14)$$

where $\mathcal{S}$ is the appropriate sum and average over final and initial spin substates. The differential cross section is

$$\frac{d\sigma}{d\Omega}(\mathbf{p}_f,f)=(p_f/p)(2\pi)^4M^2\mathcal{S}|T_d-AT_e|^2.\quad(15)$$

For present purposes, it is important to note that one may calculate the cross section using any *one* term of the sum over ρ in Eq. (13), since the wave functions have been antisymmetrized. In particular, in the next section we shall calculate the flux of emitted "0" particles, multiplying by $(A+1)$ to obtain the cross section.

[15] Since the wave function $\psi_a{}^{(0)\mathrm{sc}}$ is not properly symmetrized with respect to ξ_0, it might be felt that one should include in the sum over f nuclear states $g_f(\xi_1,\cdots,\xi_0,\cdots,\xi_A)$ which violate the Pauli principle. The reason for omitting these is that they cannot contribute to the cross section, since transitions into these states are forbidden, of course.

III. DEVELOPMENT OF THE MULTIPLE-SCATTERING PROBLEM

It is desirable to depart in certain trivial respects from the notation of I and II in order to treat the nucleons all on a equivalent basis. Referring to Eqs. (3), (4), and (5), we define

$$V_\alpha\equiv\sum_{\beta\neq\alpha}V_{\beta\alpha}\quad(\alpha=0,1,\cdots,A),\quad(16)$$

and

$$a\equiv E+i\eta-H.\quad(17)$$

(Here η is the infinitesimal positive parameter which is employed in scattering theory to define the contour of integration about the pole in such quantities as a^{-1}). We also define

$$a(\alpha)\equiv a+V_\alpha.\quad(18)$$

Evidently V_α represents the interaction of the αth nucleon with the remainder of the nucleons. Also, $a(\alpha)$ is diagonal when operating on the wave function

$$|\alpha)\equiv g_\nu(\xi_1\cdots\xi_{\alpha-1}\xi_0\xi_{\alpha+1}\cdots\xi_A)\lambda_{p,\nu}(\xi_\alpha).\quad(19)$$

Thus $[a(\alpha)]^{-1}$ is the "propagation function" for α in a plane wave state, the other nucleons being in one of their mutual eigenstates.

When the incident nucleon is α, the wave equation (5) has the formal Chew-Goldberger solution

$$\Omega(\alpha)=1_\alpha+\frac{1}{a}V_\alpha|\alpha)\quad(\alpha=0,1,\cdots,A).\quad(20)$$

for the Møller wave matrix. By 1_α we mean the result of letting the identity operator act on the state (19) when the nuclear state is $\nu=0$ (i.e., the nucleus is in its ground state). By $V_\alpha|\alpha)$ we mean the result of applying V_α to the state (19), again with $\nu=0$.

Equation (20) satisfies the Lippman-Schwinger integral equation,

$$\Omega(\alpha)=1_\alpha+\frac{1}{a(\alpha)}V_\alpha\Omega(\alpha).\quad(20')$$

Since the initial state is not antisymmetrized with respect to the coordinates of the nucleons, neither is the wave function $\Omega(\alpha)$. Using Eqs. (6) and (7), we may readily construct such a solution. This is

$$\bar{\Omega}\equiv\Omega(0)-\sum_{\alpha=1}^{A}\Omega(\alpha)=\Lambda_0\Omega(0)=\bar{I}+\frac{1}{a}\Lambda_0V_0|0)\equiv\bar{I}+\frac{1}{a}\bar{V}.\quad(21)$$

Here

$$\bar{I}\equiv\Lambda_0 I_0=1_0-\sum_{\alpha=1}^{A}1_\alpha, \tag{22}$$

and

$$\bar{V}\equiv V_0|0)-\sum_{\alpha=1}^{A}V_\alpha|\alpha). \tag{23}$$

In obtaining the final form of Eq. (21) we have used Eq. (20) and the fact that Λ_0 commutes with a^{-1}.

Equation (21) is the final formal solution to the scattering problem. It remains to show how to calculate the cross section from it. By the conclusions of Sec. II, it is necessary to calculate the outgoing flux of only one nucleon, say nucleon "0," and properly normalize it. This we proceed to do. By using the Chew-Goldberger relation,

$$\frac{1}{a}=\frac{1}{a(0)}\left[1+V_0\frac{1}{a}\right],$$

Equation (21) can be written as

$$\bar{\Omega}=\bar{I}+\frac{1}{a(0)}\left[1+V_0\frac{1}{a}\right]\bar{V}. \tag{24}$$

Since $[a(0)]^{-1}$ describes the propagation of nucleon "0" as a free particle, the quantity

$$\left[1+V_0\frac{1}{a}\right]\bar{V}$$

is essentially the amplitude for emission of the "0" nucleon. By Eq. (21), we can write $a^{-1}\bar{V}=\bar{\Omega}-\bar{I}$ and convert Eq. (24) to an integral equation:

$$\bar{\Omega}=\bar{I}+\frac{1}{a(0)}[\bar{V}-V_0\bar{I}]+\frac{1}{a(0)}V_0\bar{\Omega}. \tag{25}$$

Except for the rather complex "source term,"

$$\bar{I}+\frac{1}{a(0)}[\bar{V}-V_0\bar{I}], \tag{26}$$

this is just the integral equation (20′) for "0" as the incident particle. It is easily seen that (26) satisfies the Schrödinger equation

$$a(0)\left[\bar{I}+\frac{1}{a(0)}(\bar{V}-V_0\bar{I})\right]=0,$$

just as does 1_0. Our argument will consist of showing that for high-energy collisions the expression (26) may essentially be replaced by 1_0 so that our problem is reduced to one for which nucleon "0" is treated as distinguishable. (We suspect that there may be more general conditions for this approximation to be valid than that the incident particle be fast, but we have not found them.) Returning to Eq. (25), the quantity $V_0\bar{I}$ is

$$\begin{aligned}V_0\bar{I}&=V_0|0)-\sum_{\alpha=1}^{A}V_0|\alpha)\\&=\sum_{\beta\neq0}V_{\beta0}|0)-\sum_{\beta\neq0}\sum_{\alpha\neq0}V_{\beta0}|\alpha),\end{aligned}$$

where $V_0|\alpha)$ is the result of letting V_0 operate on the state $|\alpha)$ of Eq. (19) with $\nu=0$, etc. The difference is

$$\bar{V}-V_0\bar{I}=\sum_{\beta\neq0}\sum_{\alpha\neq0,\beta}[V_{\alpha0}|\beta)-V_{\alpha\beta}|\beta)]. \tag{27}$$

When substituted into Eq. (25), the expression

$$\frac{1}{a(0)}[\bar{V}-V_0\bar{I}]$$

describes processes in which the fast particle "β" is incident on the nucleus and the fast particle "0" is emitted after a single scattering. From the structure of Eq. (27), it is seen that these processes are all of the "target exchange" type which we agreed in the introduction to neglect. (The magnitude of these terms is estimated in the next section.) Dropping this term then, Eq. (25) becomes

$$\bar{\Omega}=\bar{I}+\frac{1}{a(0)}V_0\bar{\Omega}. \tag{28}$$

Except for the antisymmetrized identity operator $\bar{I}$, this is just the integral equation for scattering if nucleon "0" were considered to be *distinguishable* from the other nucleons.

Let us introduce explicitly the variables $\xi_0, \xi_1, \cdots, \xi_A$ into Eq. (28) for the final state, leaving the initial state unspecified. Then, because of the antisymmetry of $\bar{\Omega}$ with respect to an exchange of two ξ's

$$\begin{aligned}&\sum_{\beta,\xi_0',\xi_\beta'}(\xi_0\xi_\beta|V_{\beta0}|\xi_0'\xi_\beta')(\xi_0'\xi_1\cdots\xi_\beta'\cdots\xi_A|\bar{\Omega}\\&\quad=\sum_{\beta,\xi_0',\xi_\beta'}\tfrac{1}{2}[(\xi_0\xi_\beta|V_{\beta0}|\xi_0'\xi_\beta')-(\xi_0\xi_\beta|V_{\beta0}|\xi_\beta'\xi_0')]\\&\qquad\times(\xi_0'\xi_1\cdots\xi_\beta'\cdots\xi_A|\bar{\Omega}.\end{aligned} \tag{29}$$

We can write

$$\bar{V}_{\beta0}=(\xi_0\xi_\beta|V_{\beta0}|\xi_0'\xi_\beta')-(\xi_0\xi_\beta|V_{\beta0}|\xi_\beta'\xi_0')$$

and set

$$V_0\bar{\Omega}=\bar{V}_0\bar{\Omega}, \tag{30}$$

if we restrict the integration in the matrix product to $\xi_0'>\xi_\beta'$.[16]

From the expression (30), we note that the individual two-body scatterings are treated properly,[12] as required

[16] We suppose that we have established an ordering relation for the ξ's so that this inequality has meaning.

by the Pauli principle. That is,

$$\bar{\Omega}=\bar{I}+\frac{1}{a(0)}\bar{V}_0\bar{\Omega}. \tag{28'}$$

Our last step is to construct the submatrix of $\bar{\Omega}$ which refers to "0" as the *fast* outgoing particle.[17] For this purpose let us suppose that the set of quantum numbers, ξ_α, contains in some manner a description of the (average) kinetic energy of the nucleon. Let us also suppose that our ordering relation[16] is such that $\xi_\alpha>\xi_\beta$ if "α" has a higher (average) kinetic energy than "β." We then define the *submatrix* of $\bar{\Omega}$ which describes "0" as the fast outgoing particle in which we are interested to be $\bar{\Omega}(0)$. It is then $\bar{\Omega}(0)$ which we require to calculate the flux of outgoing nucleon "0." We shall write

$$\bar{\Omega}(0)\equiv\bar{\Omega}|_{\xi_0\gg\xi_\alpha}$$

(for most "α"), implying by the symbol on the right that "0" has a higher kinetic energy than *most* of the other nucleons and is also *the particle* which we are observing (distinguished by virtue of its momentum and direction, etc.). Then

$$\bar{\Omega}\equiv\bar{\Omega}(0)+\bar{\Omega}', \tag{31}$$

where Ω' is that submatrix of Ω for which "0" is *not* the particle of interest to us. Using the above notation, we have from Eq. (28′):

$$\bar{\Omega}(0)=\bar{I}|_{\xi_0\gg\xi_\alpha}+\frac{1}{a(0)}\bar{V}_0\bar{\Omega}|_{\xi_0\gg\xi_\alpha}. \tag{32}$$

Obviously,

$$\bar{I}|_{\xi_0\gg\xi_\alpha}=I_0 \tag{22'}$$

[see Eq. (22)], since only one particle is fast in the initial state.

Using Eq. (31), we have also

$$V_0\bar{\Omega}|_{\xi_0\gg\xi_\alpha}=\sum_{\beta,\xi_0',\xi_\beta'}(\xi_0\xi_\beta|\bar{V}_{\beta0}|\xi_0'\xi_\beta')$$
$$\times(\xi_0'\xi_1\cdots\xi_\beta'\cdots\xi_A|\bar{\Omega}(0)+\textstyle\sum_\beta\bar{V}_{\beta0}\bar{\Omega}'|_{\xi_0\gg\xi_\alpha}. \tag{33}$$

The last term refers to processes for which "0" becomes the particle of interest to us after the last scattering (represented by $\bar{V}_0$). We have already $\xi_0'>\xi_\beta'$ so this last kind of term is of the "target exchange" type. That is, these refer to processes for which "0" and "β" are both slow until a last scattering which gives them both (or at least "0") a large recoil energy, while other fast particles are "slowed down" without interaction with these (in order that energy be conserved). This process we have agreed to neglect (this will be discussed further in the next section).[18]

Combining our conclusions from Eq. (33) with Eq. (22′), we can write Eq. (32) as

$$\bar{\Omega}(0)=1_0+\frac{1}{a(0)}\bar{V}_0\bar{\Omega}(0). \tag{34}$$

This is an integral equation involving *only* the submatrix $\bar{\Omega}(0)$. Thus we can calculate $\bar{\Omega}(0)$ without knowing the complete matrix $\bar{\Omega}$. Except for the occurrence of the antisymmetrized two-body scattering interaction, $\bar{V}_0$, this is the same equation which we would have to solve if "0" were distinguishable from other nucleons.

In other words, Eq. (34) states that we may use the multiple-scattering analysis of references I and II *without* modification (other than in the use of antisymmetrized two-body amplitudes). This means that $\bar{\Omega}(0)$ is given by the equations of multiple scattering,

$$\bar{\Omega}(0)=1_0+\frac{1}{a(0)}\sum_{\alpha\neq0}t_\alpha\bar{\Omega}(0)_\alpha,$$

$$\bar{\Omega}(0)\alpha=1_0+\frac{1}{a(0)}\sum_{\beta\neq\alpha,0}t_\beta\bar{\Omega}(0)_\beta, \tag{35}$$

with

$$t_\alpha=\bar{V}_{\alpha0}+\bar{V}_{\alpha0}\frac{1}{a(0)}t_\alpha. \tag{36}$$

On making the impulse approximation, t_α is just the properly antisymmetrized two-body scattering amplitude for nucleons "0" and "α." The development of the *optical model* or study of the inelastic scattering proceeds just as in I and II.

IV. ESTIMATED SIZE OF "TARGET EXCHANGE" TERMS

From Eqs. (27) and (33), we see that a "target exchange" term of typical magnitude can be described as follows:

Nucleon "2" is initially fast and has a momentum $\mathbf{p}_2$. Nucleons "0" and "1" are initially bound. "0" and "1" scatter and are ejected, "0" being fast. To conserve energy (see the introduction, in which it was assumed that the mean free paths are sufficiently long that energy is at least roughly conserved between scatterings), particle "2" must be very nearly stopped.

Let the wave function for the "stopped" "2" be $g_2(\mathbf{z}_2)$. Let "0" and "1" be initially in the state $g_0(\bar{\mathbf{z}},\mathbf{r})$,

[17] We do not of course mean to imply that "0" is necessarily the only fast—or even the fastest—nucleon emitted. "Target exchange" involving the exchange of "0" with a nucleon which has suffered a previous scattering is expected to be especially small, since in this case *both* nucleons can be rather well localized in space. Thus the presence of fast nucleons other than "0" will not be troublesome.

[18] There are also processes for which "0" is *fast*, but not the nucleon of interest to us. Such exchange processes, as discussed in footnote 17, are expected to be even less important than those for which "0" is slow.

where

$$\mathbf{r}=\mathbf{z}_0-\mathbf{z}_1,$$
$$\bar{\mathbf{z}}=\tfrac{1}{2}(\mathbf{z}_0+\mathbf{z}_1). \tag{37}$$

Here $\mathbf{z}_0$ and $\mathbf{z}_1$ are the space coordinates of "0" and "1."[19] Also let the final momenta of these nucleons be $\mathbf{p}_0$ and $\mathbf{p}_1$, respectively. Define

$$\mathbf{K}=\mathbf{p}_0+\mathbf{p}_1,$$
$$\mathbf{p}=\tfrac{1}{2}(\mathbf{p}_0-\mathbf{p}_1).$$

Then the amplitude for this process is roughly

$$I_x=(2\pi)^{-9/2}\int \exp(-i\mathbf{K}\cdot\bar{\mathbf{z}})e^{-i\mathbf{p}\cdot\mathbf{r}}g_2^*(\mathbf{z}_2)$$
$$\times V(\mathbf{r})\exp(i\mathbf{p}_2\cdot\mathbf{z}_2)g_0(\bar{\mathbf{z}},\mathbf{r})d^3\bar{z}d^3rd^3z_2. \tag{38}$$

$V(\mathbf{r})$ is the interaction potential between nucleons "0" and "1."

Since $g_0(\bar{\mathbf{z}},\mathbf{r})$ is expected to vary rather slowly with $\bar{\mathbf{z}}$ and $\mathbf{r}$ (in comparison with the exponentials), we take it outside the integral and set

$$g_0(\bar{\mathbf{z}},\mathbf{r})\simeq 1/V_A,$$

where $V_A(=4\pi R^3/3)$ is the nuclear volume. Let

$$V_T(\mathbf{p})=(2\pi)^{-3}\int d^3re^{-i\mathbf{p}\cdot\mathbf{r}}V(\mathbf{r}), \tag{39}$$

and

$$g_T(\mathbf{p}_2)=(2\pi)^{-\frac{3}{2}}\int \exp(i\mathbf{p}_2\cdot\mathbf{z}_2)g_2^*(\mathbf{z}_2)d^3z_2$$
$$\simeq[(2\pi)^3V_A]^{-\frac{1}{2}}\int_{V_A}\exp(i\mathbf{p}_2\cdot\mathbf{z}_2)d^3z_2$$
$$\simeq(4\pi/3)[(2\pi)^3V_A]^{-\frac{1}{2}}[p_2]^{-3}. \tag{40}$$

Here we have made the reasonable assumption that $g_2(\mathbf{z}_2)$ extends over a region having the size of the nuclear volume.[20] We then have

$$I_x\simeq(2\pi)^{\frac{3}{2}}\delta(\mathbf{K})V_T(\mathbf{p})(4\pi/3)[V_A^{\frac{1}{2}}p_2]^{-3}. \tag{41}$$

We must compare this with the direct process by which "0" strikes "1" and recoils. If "1" is initially bound in the state $g_1(\mathbf{z}_1)$ and if the initial momentum of "0" is $\mathbf{p}_0$, we have for the amplitude of the direct process

$$I_d\simeq(2\pi)^{-9/2}\int \exp(-i\mathbf{K}\cdot\bar{\mathbf{z}})e^{-i\mathbf{p}\cdot\mathbf{r}}V(\mathbf{r})$$
$$\times g_1(\mathbf{z}_1)\exp(i\mathbf{p}_0\cdot\mathbf{z}_0)d^3\bar{z}d^3r. \tag{42}$$

Here $\mathbf{K}$ is the total recoil momentum of "0" and "1" and $\mathbf{p}$ is their relative momentum; $\bar{\mathbf{z}}$ and $\mathbf{r}$ are respectively their center-of-mass and relative coordinates. Again taking $g_1(\mathbf{z}_1)$ out of the integral and setting it equal to

$$g_1(\mathbf{z}_1)\simeq[V_A]^{-\frac{1}{2}},$$

we find

$$I_d\simeq(2\pi)^{-\frac{3}{2}}\delta(\mathbf{K}-\mathbf{p}_0)V_T(\mathbf{p}-\tfrac{1}{2}\mathbf{p}_0)V_A^{-\frac{1}{2}}. \tag{43}$$

Aside from the δ-functions and the factor of V_T, the ratio of Eqs. (41) and (43) is

$$I_x/I_d\simeq(4\pi/3)V_A^{-1}p_2^{-3}=[p_2R]^{-3}, \tag{44}$$

where R is the nuclear radius. This evidently decreases rapidly with increasing energy for the incoming particle. For a 100-Mev incoming nucleon, this ratio is

$$I_x/I_d\simeq 0.03/A, \tag{45}$$

where A is the mass number of the nucleus. It is clear that Eq. (45) predicts a very small contribution resulting from "target exchange" processes. (The other type of "target exchange" occurring in Eq. (27) has a comparable magnitude.) The factor of A^{-1} in Eq. (45) is removed when we sum over the $(A-1)$ nucleons, "2," "3," $\cdots$ "A," as incident particles. The correction for "target exchange" is, however, small at $E=100$ Mev, as predicted by Eq. (45), and decreases rapidly with increasing energy.[21]

V. ELASTIC SCATTERING OF FAST NUCLEONS AND THE OPTICAL POTENTIAL

In Sec. III we have shown that the problem of the scattering of a high-energy nucleon by a target nucleon reduces essentially to solving Eq. (34), in which the incident nucleon can be "distinguished" from the others. This equation can be applied both to the elastic and the inelastic scatterings. When it is applied to the latter problem we have a transport equation which describes how the incident energy will be transferred to nucleons in the target nucleus by successive collisions between pairs of nucleons. On the other hand, the first problem can be reduced to the scattering of a nucleon in the optical potential where the form of the optical potential can be connected with more fundamental quantities in the two-body problem.[2]

Here, as an example of the application of our method, we shall construct the optical potential for the elastic scattering of a nucleon according to II.

If one assumes the invariance of the scattering amplitude for a collision between two particles under space rotation, reflection, and time reversal, its most general form has been given by Wolfenstein and Ashkin.[22] When two particles are both nucleons, the scattering amplitude is antisymmetric for an interchange of their coordinates before the scattering and for an interchange of those after the scattering. Furthermore, the scattering amplitude is assumed to have the charge invariance

[19] For simplicity we neglect the other nucleons and all spin variables.

[20] "2" is certainly not expected to be localized in a region much less than the size of the nuclear volume at the termination of the scattering event.

[21] Our estimate depends really on the assumption that the momentum of the incident particle is higher than the bulk of the momentum spectra of bound nucleons.

[22] L. Wolfenstein and J. Ashkin, Phys. Rev. 85, 947 (1952).

property, except for a small correction due to Coulomb forces.

Under these additional conditions the scattering amplitude in the center-of-mass system for a collision in which a nucleon "1" with a momentum $\mathbf{q}$ and a nucleon "2" with a momentum $(-\mathbf{q})$ scatter into states with momenta $\mathbf{q}'$ and $(-\mathbf{q}')$, respectively, is given by the following expression:

$$\begin{aligned}(\mathbf{q}', -\mathbf{q}'|t_{12}|\mathbf{q}, -\mathbf{q}) &= \frac{3+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{4}\bigg\{A_{3,3}\frac{3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{4}+A_{3,1}\frac{1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{4} \\ &+B_3\frac{[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot(\mathbf{q}\times\mathbf{q}')]}{qq'}+C_3\left(\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{Q})(\boldsymbol{\sigma}_2\cdot\mathbf{Q})}{Q^2}-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right) \\ &+D_3\left(\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{P})(\boldsymbol{\sigma}_2\cdot\mathbf{P})}{P^2}-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right)\bigg\} \\ &+\frac{1-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{4}\bigg\{A_{1,3}\frac{3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{4}+A_{1,1}\frac{1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{4} \\ &+B_1\frac{[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot(\mathbf{q}\times\mathbf{q}')]}{qq'}+C_1\left(\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{Q})(\boldsymbol{\sigma}_2\cdot\mathbf{Q})}{Q^2}-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right) \\ &+D_1\left(\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{P})(\boldsymbol{\sigma}_2\cdot\mathbf{P})}{P^2}-\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right)\bigg\}, \quad (46)\end{aligned}$$

where

$$\begin{aligned}\mathbf{Q}&=\mathbf{q}'-\mathbf{q}, \\ \mathbf{P}&=(\mathbf{q}'\times\mathbf{q})\times\mathbf{Q}.\end{aligned} \quad (47)$$

A, B, C, and D are complex functions of $|q|$ and $\cos\theta$ where θ is the scattered angle in the center-of-mass system. Terms containing a factor $(3+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)/4$ or a factor $(1-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)/4$ correspond to the scattering in the state with the isotopic spin $T=1$ or $T=0$, respectively. $A_{3,1}$ and $A_{1,1}$ are the singlet state scattering amplitudes and the remaining terms are those for the triplet state. In principle, A, B, C, and D are obtainable from experimental results for p-p and p-n scattering.[23]

We have already shown that the scattering of fast nucleons by atomic nuclei can be treated according to the method of references I and II, with slight modifications due to the Pauli principle. The "optical potential" v_c for $\bar{\Omega}(0)$ (the part of the wave function with the nucleon "0" as the *fast* nucleon) can be obtained by averaging $\sum_{\alpha\neq 0}t_{0\alpha}$ over the wave function $g_0(\xi_1, \cdots, \xi_A)$ of the nucleon. $t_{0\alpha}$ is the scattering amplitude given by Eq. (46) when it is transformed to the laboratory system.

If we take, for simplicity, a self-conjugate $(N=Z)$ nucleus with the total angular momentum $J=0$, the matrix element of the optical potential v_c for an elastic scattering of the nucleon "0" from a state of relative momentum $\mathbf{p}$ to a state $\mathbf{p}'$ is given by

$$\begin{aligned}(\mathbf{p}'|v_c|\mathbf{p}) &= \int d^3z\rho(\mathbf{z})\exp[i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{z}] \\ &\times\bigg\{\frac{9A_{3,3}+3A_{3,1}+3A_{1,3}+A_{1,1}}{16} \\ &+\frac{3B_3+B_1}{4}\frac{2[\boldsymbol{\sigma}_0\cdot(\mathbf{p}\times\mathbf{p}')]}{p^2}\bigg\}. \quad (48)\end{aligned}$$

Terms of $t_{0,\alpha}$ linear in $\boldsymbol{\tau}_\alpha$ and $\boldsymbol{\sigma}_\alpha$ average to zero because $Z=N$ and $J=0$. $\rho(\mathbf{z})$ is the density of nucleons in the nucleus. The term containing $\boldsymbol{\sigma}_0$ shows the presence of spin-orbit coupling in the optical potential.[24]

There are, of course, corrections to the optical potential because of successive inelastic scatterings which bring the nucleus into the initial state after the last inelastic scattering.[25] The magnitude of this correction depends on the strength of correlations among nucleons in the nucleus. For high-energy nuclear scattering this can be shown to be small. We should like to emphasize that a solution of the scattering equation for the potential thus obtained is in general quite different from and has a wider validity than solutions obtained in the ordinary Born approximation.[26]

If we limit our attention to the forward scattering, we obtain Eq. (3) of the reference 3 from Eq. (48) under the assumption that $A_{3,3}$, $A_{3,1}$, etc., can be considered to be constant in this small region of the scattering angle. When the scattering angle becomes large, the corresponding matrix element of v_c becomes rapidly small because the density $\rho(\mathbf{z})$ has only a small amount of high-momentum components comparable with $|\mathbf{p}'-\mathbf{p}|$. This indicates that the scattering is mostly inelastic for large angles.

In conclusion, the impulse and multiple-scattering approximations hold for scatterings of fast nucleons by nuclei even if we take into account the Pauli principle. Through these approximations, amplitudes for scattering fast nucleons by nuclei can be expressed in terms of those for p-p and p-n scattering together with the knowledge of nucleon density and correlation functions.

We should finally emphasize that our conclusions apply to a much wider class of multiple-scattering problems than just those involving atomic nuclei. A simple example is, for instance, provided by the double (or triple) scattering of a beam of protons by successive hydrogen targets.

[23] The D term in Eq. (9) of reference 22 vanishes because of the Pauli principle for two nucleons. Our A, B, C, and D are linear combinations of their A, B, C, E, and F.

[24] If a nucleus has J different from 0, there are other types of terms which will flip the spin of the incoming particle by changing the spin direction of the nucleus at the same time. This part of the optical potential might be sensitive to the value of J. The magnitude of this correction relative to the remainder of the potential is of the order of J/A.

[25] See the reference II, Eq. (75).

[26] S. Tamor, Phys. Rev. 93, 227 (1954).