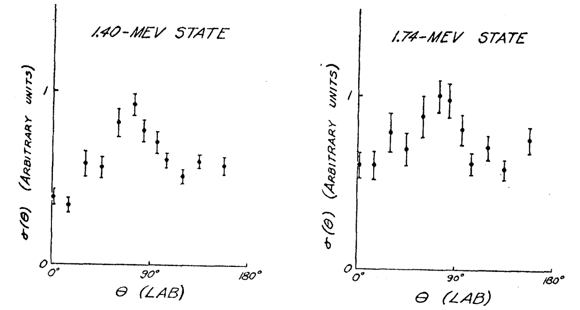

FIG. 2. Angular distribution of protons from the (d,p) reaction leading to excited states in Ti^{49} at 1.40 Mev and 1.74 Mev. The errors indicated are statistical standard deviations.

The angular distributions of Groups 1 and 2 are shown in Fig. 2. The statistical standard deviation in cross section values is of the order of 10 percent; and the angular resolution is of the order of 3°. Center-of-mass corrections are small compared with these values and have been neglected. It is apparent that the statistical theory of compound nucleus formation,[2] which is characterized by a symmetry of the angular distribution about 90°, is not applicable to these reactions. The stripping process, however, is known[7,8] to give rise to an angular distribution which is of the same general character as the data in Fig. 2. Although it is not possible to fit the data with the Butler theory,[7] the fact that the deuteron energy in this experiment is well below the Coulomb barrier of the target implies that Coulomb effects not accounted for in the Butler theory may be particularly important in this case. Computations have recently been carried out by Tobocman and Kalos,[8] which take into account Coulomb effects as well as proton absorption. These computations, for a particular choice of the proton scattering amplitudes β_l, result in an angular distribution which does indeed reproduce the general behavior of the experimental points.

From the known values of source thickness, integrated beam current, and solid angle observed, it is possible to determine the absolute values of the reaction cross sections. The total cross sections are found to be 5.7×10^{-28} cm^2 for the transition to the 1.40-Mev state of Ti^{49} and 3.5×10^{-28} cm^2 for the transition to the 1.74-Mev state. These values are believed to be accurate to within a factor of 2. The cross sections for the transitions to the ground state and to the 2.45-Mev state of Ti^{49} are estimated to be of the order of 0.5×10^{-28} cm^2 and 2×10^{-28} cm^2, respectively.

We are indebted to Dr. W. Tobocman and Dr. M. H. Kalos for making their calculations available to us prior to publication.

[7] S. T. Butler, Proc. Soc. (London) **A208**, 559 (1951).
[8] W. Tobocman and M. H. Kalos, following paper, Phys. Rev. **97**, 132 (1955).

PHYSICAL REVIEW VOLUME 97, NUMBER 1 JANUARY 1, 1955

Numerical Calculation of (d,p) Angular Distributions*

W. TOBOCMAN† AND M. H. KALOS
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received September 20, 1954)

The (d,p) cross section has been calculated by taking into account the Coulomb interaction, the interaction of the liberated protons with the residual nucleus, and the scattering of the incident deuteron beam. These effects were found to be important in all the cases considered.

I. INTRODUCTION

WHEN a thin film of matter is bombarded with a monoenergetic beam of deuterons, one observes among the reaction products energy groups of protons. Each energy group can be identified with a level of the residual nucleus. By measuring the Q of such a reaction one can determine the energies of the associated levels of the residual nucleus. It has been shown in recent years that the shapes of the angular distributions of the proton groups can be used to fix the parities and learn something about the spins of the levels of the residual nucleus.[1] The intensities of the proton groups have been shown to be proportional to the reduced widths of these levels.[2] Thus the (d,p) reaction is a powerful tool for nuclear spectroscopy.

Previous theoretical treatments of the (d,p) reaction have been based on rather sweeping approximations. Nevertheless, these theories, in particular that due to S. T. Butler, have been able to describe the main aspects

* Assisted by the joint program of the U. S. Office of Naval Research and the U. S. Atomic Energy Commission.
† Now at the Princeton Institute for Advanced Study, Princeton, New Jersey.

[1] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951); A. B. Bhatia *et al.*, Phil. Mag. **43**, 485 (1952); R. Huby, Nature **166**, 552 (1950).
[2] R. Huby, Proc. Roy. Soc. (London) **A215**, 385 (1952); S. Yoshida, Progr. Theoret. Phys. Japan **10**, 1, 370 (1953); F. L. Friedman and W. Tobocman, Phys. Rev. **92**, 93 (1953); Fujimato, Kikuchi, and Yoshida, Progr. Theoret. Phys. Japan **11**, 3, 264 (1954).

of the (d,p) angular distributions, and the spins and parities determined in this way have agreed for the most part with the results of other methods. But the Butler theory has not been uniformly successful in matching angular distributions, and the reduced widths measured by means of the Butler theory have differed from those determined by other methods by large factors.

In an article by one of the authors[3] an approximate expression for the (d,p) cross section was derived which was suitable for numerical evaluation and which included effects neglected in previous treatments. The effects referred to are those caused by the Coulomb interaction, the specifically nuclear interaction between the liberated proton and the residual nucleus, and the scattering of the incident deuteron beam.

This expression for the (d,p) cross section has the following form:[4]

$$\sigma_L(\theta_P)=\gamma_L\frac{(2J+1)\hbar^3\epsilon^{\frac{1}{2}}(1+M_N/M_I)(1+M_D/M_I)^{-1}(1+M_P/M_F)^{-1}}{(2I+1)R|E_N|E_D^{\frac{1}{2}}E_P^{\frac{1}{2}}M_NM_D^{\frac{1}{2}}}\sum_{m=-L}^{L}|B^{Lm}|^2, \tag{1}$$

$$B^{Lm}=\sum_{l=0}^{\infty}P_l^{|m|}(\cos\theta_P)\sum_{\lambda}f_{l\lambda}{}^L\Gamma_{l\lambda}{}^{Lm}\exp\{i[\sigma_l(\eta_P)+\sigma_\lambda(\eta_D)+(\pi/2)(\lambda-l)]\}, \tag{2}$$

$$\Gamma_{l\lambda}{}^{Lm}=\frac{(2\lambda+1)(2l+1)}{(2L+1)}\left[\frac{(l-|m|)!}{(l+|m|)!}\right]^{\frac{1}{2}}(l\lambda00|l\lambda L0)(l\lambda m0|l\lambda Lm), \tag{3}$$

$$f_{l\lambda}{}^L=\frac{|K_N|}{h_L^{(1)}(i|K_N|R)}\int_{R(1+M_N/M_I)}^{\infty}drh_L^{(1)}(i|K_N|R)[F_\lambda(\eta_D,K_Dr)-\alpha_\lambda H_\lambda(\eta_D,K_Dr)]$$
$$\times\left[F_l\left(\eta_P,\frac{K_Pr}{1+M_N/M_I}\right)-\beta_lH_l\left(\eta_P,\frac{K_Pr}{1+M_N/M_I}\right)\right], \tag{4}$$

where θ_P = angle of the liberated proton with respect to the incident deuteron,
E_D = energy of the incident deuteron (center-of-mass),
E_P = energy of the liberated proton (center-of-mass),
ϵ = binding energy of the deuteron,
$E_N=E_D-E_P-\epsilon=-Q-\epsilon$,
R = radius of the target nucleus,
γ_L = reduced level width of the residual nucleus,
J = spin of the residual nucleus,
I = spin of the target nucleus,
M_P = proton mass,
M_N = neutron mass,
M_D = mass of the deuteron,
M_I = mass of the target nucleus,
M_F = mass of the residual nucleus,
L = orbital angular momentum of the captured neutron,
$(l\lambda m0|l\lambda Lm)$ = the Clebsch-Gordan coefficient,
$F_l(\eta,\rho)$ = regular radial Coulomb wave function,
$G_l(\eta,\rho)$ = irregular radial Coulomb wave function,
$H_l=F_l-iG_l$,
$\sigma_l(\eta)=\arg\Gamma(1+l+i\eta)$,
Z = atomic number of target nucleus,

$$\eta_D=\frac{Z}{137.04}\left[\frac{M_Dc^2}{2E_D}\left(1+\frac{M_D}{M_I}\right)^{-1}\right]^{\frac{1}{2}},$$

$$\eta_P=\frac{Z}{137.04}\left[\frac{M_Pc^2}{2E_P}\left(1+\frac{M_P}{M_F}\right)^{-1}\right]^{\frac{1}{2}},$$

$$K_D^2=2E_D\hbar^{-2}M_D(1+M_D/M_I)^{-1},$$
$$K_P^2=2E_P\hbar^{-2}M_P(1+M_P/M_F)^{-1},$$
$$|K_N|^2=2|E_N|M_N\hbar^{-2}(1+M_N/M_I)^{-1}.$$

The α_λ and β_l appearing in the expression for $f_{l\lambda}{}^L$ are the partial wave scattering amplitudes appropriate

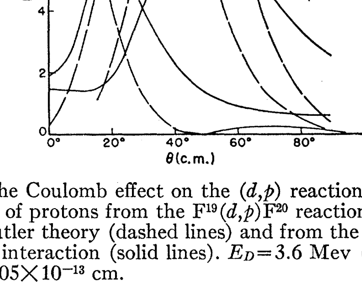

FIG. 1. The Coulomb effect on the (d,p) reaction. The angular distribution of protons from the $F^{19}(d,p)F^{20}$ reaction is calculated from the Butler theory (dashed lines) and from the Butler theory +Coulomb interaction (solid lines). $E_D=3.6$ Mev (lab), $Q=4.37$ Mev, $R=5.05\times10^{-13}$ cm.

[3] W. Tobocman, Phys. Rev. **94**, 1655 (1954).

[4] In reference 3 the σ_L and σ_λ appear in Eqs. (26), (30), and (31) with the wrong sign.

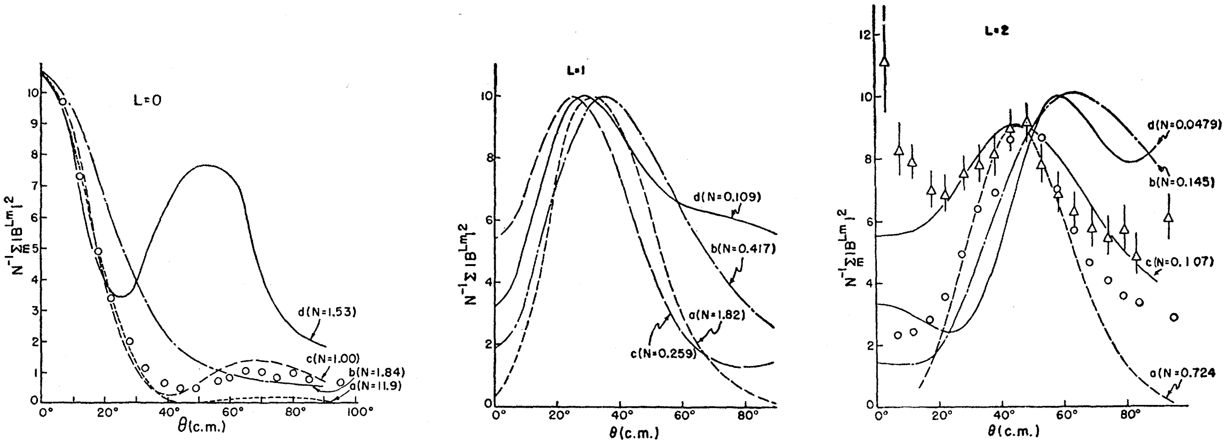

FIG. 2. The Coulomb and nuclear effects on the (d,p) reaction, for $L=0$, 1, 2. The angular distribution of protons from the $F^{19}(d,p)F^{20}$ reaction calculated from (*a*) Butler theory, (*b*) Butler+Coulomb, (*c*) Butler+Coulomb+absorption of protons with $l\leqslant 2$, (*d*) Butler +Coulomb+hard sphere scattering of protons. $E_D=3.6$ Mev (lab), $Q=4.37$ Mev, $R=5.05\times 10^{-13}$ cm. Experimental points shown are due to Bromley *et al.* (see reference 5). The Q values corresponding to these points are $L=2$, triangles, $Q=4.37$ Mev; $L=2$, circles, $Q=3.72$ Mev; $L=0$, $Q=0.88$ Mev.

to the interaction of deuterons with the target nucleus and the interaction of protons with the residual nucleus respectively.

If one assumes that $0=Z=\alpha_\lambda=\beta_l$, then Eq. (2) becomes

$$B^{Lm}=\frac{-\delta_{m0}i^L R^2 K_P K_D(1+M_N/M_I)}{|K_N|(1+|K_N|^{-2}\kappa^2)}\left[\frac{d}{dr}j_L(\kappa r) - j_L(\kappa r)\frac{d}{dr}\ln h_L^{(1)}(i|K_N|r)\right]_{r=R(1+M_N/M_I)}, \quad (5)$$

where $\boldsymbol{\kappa}=\mathbf{K}_D-(1+M_N/M_I)^{-1}\mathbf{K}_P$, $j_L(\rho)=$ the spherical Bessel function, and $h_L^{(1)}(\rho)=$ the spherical Hankel function of the first kind. This expression, introduced into Eq. (1), gives the Butler (d,p) cross section.

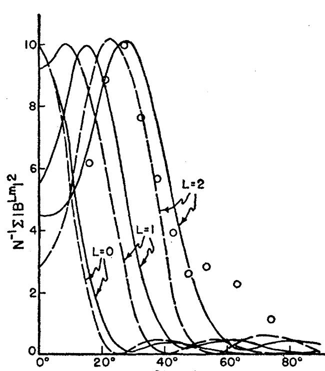

FIG. 3. The Coulomb effect on the (d,p) reaction. The angular distribution of protons from the $F^{19}(d,p)F^{20}$ reaction is calculated from the Butler theory (dashed lines) and from the Butler theory +Coulomb interaction (solid lines). $E_D=14.3$ Mev (lab), $Q=4.37$ Mev, $R=5.05\times 10^{-13}$ cm. The experimental points are due to Black. (See reference 6.)

II. CALCULATION

We have used an I.B.M. card-programmed computer to evaluate the quantity $\sum_m|B^{Lm}|^2$ appearing in Eq. (1). This quantity is dimensionless and contains the angular dependence of the (d,p) cross section $\sigma_L(\theta_P)$. The computation was a direct numerical evaluation and involved no approximations. The results of our computations are displayed in Figs. 1–7 together with some experimental points.

A factor N is introduced to provide convenient normalization. The experimental points appearing on Fig. 2 were taken from the work of Bromley *et al.*[5] The Q-value 4.373 Mev is appropriate only to the points represented by the triangles which correspond to stripping due to the ground state of F^{20}. The two angular distributions represented by circles are due to two excited states of F^{20}. Since our expression for the (d,p) cross section is not very sensitive to variations in the value of Q, we thought it would be of interest to display the latter points. The points shown on Figs. 3 and 4 are due to Black,[6] and the points displayed on Fig. 5 are the results of work by Pratt.[7]

The computations involved various choices for the parameters Z, α_λ, and β_l. In the accompanying figures we identify the choice in the following manner.

(1) $Z=0$, $\alpha_\lambda=0$, $\beta_l=0$: "Butler theory."

(2) $Z=Z$ of target nucleus, $\alpha_\lambda=0$, $\beta_l=0$: "Butler +Coulomb."

(3) $Z=Z$ of target nucleus, $\alpha_\lambda=0$, $\beta_l=\frac{1}{2}$ for $l\leqslant l_1$, and $\beta_l=0$ for $l>l_1$: "Butler+Coulomb+absorption of protons with $l\leqslant l_1$."

(4) $Z=Z$ of target nucleus, $\alpha_\lambda=0$, $\beta_l=F_l(\eta_P,K_PR)/$

[5] Bromley, Bruner, and Fulbright, Phys. Rev. **89**, 396 (1952).
[6] C. F. Black, Ph.D. thesis, Massachusetts Institute of Technology 1953 (unpublished).
[7] W. W. Pratt, preceding paper [Phys. Rev. **97**, 131 (1954)].

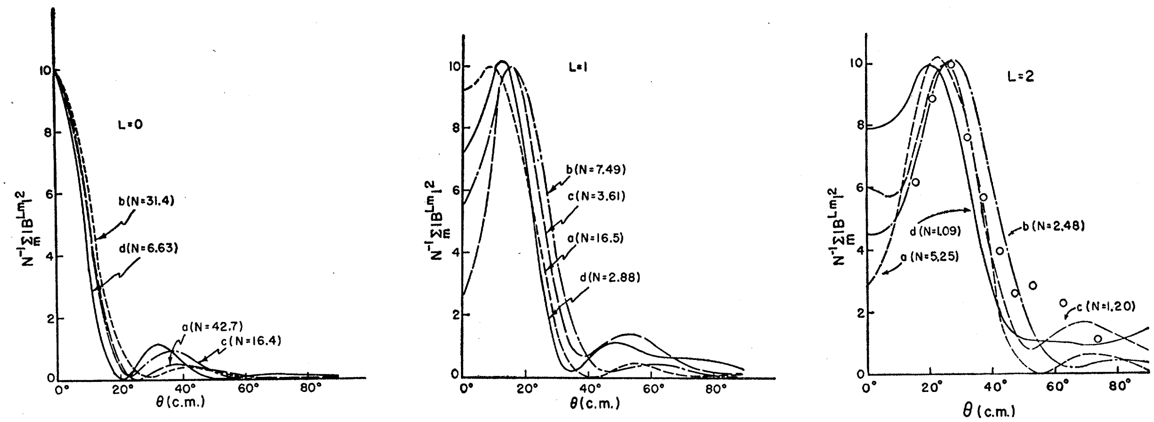

FIG. 4. The Coulomb and nuclear effects on the (*d*,*p*) reaction, for $L=0, 1, 2$. The angular distribution of protons from the $F^{19}(d,p)F^{20}$ reaction is calculated from (*a*) Butler theory, (*b*) Butler+Coulomb, (*c*) Butler+Coulomb+absorption of protons with $l\leqslant 4$, (*d*) Butler+Coulomb+hard sphere scattering of protons. $E_D=14.3$ Mev (lab), $Q=4.37$ Mev, $R=5.05\times 10^{-13}$ cm. The experimental points are due to Black. (See reference 6.)

$H_l(\eta_P,K_PR)$: "Butler+Coulomb+hard sphere scattering for protons."

(5) $Z=Z$ of target nucleus, $\alpha_\lambda=\frac{1}{2}$ for $\lambda\leqslant\lambda_1$ and $\alpha_\lambda=0$ for $\lambda>\lambda_1$, $\beta_l=[F_l'(\eta_P,K_PR)+iKRF_l(\eta_P,K_PR)]/[H_l'(\eta_P,K_PR)+iKRH_l(\eta_P,K_PR)]$, $K=(K_P^2+1\times 10^{-26}\text{ cm}^{-2})^{\frac{1}{2}}$: "Butler+Coulomb+continuum theory for protons+absorption of deuterons with $\lambda\leqslant\lambda_1$," etc.

In Figs. 6 and 7 we have examples of how strongly the (*d*,*p*) cross section can be influenced by the nuclear effects represented by the scattering amplitudes α_λ and β_l. In Fig. 6, various combinations of models were employed as a basis for the choice of the scattering amplitudes. In Fig. 7 we investigate the effect of a resonance in one of the proton scattering amplitudes. To be specific, we assume no deuteron scattering and absorption of protons with l smaller than 3 except for one proton scattering amplitude which is given the resonance value

$$\beta_l=-2\bar{\gamma}\Gamma^{-1}K_PRH_l(\eta_P,K_PR)^{-2}=-10H_l(\eta_P,K_PR)^{-2},$$

Γ being the reaction width and $\bar{\gamma}$ being the reduced width of the proton resonance. The value $2\bar{\gamma}\Gamma^{-1}K_PR=10$ is chosen arbitrarily.

III. DISCUSSION

Inspection of the calculated angular distribution discloses that the effect of introducing the Coulomb interaction into the theory is to displace the angular distribution toward larger angles and to broaden the peaks and fill in the valleys. The value of the total cross section is reduced. In extreme cases, such as the one shown in Fig. 5, these Coulomb effects can be so

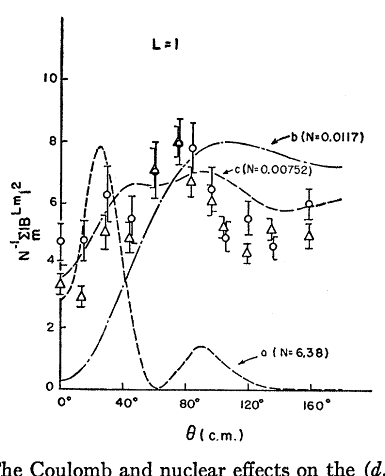

FIG. 5. The Coulomb and nuclear effects on the (*d*,*p*) reaction. The angular distribution of protons from the $Ti^{48}(d,p)Ti^{49}$ reaction is calculated from (*a*) Butler theory, (*b*) Butler+Coulomb, (*c*) Butler+Coulomb+absorption of protons with $l\leqslant 1$. $E_D=2.60$ Mev (lab), $Q=4.46$ Mev, $R=6.49\times 10^{-13}$ cm. The experimental points are due to Pratt. (See reference 7.) The triangles represent the reaction with $Q=4.46$ Mev and the circles represent the reaction with $Q=4.11$ Mev.

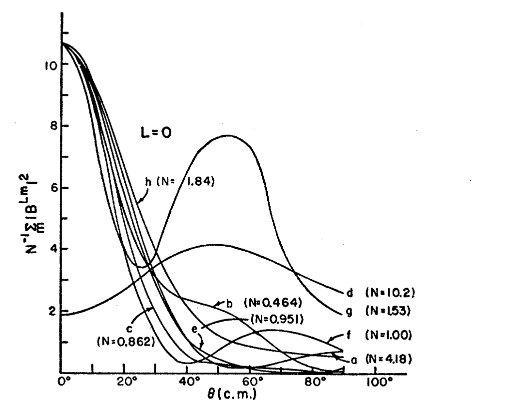

FIG. 6. The Coulomb effect, nuclear effect, and the effect due to the scattering of the incident deuteron beam on the (*d*,*p*) reaction. The angular distribution of protons from the $F^{19}(d,p)F^{20}$ reaction is calculated from Butler+Coulomb+(*a*) absorption of deuterons with $\lambda\leqslant 2$, (*b*) continuum theory for protons, (*c*) absorption of deuterons with $\lambda\leqslant 1$+absorption of protons with $l\leqslant 2$, (*d*) absorption of deuterons with $\lambda\leqslant 1$+hard sphere scattering for protons, (*e*) absorption of deuterons with $\lambda\leqslant 1$+continuum theory for protons, (*f*) absorption of protons with $l\leqslant 2$, (*g*) hard sphere scattering for protons, (*h*) nothing. $E_D=3.6$ Mev (lab), $Q=4.37$ Mev, $R=5.05\times 10^{-13}$ cm.

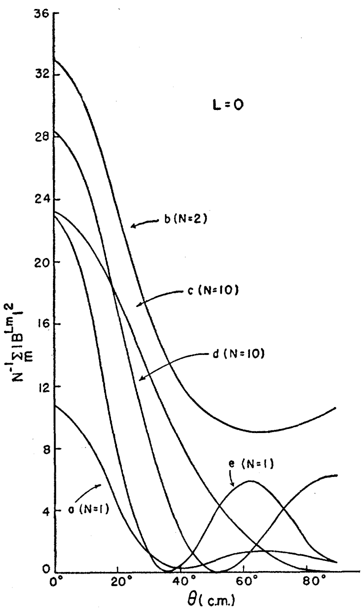

FIG. 7. The effects of resonances in the nuclear interaction on the (d,p) cross section. The angular distribution of protons from the $F^{19}(d,p)F^{20}$ reaction is calculated from Butler+Coulomb+absorption of protons with $l \leqslant 2+(a)$ no resonance, (b) resonance on the protons with $l=0$, (c) resonance on the protons with $l=1$, (d) resonance on the protons with $l=2$, (e) resonance on the protons with $l=3$. $E_D=3.6$ Mev (lab), $Q=4.37$ Mev, $R=5.05 \times 10^{-13}$ cm.

great as to distort completely the angular distribution. Even in a most favorable case where 14.3-Mev deuterons are incident on F^{19} (Fig. 3), we find that the Coulomb effects are not negligible although they are not great enough to make the choice of L uncertain.

One observes that the effect of introducing the nuclear interactions by giving α_λ and β_l nonzero values is in general opposite to the Coulomb effect—peaks are displaced toward smaller angles and become less broad. The nuclear effect on the total cross section, however, is in the same direction as the Coulomb effect, i.e., the total cross section is further reduced.

The nuclear effect on the (d,p) cross section is enhanced as the Coulomb effect becomes more important. This behavior can be explained as follows. When a neutron is captured with angular momentum $L\hbar$ from a deuteron of angular momentum $\lambda\hbar$, then the liberated proton has an angular momentum between $(\lambda+L)\hbar$ and $|\lambda-L|\hbar$. Since we are dealing with small values of L, we can say that the proton emerges with an angular momentum approximately equal to that of the incident deuteron. But most of the deuterons which interact with the target nucleus have angular momenta of about $R[2M_D(E_D-Ze^2/R)]^{\frac{1}{2}}=\lambda_0\hbar$ or less. Thus most of the protons emerge with angular momenta about equal to or less than $\lambda_0\hbar$. Now the proton partial waves with angular momenta below $R[2M_P(E_D+Q-Ze^2/R)]^{\frac{1}{2}}=l_0\hbar$ are distorted by the nuclear effect. We see that λ_0 decreases more rapidly with Z than l_0. This difference in behavior becomes less important as E_D becomes much larger than Q. Thus increasing Z increases the proportion of proton partial waves distorted by the nuclear effect.

We conclude that in order to determine the angular momentum of the captured particle from the deuteron-stripping angular distribution without making hypotheses about the scattering amplitudes for the incident deuterons and the emitted protons, the experiment must be performed with deuterons having energies well above the Coulomb barrier. However, even at such elevated energies the normalization of the cross section is still sufficiently sensitive to the Coulomb and nuclear effects to permit nothing better than a gross order of magnitude estimate of the reduced width when these effects are neglected.

Thus the presence of Coulomb and nuclear effects render the use of deuteron stripping to gain information about the spins, parities, and reduced widths of nuclear levels more cumbersome. On the other hand, deuteron stripping presents us with a new tool for the study of proton and neutron interactions with nuclei. For instance, it would be very interesting to study proton elastic scattering and the corresponding (d,p) reaction side by side, trying to fit the angular distributions from both experiments with the same set of scattering amplitudes.

IV. ACKNOWLEDGMENTS

The authors are indebted to H. Bethe for his help and encouragement. We thank W. W. Pratt for making his data available to us prior to publication. We also wish to thank R. Lesser and the staff of the Cornell Computing Center for their assistance in carrying out the computations.