Independent-Particle Model and the Nuclear Photoeffect. II*

J. S. Levinger
Louisiana State University, Baton Rouge, Louisiana

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A simple harmonic oscillator independent-particle model is used for sum-rule calculation of electric dipole transitions in the nuclear photoeffect. First we find the level spacing $\hbar\omega = 42A^{-1/3}$ Mev for nuclear radius parameter $r_0 = 1.2$. Combining this result with the integrated cross section, we find the bremsstrahlung-weighted cross section $\sigma_b = \int (\sigma/W)dW = 0.36A^{4/3}$ millibarns. The calculated σ_b is not inconsistent with a preliminary analysis of experimental measurements for He, Be, C, Al, Cu, Mo, Ag, Ta, Pb, and U. We also use the simple harmonic oscillator independent-particle model to calculate the increase in the integrated cross section due to neutron-proton exchange forces. We find that the relative increase is not far from the Levinger-Bethe value of $0.8x$ (where x is the fraction of exchange force) for the closed-shell nuclei He^4, O^{16}, and Ca^{40}, for both Gaussian and Yukawa neutron-proton potentials, and for two values of the radius parameter r_0: 1.2 or 1.5.

I. INTRODUCTION

In a previous paper[1] (here designated by I), the author and D. C. Kent used sum-rules to calculate various moments of the $\sigma(W)$ curve for nuclear absorption cross section *vs* photon energy. We used nuclear wave functions for an independent-particle model (IPM) in a finite square well, for two nuclei of mass number 68 and 184, respectively. We found that the calculations were not inconsistent with present experimental data on Cu and Ta, provided: (i) that in the calculations we used a rather small nuclear radius, with radius parameter $r_0 = 1.2$; and (ii) that we also included the effects of exchange forces in increasing the harmonic mean energy for photon absorption.

In this paper we would like to study further the applicability of the IPM for the nuclear photoeffect. To simplify calculations we have changed from the finite square well IPM used in I to a simple harmonic oscillator IPM. The parameter ω, defined using the IPM potential $\frac{1}{2}M\omega^2 r^2$, is evaluated in terms of the radius parameter r_0 in the following section. In Sec. III we use the simple harmonic oscillator IPM to evaluate the bremsstrahlung-weighted cross section,

$$\sigma_b = \int (\sigma/W)dW; \tag{1}$$

σ_b is the electric dipole cross section for the nuclear photoeffect weighted by the dW/W approximation to the bremsstrahlung spectrum. Thus σ_b is rather easily compared with measured bremsstrahlung yields for photonuclear processes. Also the calculated σ_b is not changed by the presence of neutron-proton exchange forces. The calculated σ_b is proportional to the square of the nuclear radius, and we can obtain reasonable agreement between calculations and experiment if we use $r_0 = 1.2$ for the radius parameter. (The radius is $R = r_0 A^{1/3} \times 10^{-13}$ cm.)

In Sec. IV we calculate the increase in the integrated cross section $\sigma_{\rm int}$ due to the neutron-proton exchange force. Levinger and Bethe[2] (here referred to as LB) calculated

$$\sigma_{\rm int} = \int \sigma dW = 0.015A(1+0.8x) \text{ Mev-barns.} \tag{2}$$

Here x is the fraction of the neutron-proton force that has an exchange character. The coefficient 0.8 depends on the nucleon wave functions that are assumed. LB used a model of a degenerate Fermi gas; while we shall use nucleon wave functions for a simple harmonic oscillator IPM. We find that the simple harmonic oscillator IPM calculation gives a coefficient with a value rather close to that found in the LB calculation. These phenomenological calculations of $\sigma_{\rm int}$ are being superseded by the more fundamental dispersion-theory calculations of Gell-Mann, Goldberger, and Thirring.[3] However it seems worthwhile to repeat the phenomenological calculation for a somewhat different nuclear model, to provide a firmer number to compare with the dispersion theory result.

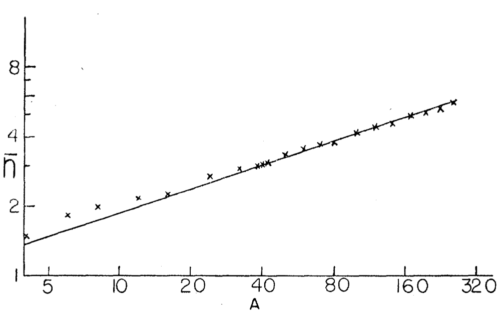

Fig. 1. The graph shows that the average quantum number $\bar{n}$ [defined by Eq. (5)] for a simple harmonic oscillator independent-particle model can be approximately represented by the relationship $\bar{n} = 0.87A^{1/3}$. The line shows this equation; the points are shown for calculations for individual nuclei.

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[1] J. S. Levinger and D. C. Kent, Phys. Rev. **95**, 418 (1954), designated by I.

[2] J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950), designated by LB.

[3] Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954).

II. SIMPLE HARMONIC OSCILLATOR POTENTIAL

In this section we shall evaluate the classical angular frequency ω for the simple harmonic oscillator potential in terms of the nuclear radius. This evaluation has been done by Wu[4] for several light nuclei. Wu uses an oscillatory wave function for the least bound nucleon for $r<R$, and a damped wave function for $r>R$. Matching at the nuclear radius R determines the parameter ω. In this paper we shall match the radius R in a different manner, which gives us a rather smaller value of ω.

Our method consists in calculating the expectation value $\langle r^2\rangle$ for each nucleon in a simple harmonic oscillator potential. This mean square radial distance is then averaged over all nucleons in the nucleus, giving $\langle r^2\rangle_A$ for the nucleon density distribution, in terms of ω and A. We equate $\langle r^2\rangle_A$ to the value $\frac{3}{5}R^2$ for a sphere of uniform density out to radius $R=r_0A^{\frac{1}{3}}\times 10^{-13}$ cm. Thus we determine ω in terms of A and r_0.

Our present method of using r_0 implies that the values given (e.g., Hofstadter *et al.*[5] and Fitch and Rainwater[6]) are based on the mean square radius for the nucleon density distribution. Actually different measurements weight the density distribution with different factors,[7] and also measure the nuclear charge distribution rather than the possibly different nucleon density distribution.

For a nucleon with quantum number n, the mean square radial distance

$$\langle r^2\rangle=(\hbar/M\omega)(n+3/2). \tag{3}$$

The nuclear mean square radial distance $\langle r^2\rangle_A$ is found averaging over all A nucleons:

$$\langle r^2\rangle_A=(\hbar/M\omega)\sum_n c_n(n+3/2)/A=(\hbar/M\omega)\bar{n}, \tag{4}$$

where c_n is the number of nucleons with quantum number n. (We are neglecting any differences among nucleons due to spin or charge.) Figure 1 shows that $\bar{n}$, the average value of n, is well represented by

$$\bar{n}=\sum_n c_n(n+3/2)/A=0.87A^{\frac{1}{3}}. \tag{5}$$

(The "breaks" at the magic numbers are quite small.) Matching the nuclear mean radial distance with $\frac{3}{5}R^2$, we solve for $\hbar\omega$:

$$\hbar\omega=1.45(\hbar^2/Mr_0^2)A^{-\frac{1}{3}}=42A^{-\frac{1}{3}}\text{ Mev}. \tag{6}$$

Here the numerical result is based on $r_0=1.2$. The proportionality of the nuclear energy scale to $r_0^{-2}A^{-\frac{1}{3}}$ was found in I for a finite square well, and is characteristic of any IPM.[8]

[4] S. S. Wu, Ph.D. thesis, University of Illinois, 1951 (unpublished).
Note added in proof.—Also see B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954).
[5] Hofstadter, Hahn, Knudsen, and McIntyre, Phys. Rev. **95**, 512 (1954).
[6] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1954).
[7] D. L. Hill and K. W. Ford, Phys. Rev. **94**, 1617 and 1630 (1954).
[8] J. S. Levinger, Ann. Revs. Nuclear Sci. **4** (1954).

TABLE I. Nuclear energies calculated from the IPM.

Nucleus	This paper[a] $r_0=1.37$	This paper[a] $r_0=1.2$	Wu[b] $r_0=1.37$	Paper I[c] $r_0=1.2$
He^4	20 Mev	—	36 Mev	—
Be^9	15	—	29	—
Ne^{20}	12	—	24	—
Ca^{40}	9.4	—	16	—
Cu	—	10	—	13
Ta	—	7.4	—	8.7

[a] The value of $\hbar\omega$ is given by Eq. (6).
[b] See reference 4.
[c] See reference 1, Tables VI and VII, for the harmonic mean energy (not including exchange effects).

We could obtain a better fit to the low-A points in Fig. 1 if we used an exponent slightly lower than $\frac{1}{3}$, giving an exponent of -0.35 in Eq. (6), and a somewhat larger coefficient. This refinement seems unwarranted at present, in view of the following approximations: (a) We have assumed that $\langle r^2\rangle_A$ is proportional to $A^{\frac{2}{3}}$ for all nuclei, even those as light as He^4; (b) we have neglected Coulomb effects on the proton; we use $A=2Z$ throughout this paper; (c) we have assumed that the simple harmonic oscillator potential continues to indefinitely large R; (d) we have neglected spin-orbit coupling, and configuration interactions.

For the alpha-particle and lighter nuclei, we can use the LB calculation[2] of the harmonic mean energy W_H, since for these nuclei there are no Pauli principle correlations to change the result. With $r_0=1.2$, the LB value is $W_H=72A^{-\frac{2}{3}}$ Mev$=29$ Mev, for He^4, in reasonable agreement with 26 Mev given by Eq. (6). For the alpha particle W_H and σ_b are determined entirely by the value of $\langle r^2\rangle_A$. (We plan in subsequent work to calculate $\langle r^2\rangle_A$ from alpha-particle wave functions. Here we use the dubious extrapolation of the nuclear radius to very small nuclei, to find approximate analytical expressions for W_H and σ_b in terms of A.)

Equation (6) gives lower values for light nuclei than those calculated by Wu. In Table I we compare the two results, for $r_0=1.37$. We also compare Eq. (6) for $\hbar\omega$ for Cu and Ta with the calculation in I for the harmonic mean energy for these two nuclei using a finite square well, with $r_0=1.2$. Our value of $\hbar\omega$ is different from Wu's, since we have interpreted the nuclear radius R in terms of a mean square radial distance, rather than as the limit of the classically allowed region. Our value of $\hbar\omega$ here is somewhat lower than the harmonic mean energy calculated in I for Cu and Ta, even though the same $\langle r^2\rangle_A$ is used.

Our present value of $\hbar\omega$ can be compared unfavorably both with photonuclear experiments and with measurements of low-lying nuclear energy levels. The former experiments show that the bulk of levels that can be reached by dipole transitions from the ground state have an excitation energy of about 17 Mev for medium nuclei such as Cu; while Eq. (6) gives only 10 Mev, even if we use a small nuclear radius. However, (see I) exchange forces increase the harmonic mean energy for photon absorption by about forty percent, greatly improving the agreement between calculations and experiment. (The argument of I is somewhat inconsistent: it is claimed that the IPM holds to a reasonable approximation for all nucleons in their ground state; but that nucleons excited by electric dipole absorption experience quite a different potential, the difference being caused by neutron-proton exchange forces. Since the potential has not yet been derived from two body forces the effect of exchange forces cannot at present be included in the nuclear potential. We believe that our method of including the effect of exchange forces on the harmonic mean energy has a heuristic value, in spite of its inconsistency.)†

On the other hand, our value $\hbar\omega=10$ Mev for medium nuclei is much larger than the values of several Mev found for the excitation energy of low-lying nuclear levels which might well be identified as the next nucleon level for a simple harmonic oscillator potential. A detailed analysis of low-lying levels to identify them with oscillator levels is desirable.

We shall not try to justify our value of $\hbar\omega$, but shall use it in this paper as an admittedly crude approximation to an IPM treatment of the nuclear photoeffect.

III. BREMSSTRAHLUNG-WEIGHTED CROSS SECTION

In this section we shall evaluate the bremsstrahlung-weighted cross section σ_b using a simple harmonic oscillator IPM, and we shall compare with experimental data.

LB[2] evaluate the bremsstrahlung-weighted cross section as

$$\sigma_b=\int_0^\infty (\sigma/W)dW$$
$$=(e^2/\hbar c)(4\pi^2)\{[(N/A)\textstyle\sum_i z_i-(Z/A)\sum_j z_j]^2\}_{00}. \quad (7)$$

Here $\sum_i z_i$ is the sum of the components of proton displacements along the direction of polarization of the photon; while $\sum_j z_j$ refers to neutrons. The expectation value is taken for the ground state of the nucleus. LB assumed that there were no correlations among the nucleons. In I we included Pauli principle correlations among pairs of protons, and among pairs of neutrons, for a square well IPM. In this paper we shall include the Pauli principle correlations for a simple harmonic oscillator IPM.

The Pauli principle correlations decrease σ_b, since due to the Pauli principle each proton is surrounded by an "exchange hole" in which there is a decreased likelihood of finding another proton.[9] There may also be significant dynamical correlations, causing a neutron and proton in a triplet S state, for example, to tend to remain close together, further decreasing the value of σ_b. These dynamical correlations, which are most marked in a sub-unit model[10] are neglected in the IPM treatment of this paper.

In I we calculated σ_b using Eq. (7). Our present calculation is done by the same method in the Appendix. However we shall find it shorter to calculate the same answer for σ_b by a round-about approach: i.e.,

$$\sigma_b=\sigma_{\rm int}/W_H. \quad (8)$$

Here we have utilized the LB definition of the harmonic mean energy W_H. For the simple harmonic oscillator potential, with pure ordinary forces, W_H is just $\hbar\omega=42A^{-\frac{1}{3}}$ Mev since electric dipole absorption causes transitions only between adjacent levels. (The special property that the harmonic mean energy equals the mean energy only for the case of an oscillator potential is discussed in the appendix, and related to the Heisenberg uncertainty principle.) Also, for pure ordinary forces, the numerator is (LB)

$$\sigma_{\rm int}=\int \sigma dW=15A \text{ Mev-millibarns}. \quad (9)$$

Substituting in Eq. (8) we have the bremsstrahlung-weighted cross section

$$\sigma_b=15A/42A^{-\frac{1}{3}}=0.36A^{4/3} \text{ mb}. \quad (10)$$

Our value for σ_b is proportional to r_0^2, since r_0 determines the scale for proton displacement z_i in Eq. (7), and the proton displacement enters squared. (Our numerical value is based on $r_0=1.2$.) σ_b is proportional to $A^{4/3}$, rather than $A^{5/3}$ as found by LB, since the Pauli principle correlations are increasingly effective for large A in decreasing the sum in Eq. (7). Our present value for σ_b is somewhat higher than that given in I for Cu and Ta, corresponding to the comparison in Table I that our $\hbar\omega$ is somewhat lower than W_H of I. (The numerator $\sigma_{\rm int}$ is independent of the model, for pure ordinary forces.) σ_b depends on the ground state nuclear wave functions, but is not changed by the existence of neutron-proton exchange forces. Thus we can calculate σ_b for pure ordinary forces, and compare our result with experimental data on actual nuclei, where there are significant exchange forces.

In comparing Eq. (10) with experiments we shall not attempt any exhaustive treatment of the experimental data. Our purpose here is to call attention to the significance of the bremsstrahlung-weighted cross section as a useful source of information concerning nuclear wave functions; and to make only a preliminary comparison between experiment and theory. In I we integrated to an upper limit of 70 Mev in using the experi-

† *Note added in proof.*—A more consistent interpretation is that two-body exchange forces contribute to a velocity-dependent nuclear potential for the IPM. This velocity-dependence increases W_H.

[9] J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Sec. III. 3.

[10] A. Winslow, Ph.D. thesis, Cornell University, 1952 (unpublished).

TABLE II. Analysis of the experimental contributions to the bremsstrahlung-weighted cross section σ_b (in millibarns).

1. Nucleus	2. Neutrons 0–22 Mev	3. Protons 0–22 Mev	4. Neutrons 22–62 Mev[a]	5. Protons 22–62 Mev	6. 62–150 Mev[a]	σ_b[k]
He[b]	—	—	—	—	—	$1.5\leq\sigma_b\leq 3$
Be	4.3[c]	0.5[d]	$1\leq\sigma_b\leq 3$	$\sim$0.1	0.5	$5.8\leq\sigma_b\leq 8.4$
C	0.9[d,e]	2.5[f]	$2\leq\sigma_b\leq 2.5$	$\sim$6.	0.4	$5.4\leq\sigma_b\leq 12$
Al	5. [d,g]	5.1[f]	$4\leq\sigma_b\leq 7$	$\sim$5.	2.5	$14.\leq\sigma_b\leq 24$
Cu	40. [d,g]	15. [f,h]	$10\leq\sigma_b\leq 26$	$\sim$7.	6.	$65.\leq\sigma_b\leq 94$
Mo	80. [d,g]	6. [f]	$20\leq\sigma_b\leq 46$	$\sim$3.	9.	$110.\leq\sigma_b\leq 140$
Ag	100. [d,g]	small	$20\leq\sigma_b\leq 50$	small	12.	$120.\leq\sigma_b\leq 160$
Ta	190. [e,g,i]	small	$30\leq\sigma_b\leq 74$	small	20.	$220.\leq\sigma_b\leq 280$
Pb	240. [d,g,i]	small	$30\leq\sigma_b\leq 80$	small	25.	$270.\leq\sigma_b\leq 340$
U	420. [e,g,j]	small	$30\leq\sigma_b\leq 110$	small	35.	$450.\leq\sigma_b\leq 560$

[a] Jones and Terwilliger, reference 21; [b] Fuller, reference 26, Benedict and Woodward, reference 27; [c] Nathans and Halpern, reference 15; [d] Montalbetti, Katz, and Goldemberg, reference 13; [e] Nathans and Halpern, reference 12; [f] Halpern *et al.*, reference 18; [g] Price and Kerst, reference 11; [h] Byerly and Stephens, reference 19; [i] Wilkinson *et al.*, and Hanson *et al.*, reference 16; [j] Duffield and Huizenga, reference 17. [k] The last column is the sum of the contributions of the various processes to the bremsstrahlung cross section σ_b, which is defined in Eq. (1) of the text. The lower limit is found by using the lower limit of column 4 (neutron multiplicity from compound nucleus estimate) and omitting the contributions of columns 5 and 6. The upper limit is found by using the upper limit of column 4 (neutron multiplicity equals unity) and including all the other contributions. The value of σ_b is plotted in Fig. 2, and compared with theory.

mental data to determine σ_b [Eq. (1)] and σ_{int} [Eq. (9)]; but we have changed to an upper limit of 150 Mev for consistency with the value used by Gell-Mann, Goldberger, and Thirring.[3] (A real advantage of working with σ_b rather than σ_{int} is that the experimental value is rather insensitive to the cross section at high energies, where the experiments are less complete, and where we face the question of what upper limits to use in the integrals.)

The experimental data on the nuclei Be, C, Al, Cu, Mo, Ag, Ta, Pb, and U are treated in three energy ranges: medium (0 to 22 Mev); high (22 to 62 Mev) and very high (62 to 150 Mev). In the medium-energy region we can determine the contribution to σ_b directly from the yield curve[11–13] or we can determine the contribution to σ_b by numerical integration of the published excitation curves.[12,13] The yield Y is

$$Y=\int\gamma(W)\sigma(W)dW\cong\varphi(W_m)\int\sigma(W)dW/W, \quad (11)$$

where we have written the bremsstrahlung spectrum $\gamma(W)=\varphi(W)/W$; $\varphi(W)$ is a slowly varying function of W and may to a first approximation be taken out of the integral and evaluated at the energy W_m corresponding to the peak cross section. In our preliminary work we have taken $\varphi(W)$ as the Schiff thin-target spectrum, corrected for absorption, and appropriately normalized to the ionization in roentgens.[14]

Our values for σ_b using the 22-Mev bremsstrahlung yield given in column 2 of Table II are in good agreement with integrations from the $\sigma(W)$ curve. The 22-Mev yield method is inapplicable to a case such as C, where the peak cross section occurs at just about 22 Mev,[13] and also to the double-peaked cross section[15] for Be.

We shall use the data based on particle (neutron and proton) yields, even though such measurements tend to give too large an absolute cross section since multiple reactions are counted more than once. Corrections of 20 percent are made for multiple neutron emission for Ta,[16] Pb (assumed the same as Ta); and corrections of 35 percent for fission and emission of two neutrons by U.[17] Corrections for multiple particle emission are believed small in this energy region for the other nuclei.

Proton yields in the medium energy region are of importance for nuclei up to Mo as shown by column 3 of Table II. The contributions to the bremsstrahlung-weighted cross section are taken from the measurements of Montalbetti, Katz, and Goldemberg;[13] Halpern *et al*;[18] and Byerly and Stephens.[19] (Also see Chastel,[20] who finds a larger proton yield for Li gammas on Cu than would be expected from the work of Halpern or Stephens with 24-Mev bremsstrahlung.)

The neutron yields above 22 Mev have been measured by Jones and Terwilliger.[21] Their $\nu\sigma(W)$ curves have been treated in two different ways to obtain upper and lower limits for the neutron contribution to σ_b from 22 to 62 Mev given in column 4 of Table II. We find the upper limit by assuming that only one neutron is emitted (multiplicity ν equals one) and calculating $\int(\sigma/W)dW$. We obtain a lower limit for the contribution to σ_b by assuming that neutron multiplicity has the approximate upper limit of $W/\bar{E}$, where $\bar{E}$, the average energy per neutron emission, is about $\frac{3}{2}$ the neutron

[11] G. A. Price and D. W. Kerst, Phys. Rev. **77**, 806 (1950).
[12] R. Nathans and J. Halpern, Phys. Rev. **93**, 437 (1954).
[13] Montalbetti, Katz, and Goldemberg, Phys. Rev. **91**, 659 (1953).
[14] R. Nathans, Ph.D. thesis, University of Pennsylvania, 1954 (unpublished), Table IVa. Also see reference 12.
[15] R. Nathan and J. Halpern, Phys. Rev. **93**, 940 (1953).
[16] E. A. Whalin and A. O. Hanson, Phys. Rev. **89**, 324 (1953); Carver, Edge, and Wilkinson, Phys. Rev. **89**, 658 (1953).
[17] R. B. Duffield and J. R. Huizenga, Phys. Rev. **89**, 1042 (1953).
[18] E. V. Weinstock and J. Halpern, Phys. Rev. **94**, 1651 (1954), and earlier papers by Halpern *et al.*
[19] P. R. Byerly and W. E. Stephens, Phys. Rev. **83**, 54 (1951).
[20] R. Chastel, J. phys. et radium **15**, 459 (1954).
[21] L. W. Jones and K. M. Terwilliger, Phys. Rev. **91**, 699 (1953).

binding energy.[22] With this assumption the contribution to σ_b has a lower limit of $\bar{E}\int(\sigma/W^2)dW$.

Rather little is known about proton reactions in the high-energy region. We estimate this contribution to σ_b by using the neutron contribution to σ_b in this energy region and assuming the proton/neutron yield ratio is about the same in the high-energy region as in the medium-energy region. The resulting numbers, given in column 5 of Table II are most uncertain, even though, particularly for C and Al, high-energy proton yield contributes a large part of the total bremsstrahlung-weighted cross section. (Nathans and Halpern[12] have suggested a substantial contribution of the high-energy proton cross section to σ_{int} for light elements.)

The contribution of the very high-energy region from 62 to 150 Mev is estimated in column 6 of Table II. Here we find the absorption cross section at 140 Mev as done by Jones and Terwilliger.[21] Assuming that the cross section is about constant from 62 to 150 Mev, the integration for the bremsstrahlung-weighted cross section is then performed. This rather uncertain contribution to σ_b is rather small (10 percent or less). We have not included in this table the contributions from inelastic[23] or elastic[24] gamma scattering. They are probably less than 10 percent of the total.

The values for σ_b given in the last column of Table II are given as lower and upper limits. The lower limit is found by adding the values of columns 2 and 3 to the lower limit in column 4, and omitting the contributions of columns 5 and 6. For the upper limit to σ_b, we use the upper limit for the high-energy neutron yield given in column 4, and add all the other contributions to σ_b. An additional allowance of at least 20 percent (Nathans and Halpern)[12] should be made for systematic errors in the various absolute measurements.

The rather poorly known bremsstrahlung-weighted cross section for He^4 is based on work by Halpern *et al.*[25] up to 26 Mev; on Fuller's work[26] up to 36 Mev; and on Benedict and Woodward's measurements[27] at the higher energies. Halpern's data give 0.2 mb for the neutron contribution to σ_b up to 26 Mev; while Fuller's data give a contribution of 0.6 mb for the proton yield up to 36 Mev. (Fuller's proton cross sections are about 25 percent higher than Halpern's neutron cross sections, for the same photon energies.) If we take 0.2 mb as an average cross section for proton emission in the high-energy region,[27] we have a proton contribution to σ_b of 0.3 mb from 36 to 150 Mev. The total proton contribution of 0.9 mb is doubled to account for single neutron emission; and might be increased somewhat further to take account of multiple disintegration of the alpha particle. However, Halpern's neutron measurements suggest a reduction of 20 percent. We take 1.5 mb $\leq\sigma_b\leq 3$ mb.

Figure 2 compares the experimental bremsstrahlung-weighted cross sections estimated in Table II with the calculated value of σ_b given in Eq. (10). We also show a dashed line from Cu to Ta giving the finite square well IPM calculations of I. The agreement between experiment and theory is rather better than expected, considering the preliminary character of both estimates. More accurate experimental data for σ_b would be of great help in deciding both the size of the nucleus, and the extent of dynamical sub-unit correlations. The present agreement between experiment and theory (if we use a rather small radius, with radius parameter $r_0=1.2$) indicates that sub-unit correlations are not of great significance for moderate energy photonuclear reactions. (Note that σ_b weights the $\sigma(W)$ curve favoring low W.) That is, the IPM is not unsuccessful in the moderate-momentum region, for the nuclear ground state. (If we used $r_0=1.5$, the disagreement between experiment and theory would suggest sub-unit correlations significantly modifying the IPM.)

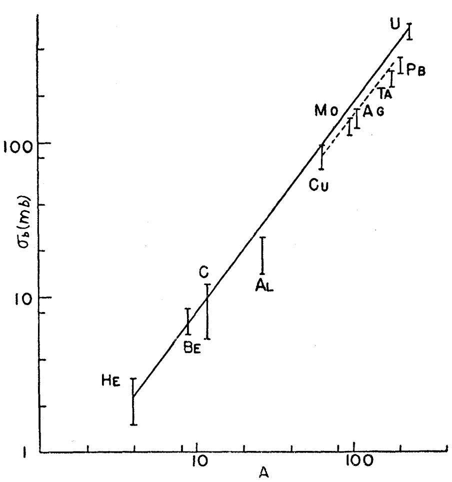

FIG. 2. Comparison of experimental and theoretical values for the bremsstrahlung-weighted cross section $\sigma_b=\int(\sigma/W)dW$. The experimental values shown by vertical bars are given in Table II. The bars show the lower and upper limits for σ_b, based on different analyses of the experiments. They do not include errors in measurement of absolute values. The solid line shows $\sigma_b=0.36A^{4/3}$ millibarns, derived from a simple harmonic oscillator independent-particle model [Eq. (10)]. The dotted line is based on calculations for Cu and Ta (reference 1) for a finite square well independent-particle model.

IV. EXCHANGE FORCE CONTRIBUTION TO INTEGRATED CROSS SECTION

The increase by a factor $(1+Cx)$ in σ_{int} due to exchange forces was calculated by Levinger and Bethe (LB)[2] using nuclear wave functions for a degenerate perfect Fermi gas, normalized to give the correct

[22] J. S. Levinger and H. A. Bethe, Phys. Rev. **85**, 577 (1952).
[23] C. S. Del Rio and V. L. Telegdi, Phys. Rev. **90**, 439 (1953).
[24] E. G. Fuller and E. Hayward, Phys. Rev. **95**, 1106 (1954).
[25] Ferguson, Halpern, Nathans, and Yergin, Phys. Rev. **95**, 776 (1954).
[26] E. G. Fuller, Phys. Rev. **96**, 1306 (1954).
[27] T. S. Benedict and W. M. Woodward, Phys. Rev. **83**, 1269 (1951).

nucleon density. (Also see Feenberg[28] and Siegert.[29]) In this paper we shall present the results of calculations using the simple harmonic oscillator IPM discussed above. We start with LB, Eq. (11), for the summed oscillator strength for electric dipole transitions:

$$\sum_n f_{0n} = NZ/A - (2M/\hbar^2)x(1/6)\int \Psi_0^* \sum_i \sum_j r_{ij}^2 \times V(r_{ij}) P_{ij} \Psi_0 d\tau. \quad (12)$$

Here x is the fraction of the neutron proton force that has an exchange character; Ψ_0 is the complete nuclear wave function; i denotes protons and j neutrons, the double sum being over all pairs of neutrons and protons; r_{ij} is the distance between proton i and neutron j, while $V(r_{ij})$ is the neutron-proton potential; and P_{ij} is the Majorana exchange operator. Three approximations were made: (i) The LB approximation of a degenerate Fermi gas model allowed replacing the double sum in Eq. (12) by integrals over the momenta of protons and neutrons. (ii) In evaluating the integral over r_{ij} the Fermi gas approximation also neglects surface effects that might be appreciable for a real nucleus. (iii) LB found that the exchange operator P_{ij} gave a significant decrease in the integral. On the other hand, for a very light nucleus such as the alpha particle, all particles have the same spatial wave function, to the extent that we neglect the spin and charge dependence of nuclear forces, as we shall do in this paper. Then for the alpha particle the exchange operator cannot decrease the integral, as it does for the Fermi gas model. These three differences suggest that a calculation with discrete IPM wave functions for a finite nucleus might give a rather different result than found using the LB approximations. However we find that the LB approximation gives results quite close to those for a harmonic oscillator IPM for closed shell nuclei. Apparently the main contribution to the integral of Eq. (12) comes from rather small neutron-proton distances, so that the details of the neutron and proton wave functions are not significant.

Evaluating Eq. (12) with a neutron-proton Gaussian potential $[-V_0 \exp(-r^2/\beta^2)]$ and simple harmonic oscillator wave functions is straightforward, though tedious, so we shall not present the details of the calculation. We perform the integrations in Cartesian coordinates, and for this reason find it convenient to use a Gaussian neutron-proton potential, which can be simply expressed in terms of the Cartesian coordinates of the two particles. The integral of Eq. (12) is evaluated for each sextet of quantum numbers for the neutron-proton pair: (n_x,n_y,n_z) for proton, and (n_x',n_y',n_z') for neutron. The double sum over neutron-proton pairs is then done algebraically. Since here we neglect the spin and charge dependence of nuclear forces, the double sum does not involve a prohibitive amount of work: thus for Ca^{40} the 400 different neutron-proton pairs amount to only 15 different sextets of neutron and proton quantum numbers, combined with appropriate weights.

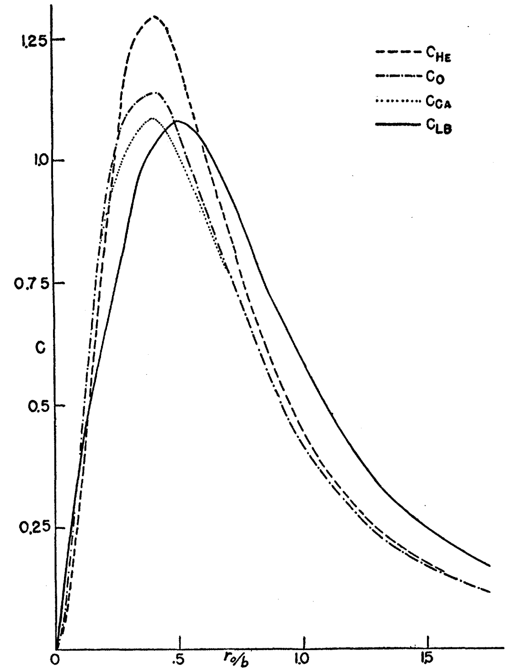

FIG. 3. The coefficient C for the increase of the integrated cross section is plotted against the parameter r_0/b for several different nuclei. The integrated cross section$=0.015A(1+Cx)$, where x is the fraction of neutron-proton exchange force; r_0 is the nuclear radius parameter, and b is the intrinsic range of the Gaussian neutron-proton potential. The four curves are C_{LB} (reference 2) for the Fermi gas model; C_{He} for He^4; C_O for O^{16}; and C_{Ca} for Ca^{40}. The last three are calculated using a simple harmonic oscillator independent-particle model. See Eqs. (20), (16), (17), and (18) for the four curves.

The neutron-proton potential is taken as a weighted average of the singlet and triplet potentials for the Gaussian case. We use an intrinsic range b of 2.2×10^{-13} cm, and a well-depth parameter $s=1.3$. We express our results for the coefficient C of x in terms of the two related parameters v and u.

$$v=1/\beta^2\alpha, \quad (13)$$

$$\alpha=M\omega/\hbar=1.46A^{-\frac{1}{3}}r_0^{-2}\times10^{26}\ \mathrm{cm}^{-2}, \quad (14)$$

$$u=r_0/b=(0.71A^{-\frac{1}{3}}v)^{\frac{1}{2}}. \quad (15)$$

Here $b=1.44\beta$ from the effective range theory; and we use $\hbar\omega$ from Sec. II. (The actual calculation is done in terms of v, giving polynomials with integral coefficients. The parameter u is useful for comparing calculations for the various nuclei.) Our equations for the coefficient C for the closed-shell nuclei He^4, O^{16}, and Ca^{40} are given below; plotted against u in Fig. 3; and evaluated in Table IV for the values $r_0=1.2$ and $r_0=1.5$. For He^4,

$$C=14.0v(1+2v)^{-5/2}=31.5u^2(1+4.48u^2)^{-5/2}. \quad (16)$$

[28] E. Feenberg, Phys. Rev. 49, 328 (1936).
[29] A. J. F. Siegert, Phys. Rev. 52, 787 (1937).

For O^{16},

$$C=3.50v(4+6v+31v^2)(1+2v)^{-9/2}$$
$$=12.4u^2(4+21.3u^2+391u^4)(1+7.1u^2)^{-9/2}. \quad (17)$$

For Ca^{40},

$$C=1.75v(8+24v+176v^2+168v^3+389v^4)(1+2v)^{-13/2}$$
$$=8.43u^2(8+116u^2+4080u^4+18\,800u^6+209\,000u^8)$$
$$\times(1+9.64u^2)^{-13/2}. \quad (18)$$

Using the Fermi gas model [LB, Eq. (19)] for this Gaussian neutron-proton potential, we find it convenient to express the coefficient C in terms of a parameter t, as well as the parameter u used above.

$$t=(\beta k)^{-2}=0.695u^2. \quad (19)$$

Here k is the maximum wave number for the Fermi gases of neutrons and protons. The LB calculation for the Fermi gas model gives

$$C=3.95t^{\frac{3}{2}}[(2t+1)e^{-1/t}+1-2t]$$
$$=(4.58u^3+3.29u)e^{-1.44/u^2}+3.29u-4.58u^3. \quad (20)$$

This equation is also plotted in Fig. 3, and evaluated in Table III.

In Table III we also present the LB results for square and Yukawa neutron-proton potentials. (See LB, Table I.) We have calculated C for a quasi-Yukawa potential by expressing the contribution to C for each of the Gaussians. We use a Yukawa potential for an intrinsic range b of 2.5×10^{-13}, and a depth parameter of 1.3:

$$V_Y=1.57(\hbar^2/M\times10^{-26})e^{-r/1.18}/(r/1.18). \quad (21)$$

Here the radial distance r is given in units of 10^{-13} cm. We fit V_Y within 10 percent over a range of r from 0.3 to 4×10^{-13} cm (a factor of almost 1000 in V_Y) with the sum of 4 Gaussian potentials:

$$V_Y'=(\hbar^2/M\times10^{-26})(0.12e^{-r^2/7.75}+0.55e^{-r^2/2.23}$$
$$+2.6e^{-r^2/0.53}+8.4e^{-r^2/0.065}). \quad (22)$$

TABLE III. Summed oscillator strength (times A/NZ).

Nucleus	Neutron-proton potential	Nuclear radius parameter r_0 $r_0=1.2$	$r_0=1.5$
He^4, SHO	Gaussian	$1+1.12x$	$1+0.87x$
O^{16}, SHO	Gaussian	$1+0.99x$	$1+0.80x$
Ca^{40}, SHO	Gaussian	$1+0.94x$	$1+0.77x$
Fermi gas (LB)	Gaussian	$1+1.07x$	$1+0.96x$
Fermi gas (LB)	Square	$1+1.06x$	$1+0.80x$
Fermi gas (LB)	Yukawa	$1+0.76x$	$1+0.69x$
He^4, SHO	Quasi-Yukawa	$1+0.84x$	$1+0.72x$
O^{16}, SHO	Quasi-Yukawa	$1+0.73x$	$1+0.63x$
Ca^{40}, SHO	Quasi-Yukawa	$1+0.70x$	$1+0.61x$

In Table III x is the fraction of the neutron-proton force that has an exchange character. The coefficient of x gives the relative change in the summed oscillator strength, or in σ_{int}, due to complete exchange force. This coefficient C is plotted in Fig. 3, and given in Eqs. (16), (17), (18) for a calculation with simple harmonic oscillator (SHO) independent-particle model for He^4, O^{16}, and Ca^{40} respectively. The LB calculation (reference 2) for a Fermi gas model is given in Eq. (20) and Fig. 3 for a Gaussian neutron-proton potential, and in LB [Eqs. (20) and (21) and Table I] for the square and Yukawa neutron-proton potentials. The calculations for He, O, and Ca for a quasi-Yukawa potential are discussed following Eq. (22).

The 4 terms in V_Y' correspond to intrinsic ranges b for the 4 different Gaussians of 4.0, 2.1, 1.1, and 0.37 $\times10^{-13}$ cm respectively. These 4 Gaussians should be used with Fig. 3 with weights of 0.26, 0.35, 0.40, and 0.16 respectively. (Note that Fig. 3 is for a Gaussian potential of depth V_0 proportional to b^{-2} and that we are using Fig. 3 for different values of b.) We find the results for He, O, and Ca for a quasi-Yukawa potential given in the last three rows of Table III.

As a check on this method of calculating with a Yukawa potential we have used the curve marked LB in Fig. 3 for the LB Fermi gas model, together with V_Y' as a sum of 4 Gaussians to find the coefficient C for a Yukawa potential with the LB model. Our values of 0.78 (for $r_0=1.2$) and 0.71 (for $r_0=1.5$) are in good agreement with those calculated directly for V_Y by LB for the Fermi gas model: 0.76 (for $r_0=1.2$) and 0.69 (for $r_0=1.5$).

LB found that the coefficient C was not sensitive to the shape of the neutron-proton potential, or to the radius parameter r_0. We have now found that C is also not sensitive to the particular nuclear wave functions used: the Fermi gas model, or simple harmonic oscillator IPM for the closed-shell nuclei He, O, or C. The extreme range in the coefficient C for the 18 examples given in Table III is from 0.61 to 1.12. The LB rough average of 0.8 seems about as good an average as any. (It agrees quite well with the calculated values ranging from 0.70 to 0.84 for a Yukawa potential, with $r_0=1.2$, which we believe is the most appropriate set of parameters.) In any case, even a 20 percent change in C changes the integrated cross section $\sigma_{int}=0.015A(1+Cx)$ by only 7 percent. (Here we have taken the value $x=0.5$ for the approximate fraction of exchange force.)

We have here used the simple harmonic oscillator IPM for closed-shell nuclei filling the lowest, first, and second shells. It is very probable that heavier nuclei treated on this model would show quite good agreement with the results found here for Ca^{40}. (The curves for O and Ca in Fig. 3 agree quite closely.) Nuclei intermediate between closed shells might give a somewhat lower value of C, as the interference effects of the P_{ij} term in Eq. (12) would become relatively more important. Preliminary calculations on the B^{10} nucleus with simple harmonic oscillator IPM and Gaussian neutron-proton potential give a value of C about 25 percent lower than that for the closed-shell nuclei He and O. This calculation made in Cartesian coordinates, is not completely reliable, as arbitrary assumptions were made as to the filling of the neutron and proton shells in this coordinate system.

For large u, all the curves in Fig. 3 [Eqs. (16), (17), (18), and (20)] are proportional to u^{-3}. This result corresponds to the treatment without interference [LB, Eq. (14)] for the square-well neutron-proton potential. The u^{-3} dependence for fixed intrinsic range b corresponds to an r_0^{-3} dependence. That is, the average value of $r_{ij}^2V(r_{ij})$ in Eq. (12) is proportional to the mean nucleon density. For small u, the IPM curves are proportional to u^2. For fixed b, small u corresponds to small r_0, or a small nucleus. The finite nuclear size then limits the average value of $r_{ij}^2V(r_{ij})$ to proportionality with r_0^2; i.e., proportionality to u^2. In the LB Fermi gas model, the coefficient is proportional to u for small u. Here LB used an infinite nucleus, the interference effect of the P_{ij} term greatly decreasing the value of the coefficient for small u, though the decrease due to the interference term is not as great as that considered here due to the finite nuclear size.

This discussion for large u and small u is not directly applicable to the actual case of u about 0.6. However the discussion indicates the similarity of the LB Fermi gas treatment and IPM treatment at least in the extreme cases of large u and small u.

Preliminary numerical results by Gell-Mann, Goldberger, and Thirring[3] (GGT) for the integrated *total* photonuclear cross section give an increase of about forty percent over the standard sum-rule result of $0.015A$ Mev-barns. Their numerical result agrees surprisingly well with the LB result (and that of this paper) of $0.8x$, using x about $\frac{1}{2}$. We might speculate briefly that these quite different calculations give similar results because the approximations used are more closely related than appears on the surface. The fundamental approximation of the GGT calculation is the assumption that the forward-scattering amplitude for extremely high-frequency electromagnetic waves is the same for A nucleons bound in a nucleus as for the A free nucleons. (Given this one assumption, the GGT calculation then proceeds using dispersion theory relations between the forward-scattering amplitude and the total photonuclear cross section. The numerical results are based on the difference of pion photoproduction cross sections for free and bound nucleons.) The GGT assumption of the high-frequency forward-scattering amplitude appears to be closely related to the assumption that the nucleus is essentially a nonrelativistic system. (For example, in atomic physics *if* we can calculate the forward-scattering amplitude using the atomic form factor, as is appropriate for a nonrelativistic system for frequencies far from those giving any appreciable absorption, we find that the atomic forward scattering is just that of Z free electrons.) The assumption that a nucleus is essentially a nonrelativistic system is, in turn, needed to justify the several approximations of the LB treatment (or that of this paper): (a) the use of a phenomenological treatment of the meson exchange currents, as in Siegert's theorem;[28] (b) the calculation of electric dipole processes only; and (c) the neglect of retardation in calculation of the matrix element. (Several theorists have suggested that the last two approximations tend to cancel.)

Thus it seems possible that the agreement of the GGT result with that of LB and this paper is caused by both calculations resting on the same basic assumption of neglect of relativistic effects in a nucleus. On the other hand, it is quite possible that any numerical agreement between the GGT and LB calculations is fortuitous, and may disappear completely when more accurate numerical calculations are made.

V. DISCUSSION

In I we concentrated on the calculation of the harmonic mean energy W_H for two nuclei. In this paper Eq. (6) gives $W_H=42A^{-\frac{1}{3}}$ Mev for a simple harmonic oscillator IPM with $r_0=1.2$, and pure ordinary forces. A Serber force gives about 40 percent increase in σ_{int}, and therefore in W_H, giving the value $W_H=60A^{-\frac{1}{3}}$ Mev.

This value of the harmonic mean energy, corrected for exchange effects, is somewhat low when compared with experiment, though it is not in serious disagreement with experimental results.

Since we have shown in Fig. 2 that our calculated bremsstrahlung-weighted cross section σ_b is in reasonable agreement with our preliminary interpretation of photonuclear experiments, the disagreement between the calculated and experimental harmonic mean energies implies a disagreement between the calculated and experimental values of σ_{int}, the experimental values being somewhat higher. Recent summaries,[12,13] give rather good agreement for medium and heavy nuclei between calculated and experimental σ_{int} (using $x=\frac{1}{2}$ for a Serber force). We believe that if the integrations were done up to an energy of 150 Mev (in accord with the GGT calculation) that the experimental cross sections would be somewhat higher than the calculated value for the whole range of nuclei. (See, for example, the Jones-Terwilliger value of σ_{int} for Ta.) Since the high-energy cross sections are still poorly known, we cannot do more than speculate on this question. Neutron-proton correlations, as for example in the quasi-deuteron model which may be a valid approximation for high-energy photons[30] would tend to decrease σ_b and to increase σ_{int}. Both effects would increase W_H.

We believe we have shown that a crude independent-particle model for the nuclear ground state gives results for the bremsstrahlung-weighted cross section σ_b, for the integrated cross section, and for the harmonic mean energy that are not inconsistent with present experimental data. More experimental data are needed, particularly in the energy region from 25 to about 100 Mev. Also an analysis of the experimental data to give the quantities σ_b and W_H as well as σ_{int} would be of use in comparing experiments with sum-rule calculations. The independent-particle model calculation should be refined, and deviations from the IPM (as in the quasi-deuteron model) should be studied.

[30] M. Q. Barton and J. H. Smith, Phys. Rev. **95**, 573 (1954); and Myers, Odian, Stein, and Wattenberg, Phys. Rev. **95**, 576 (1954).

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APPENDIX

Harmonic Mean Energy

Here we derive certain well-known results in a different manner. First we wish to shown the relation between the Heisenberg uncertainty principle and the delta-function shape of the $\sigma(W)$ curve for simple harmonic oscillator wave functions. For simplicity we shall work with only one particle in a simple harmonic oscillator potential.

Using the notation and some equations from LB,[2] we have

$$\sum_n f_{0n}=1, \tag{A1}$$

$$\sum_n (E_n-E_0)f_{0n}=(2/M)(p^2)_{00}, \tag{A2}$$

$$\sum_n f_{0n}/(E_n-E_0)=(2M/\hbar^2)(z^2)_{00}. \tag{A3}$$

Here f_{0n} is the oscillator strength, z is the component of the displacement, and p of the momentum, along the polarization direction. Since $\bar{p}=\bar{z}=\text{zero}$ for the ground (0) state, we write

$$(\Delta p)^2=(p^2)_{00};\quad (\Delta z)^2=(z^2)_{00}. \tag{A4}$$

Using these equations, we find the mean energy $\bar{W}$ and the harmonic mean energy W_H:

$$\bar{W}=\sum_n(E_n-E_0)f_{0n}/\sum_n f_{0n}=(2/M)(\Delta p)^2, \tag{A5}$$

$$W_H=\sum_n f_{0n}/[\sum_n f_{0n}/(E_n-E_0)]=\hbar^2/2M(\Delta z)^2. \tag{A6}$$

For any shape of the photon absorption curve we must have $\bar{W}\geq W_H$. Using this inequality together with Eqs. (A5) and (A6), we find:

$$(2/M)(\Delta p)^2\geq \hbar^2/2M(\Delta z)^2,$$
or
$$(\Delta p)^2(\Delta z)^2\geq \hbar^2/4. \tag{A7}$$

This is a rather indirect method of deriving the Heisenberg uncertainty principle. Further, the equality between $\bar{W}$ and W_H holds only for a delta-function shape of the $\sigma(W)$ curve; while the equality in the Heisenberg uncertainty principle holds only for a simple harmonic oscillator potential. Thus we have related these two properties of simple harmonic oscillator wave functions.

For many particles in a simple harmonic oscillator IPM, we want to show that $W_H=\hbar\omega$. Using the notation of I,[1] we write the harmonic mean energy:

$$W_H=(\hbar^2/2M)A/(D+B), \tag{A8}$$

where the diagonal term is

$$D=(\sum_i z_i^2)_{00}=(\hbar/M\omega)\sum_n(n+\tfrac{1}{2})c_n, \tag{A9}$$

and the off-diagonal term B due to the Pauli principle correlations between protons is given by

$$B=-\sum_k\sum_l(z_{kl})^2. \tag{A10}$$

In these equations n is the quantum number for motion in the z direction; c_n is the number of protons that have quantum number n; and z_{kl} is the dipole matrix element between occupied states k and l. Using the value $(z_{kl})^2=(\hbar/2M\omega)$ (the larger quantum number) where $n_k=n_l\pm 1$, we find that for all but the highest occupied level the contribution to B exactly cancels the contribution to D. For this highest level we have

$$D+B=(n+1)\hbar/2M\omega \text{ per nucleon.} \tag{A11}$$

We must multiply this result by the number of nucleons in the highest level with a given value of n, and sum. (For simplicity we have taken the case where the highest occupied level is completely full.)

$$D+B=(2\hbar/M\omega)\sum_{n=0}^{N}(n+1)(N-n+1)$$
$$=(\hbar/3M\omega)(N^3+6N^2+11N+6). \tag{A12}$$

Here N is the total quantum number for the highest occupied level.

We wish to compare this summation with the mass number A in the numerator of Eq. (A8), expressed as

$$A=2\sum_n(n+1)(n+2)$$
$$=\tfrac{2}{3}(N^3+6N^2+11N+6).$$

We find from Eqs. (A8), (A12), and (A13) that the harmonic mean energy is indeed $\hbar\omega$, as we knew it must be for a simple oscillator.

A completely analogous calculation has shown that the mean energy $\bar{W}$ also equals $\hbar\omega$.