Radiative Corrections to Electron Scattering*

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The techniques and results of a previous paper are applied to the calculation of the second Born approximation of the one-photon radiative corrections to the scattering of electrons by nuclei. Nonrelativistic and high-energy approximations are calculated explicitly for pure Coulomb scattering. Modifications due to the finite extension of the nucleus are discussed. No definite conclusion is reached concerning the change of the correction due to this extension, but it is shown that at extremely high energies the relative radiative correction to the second Born approximation is independent of the nature of the charge distribution. It is also shown that the fictitious zeros of the first Born approximation disappear first in the *third* Born approximation. The shape factor occurring there is briefly discussed.

1. INTRODUCTION

THE nuclear scattering of electrons is of fundamental importance because it is, at high energies, a sensitive tool for probing the charge distribution in the nucleus. It is, moreover, a tool whose properties are relatively well known, since the theoretical interpretation of the experiments is unencumbered by the complications of meson theory. For this reason there has been much recent interest in the high-energy scattering of electrons, experimentally[1–7] as well as theoretically.[8–18] The fact that at today's experimentally accessible energies (of the order of 100 Mev) scattering is beginning to become sensitive to the nuclear charge distribution makes it, of course, imperative to use all the theoretical tools available to describe the differential cross section as precisely as possible according to the present theory. That means that the quantum electrodynamic corrections to the scattering cross section ought to be taken into account whenever known, unless they are shown to be negligible.

The lowest-order radiative correction to the relativistically modified Rutherford scattering cross section was first derived by Schwinger.[19] It is a first Born approximation (hereafter abbreviated as BA) with respect to the interactions both with the radiation field and with the Coulomb potential, i.e., it assumes separately both $\alpha \ll 1$ and $Z\alpha \ll 1$ (where α is the fine structure constant and Z, the nuclear charge number). The second of these assumptions is no longer valid as the heavier elements are approached for scattering targets, and better approximations are required.

The present calculation is one of the next higher order compared to Schwinger's in the sense that it retains the next power of $Z\alpha$. It is a first Born approximation with respect to the radiation field and its relevant parameter α; that is to say that the emission and reabsorption of never more than one photon is taken into account; a second BA as far as the scattering potential and its relevant parameter $Z\alpha$ is concerned, i.e., at most two virtual interactions with the static potential are contemplated. In other words it will yield corrections of the order $Z\alpha$ to Schwinger's result, or of the order α to the result of McKinley and Feshbach.[9] For light elements, $Z\sim 1$, this correction is, of course, very small (at energies that are neither extremely small nor extremely large), while for very heavy elements the expansion in powers of $Z\alpha$ becomes quite useless, even if it should still be correct. It is for elements of intermediate position, say in the neighborhood of $Z=35$, that one may expect such corrections to be appreciable and the expansion still to be useful (perhaps it is too much to hope that it converges too?).

Section 2 will be concerned with the calculation of the elastic cross section, in (2a) of the terms arising from the mass operator, in (2b) of those from the vacuum polarization. The notation developed in (2a) will be carried through from then on. Section 3 deals with the contribution from the inelastic cross section, necessary for the removal of infrared divergencies. In Sec. 4 the infrared and the Coulomb divergencies are shown to cancel and the integration over the Fourier-transformed Coulomb fields is performed. We then make the two essential approximations: in Sec. 5, the nonrelativistic approximation, and in Sec. 6, the extreme relativistic case. Both are discussed in their respective sections. There are five appendices: A lists

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the traces necessary in the work; B rederives the second BA, including the form factor in integral form; C performs the integration over the Fourier transform variables of the Coulomb field; D treats the integrals occurring in the form factors of the second and third BA; and E elaborates on the statement that, where the first BA vanishes, the first nonzero term occurs in the *third* BA. The appropriate form factors for the third BA is exhibited in that appendix.

The technique used in this calculation is based on Schwinger's "mass operator"[20] method and has been developed in as much detail as is needed for the present in a previous paper.[21] Rather than repeat the preliminaries here, we refer the reader who is unfamiliar with the technique to references 20 and I.

The cross section for scattering from the initial momentum p to the final momentum q is given by I (3.14) as[22]

$$d\sigma/d\Omega=2\pi^4(|\mathbf{q}|/|\mathbf{p}|)\,\mathrm{tr}(m-\gamma p)(\mathbf{p}|H|\mathbf{q})\times(m-\gamma q)(\mathbf{q}|\gamma_0H^\dagger\gamma_0|\mathbf{p}) \quad (1.1)$$

where

$$H=(1+\mathcal{H}G_0)^{-1}\mathcal{H}, \quad (1.2)$$

G_0 is the free-particle Green's function, and

$$\mathcal{H}=\Delta M-e\gamma A \quad (1.3)$$

with the renormalized mass operator ΔM.

2. ELASTIC CROSS SECTION

The radiative corrections to the elastic cross section are obtained from (1.1) by setting $|\mathbf{p}|=|\mathbf{q}|$ and

$$\mathcal{H}=-e\gamma A-e\gamma A'+\Delta M, \quad (2.1)$$

where A is the static scattering potential; A', the vacuum polarization potential induced by A, and ΔM the mass operator as a function of A. Since we are calculating the first BA in the radiation field, all terms in the expansion of H in powers of $\mathcal{H}$ are dropped except those that are linear in either ΔM or $\gamma A'$. Among the ones thus retained ΔM and A' too are expanded in powers of A. Then all terms containing more than a total of two occurrences of A are discarded, since this is a second BA in the external field. In view of the facts that the cross section, Eq. (1.1), is quadratic in H, and that the vacuum polarization potential is an odd function of the inducing field A and hence for our purposes proportional to it, we may write the remaining terms schematically as follows:

$$(d\sigma/d\Omega)_2\approx[-e\gamma A-e\gamma A'+e\Delta M_1+e^2\Delta M_2 -e^2\gamma AG_0\gamma A-e^2\gamma A'G_0\gamma A-e^2\gamma AG_0\gamma A' +e^2\gamma AG_0\Delta M_1+e^2\Delta M_1G_0\gamma A]^2. \quad (2.2)$$

[20] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452, 455 (1951).
[21] R. G. Newton, Phys. Rev. **94**, 1773 (1954). We shall refer to this paper as I.
[22] "Natural units" will be used, with $\hbar=c=1$, and $\alpha=e^2/4\pi$. Bold-face type indicates three-vectors, while four-vectors are not specially marked. When the latter's indices are not suppressed, the usual summation convention is, of course, in force.

When the square is carried out, one obtains: (1) the lowest-order term, which yields the relativistically corrected Rutherford formula; (2) the terms with three powers of γA, which yield the ordinary second BA, whose value was first derived by McKinley and Feshbach[9] and Schwinger, and which will be easy to check with the present tools (see Appendix B); (3) the terms with one γA in addition to either $\gamma A'$ or ΔM_1, which, as was shown in I, yield Schwinger's result; (4) the following terms which are the ones to be calculated here:

$$-e^3[(\gamma A)(\Delta M_2+\gamma AG_0\Delta M_1+\Delta M_1G_0\gamma A) +(\gamma AG_0\gamma A)(\Delta M_1)+\circlearrowleft]+e^3[(\gamma A')(\gamma AG_0\gamma A) +(\gamma A)(\gamma A'G_0\gamma A+\gamma AG_0\gamma A')+\circlearrowleft]. \quad (2.3)$$

a. Mass-Operator Terms

We shall consider the first line of (2.3) in this subsection. It was shown in I[23] that $\gamma AG_0\Delta M_1+\Delta M_1G_0\gamma A$ cancels a distinct part of ΔM_2, there called $\Delta M_2^{2'}$. Therefore

$$(d\sigma/d\Omega)_{2M}=-4\pi^4e^3\,\mathrm{Re}\,\mathrm{tr}(m-\gamma p)(\mathbf{p}|\Delta M_2-\Delta M_2^{2'}|\mathbf{q}) \times(m-\gamma q)(\mathbf{q}|\gamma A|\mathbf{p})-4\pi^4e^3\,\mathrm{Re}\,\mathrm{tr}(m-\gamma p) \times(\mathbf{p}|\Delta M_1|\mathbf{q})(m-\gamma q)(\mathbf{q}|\gamma AG_0\gamma A|\mathbf{p}). \quad (2.4)$$

Matrix elements of the second-order mass operator, given by I, Eqs. (2.10) to (2.17), are obtained via the simple equation:

$$\int\frac{(dk_1)^3}{(2\pi)^{\frac{3}{2}}}\int\frac{(dk_2)^3}{(2\pi)^{\frac{3}{2}}}(\mathbf{p}|e^{i\mathbf{k}_1\cdot\mathbf{r}}\mathfrak{M}_2(\mathbf{k}_1,\mathbf{k}_2,\mathbf{p}_{op})e^{i\mathbf{k}_2\cdot\mathbf{r}}|\mathbf{q}) =(1/8)m^3\lambda^3\int(dk/2\pi)^3\mathfrak{M}_2(m\lambda(\mathbf{n}+\tfrac{1}{2}\mathbf{k}), m\lambda(\mathbf{n}-\tfrac{1}{2}\mathbf{k}),\,\mathbf{p}+\mathbf{q}-m\lambda\mathbf{k}), \quad (2.5)$$

where the substitution $\mathbf{k}_1-\mathbf{k}_2=m\lambda\mathbf{k}$, was made, and

$$\boldsymbol{\lambda}\equiv(\mathbf{p}-\mathbf{q})/2m\equiv\lambda\mathbf{n}, \quad (2.6)$$

$$\lambda\equiv(|\mathbf{p}|/m)\sin\tfrac{1}{2}\vartheta, \quad (2.7)$$

ϑ being the scattering angle, $\mathbf{p}\cdot\mathbf{q}\equiv\mathbf{p}^2\cos\vartheta$.

The matrix element of γA is immediate:

$$(\mathbf{q}|\gamma A|\mathbf{p})=(2\pi)^{-\frac{3}{2}}\gamma A(-2m\boldsymbol{\lambda}), \quad (2.8)$$

where the A on the right-hand side is the Fourier transform of the one on the left.

Equation (2.4) can thus be written

$$(d\sigma/d\Omega)_{2M}=-2^{-8}\pi^{-5}e\alpha^2m^3\lambda^3p_0A_0(-2m\boldsymbol{\lambda}) \times\int(dk)^3A_0(m\lambda\mathbf{k}_1)A_0(m\lambda\mathbf{k}_2)\Gamma(\mathbf{k},p_0,\vartheta)_{2M}, \quad (2.9)$$

[23] Equation (2.19).

and its first line yields

$$\Gamma_{2M1}=\mathrm{Im}\int_0^\infty ds\int_0^1 du\,\exp(-isum^2)$$
$$\times\frac{1}{4}\frac{\mathrm{tr}\gamma_0(\gamma p-m)(\mathfrak{M}_2-\mathfrak{M}_2{}^{2'})(m-\gamma q)}{p_0A_0(m\lambda\mathbf{k}_1)A_0(m\lambda\mathbf{k}_2)},\quad(2.10)$$

where the arguments in $\mathfrak{M}_2-\mathfrak{M}_2{}^{2'}$ are those indicated in (2.5) and we have collected all factors of (2π), so that at this point we mean

$$A(\mathbf{k})\equiv\int(d\mathbf{r})A(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}.\quad(2.11)$$

Also,

$$\mathbf{k}_1\equiv\mathbf{n}+\tfrac{1}{2}\mathbf{k},\quad \mathbf{k}_2\equiv\mathbf{n}-\tfrac{1}{2}\mathbf{k}.\quad(2.12)$$

Use has been made of a gauge in which all components but A_0 vanish.

Before calculating Γ_{2M1} we may now put (2.9) into somewhat more transparent form. No assumption concerning the special external field has as yet been made, except that it be a static, spherically symmetric electric field. Suppose that this field is due to a spherically symmetric charge distribution of finite extension R:

$$\begin{aligned}J_0(\mathbf{r})&=-(Ze/4\pi)\rho(r/R)R^{-3},\quad &r<R\\&=0,&r>R,\end{aligned}\quad(2.13)$$

where ρ is normalized so that

$$\int_0^1 v^2dv\rho(v)=1.\quad(2.14)$$

Then

$$A_0(\mathbf{k})=-Zek^{-2}\int_0^1 dvv^2\rho(v)\left(\frac{\sin vkR}{vkR}\right),\quad(2.15)$$

where $k=|\mathbf{k}|$. For such an arbitrary spherically symmetric charge distribution the scattering cross section may be written

$$(d\sigma/d\Omega)_2=\mathfrak{R}S_1[S_1(1-\beta^2\sin^2\vartheta/2)(1-\delta_1)+B_2(S_2-\delta_2)].\quad(2.16)$$

Here

$$\mathfrak{R}=\left[\frac{\hbar}{mc}\cdot\frac{Z\alpha(1-\beta^2)^{\frac{1}{2}}}{2\beta^2}\csc^2\tfrac{1}{2}\vartheta\right]^2\quad(2.17)$$
$$(\beta=v/c)$$

is the Rutherford cross section;

$$B_2=\pi Z\alpha\beta\sin\tfrac{1}{2}\vartheta(1-\sin\tfrac{1}{2}\vartheta)\quad(2.18)$$

is the second BA[9] to pure Coulomb scattering;

$$S_1=\int_0^1 dvv^2\rho(v)\left(\frac{\sin 2\lambda vK}{2\lambda vK}\right)\quad(2.19)$$

is the "form factor" for an arbitrary nuclear charge distribution,

$$K\equiv\frac{\text{nuclear radius}}{\text{electron's Compton wavelength}};\quad(2.20)$$

S_2 is the special form factor for B_2 which we shall be able to write down later (B.7); δ_1 is the first radiative correction;[19] and δ_2 is the correction we are presently concerned with:

$$\delta_2=(\alpha/8\pi^2)(1-\sin\tfrac{1}{2}\vartheta)^{-1}\int_0^1 dv_1v_1^2\rho(v_1)\int_0^1 dv_2v_2^2\rho(v_2)$$
$$\times\int[dk]\left(\frac{\sin v_1\lambda Kk_1}{v_1\lambda Kk_1}\right)\left(\frac{\sin v_2\lambda Kk_2}{v_2\lambda Kk_2}\right)\Gamma(\mathbf{k},p_0,\vartheta),\quad(2.21)$$

$$\int[dk]f(\mathbf{k})\equiv\tfrac{1}{2}\pi^{-2}\int(dk)^3k_1^{-2}k_2^{-2}f(\mathbf{k}).\quad(2.22)$$

Because of the normalization (2.14), the limits as we approach a point nucleus, or very low energies $(K\lambda\to 0)$, are

$$S_1\to 1,$$
$$\delta_2\to(\alpha/8\pi^2)(1-\sin\tfrac{1}{2}\vartheta)^{-1}\int[dk]\Gamma.\quad(2.21\text{a})$$

We may now return to the contribution to Γ from the first line of (2.4), Γ_{2M1}. Equations I (2.14), I (2.16), and I (2.17) are substituted in (2.10). The s-integration is then easily carried out. The following type of integral deserves special mention:

$$\mathrm{Im}\int_0^\infty ds\,\exp(-isum^2)\cdots G_0\cdots$$
$$=\mathrm{Im}\int_0^\infty ds\,\exp(\quad)\cdots\mathrm{Re}G_0\cdots$$
$$+\mathrm{Re}\int_0^\infty ds\,\exp(\quad)\cdots\mathrm{Im}G_0\cdots.\quad(2.23)$$

Terms like the second one on the right-hand side yield $\cdots\delta(u)$. But as discussed in I, Sec. IV A, $\exp(-isum^2)$ is to be replaced by $\exp[-isum^2(1+u^{-2}(\epsilon/m)^2)]$, where ϵ is a small photon mass, in order to cut off infrared divergencies. If this replacement is made consistently, then $\delta(u)\to\delta(u+(\epsilon/m)^2u^{-1})=0$ and such terms do not contribute.[24] Integrations over the δ function starting at zero may frequently be considered spurious because they are quite discontinuous functions of their lower limit.

[24] Another way of deriving the same result is the following: The factor u^{-1} stems from the proper time integration of $\exp(-isum^2)$, which originates from the zero order Green's function. The definition of the +-Green's function implies that m^2 should be replaced by $m^2-i\epsilon$. The real part of the proper time integration in (2.12) therefore does not result in $m^{-1}\delta(u)$, but in $u^{-1}\delta(m^2)=0$.

The traces needed in (2.10) are listed in Appendix A. In terms from $\mathfrak{M}_2^3$ we may conveniently change variables according to $v=v_1-v_2$, $w=\frac{1}{2}(v_1+v_2)/(1-v)$. The operation (2.5) then results in the following replacements [in the notation of I (2.8)]:

$$\not{p}E\exp(-isum^2)\to\exp(-ism^2\eta_1), \quad (2.24)$$

$$\eta_1\equiv\lambda^2v(1-u)\eta+u[1+\lambda^2(1-v)^2(1-w^2)+\lambda^2v(1-v)(1+\tfrac{1}{4}k^2)-wv(1-v)\mathbf{n}\cdot\mathbf{k}], \quad (2.25)$$

$$\eta\equiv-1-\mathbf{Q}\cdot\mathbf{k}+\tfrac{1}{4}k^2, \quad (2.26)$$

$$\mathbf{Q}\equiv(\mathbf{p}+\mathbf{q})/2m\lambda,\quad Q=\cot\tfrac{1}{2}\vartheta, \quad (2.27)$$

$$\bar{\psi}\exp(-isum^2)\to\exp(-ism^2\eta_2), \quad (2.28)$$

$$\eta_2\equiv u[1+4\lambda^2v(1-v)]+(\epsilon/m)^2u^{-1}, \quad (2.29)$$

$$\psi E\exp(-isum^2)\to\tfrac{1}{2}[\exp(-isum^2\eta_3{}^+)+\exp(-isum^2\eta_3{}^-)], \quad (2.30)$$

$$\eta_3{}^\pm\equiv\lambda^2v(1-u)\eta+u[1+\lambda^2v(1-v)\mathbf{k}_{1,2}{}^2], \quad (2.31)$$

$$E\exp(-isum^2)\to\exp(-isum^2\eta_4), \quad (2.32)$$

$$\eta_4\equiv\lambda^2v(1-u)\eta+u. \quad (2.33)$$

Somewhat lengthy calculation, involving partial integrations of the kind shown below I (2.9), yields the following contributions to Γ (where the u-integration is understood everywhere to extend from zero to one):

$$\Gamma_a=\int_0^1dv(1-v)\int_{-1}^1\tfrac{1}{2}dw\eta_1{}^{-2}\{\lambda^4\mathbf{Q}\cdot\mathbf{k}[4(u-2)+2uv(3-2u)^2-\tfrac{1}{2}uvk^2+8u(1-u)^2v\gamma]+\lambda^2[16(1-u)(2-u)\gamma\lambda^2-(1+u)k^2+4(1+5u-8u^2+4u^3)+16(1-u)(1-u^2)\gamma]\}+4\int_0^1dv(1-v)\int_{-1}^1\tfrac{1}{2}dw\eta_1{}^{-1}\{4u(1-u)\gamma\lambda^2-2u(1+u)-u^2v\lambda^2\mathbf{Q}\cdot\mathbf{k}\}, \quad (2.34)$$

$$\Gamma_b=4(1+u)\int_0^1dv\eta_2{}^{-1}, \quad (2.35)$$

$$\Gamma_c=2\lambda^2\int_0^1dv\eta_3{}^{-1}[\mathbf{Q}\cdot\mathbf{k}(2-u+u(1-u)(1-2v))-4u(1-u)\gamma], \quad (2.36)$$

$$\Gamma_d=2\int_0^1dv\eta_3{}^{-1}\eta^{-1}\{4(1-u^2)(4\gamma+1-\tfrac{1}{4}k^2)+k_1{}^2[4u(1-u)+(4\gamma-\mathbf{Q}\cdot\mathbf{k})\lambda^2(2-u-u^2(1-2v)^2)]\}, \quad (2.37)$$

$$\Gamma_e=4(1-u^2)\int_0^1dv\eta_4{}^{-1}\eta^{-1}(\tfrac{1}{4}k^2-1-4\gamma), \quad (2.38)$$

$$\Gamma_f=4\frac{u(1-u^2)}{u^2+(\epsilon/m)^2}(\mathbf{Q}\cdot\mathbf{k}-4\gamma)\eta^{-1}, \quad (2.39)$$

where

$$\gamma=\lambda^{-2}+\csc^2\tfrac{1}{2}\vartheta-1. \quad (2.40)$$

The first-order mass operator contributions, the second line of (2.4), are calculated in the same manner as those of the second-order one. Terms containing $\mathrm{Im}G_0$ are dropped by the same reasoning as previously. The result is the following contribution to Γ:

$$\Gamma_g=4\int_0^1dv\eta^{-1}\left\{\frac{4(1-u)}{1+\lambda^2(1-v^2)}+\lambda^2(\mathbf{Q}\cdot\mathbf{k}-4\gamma)\left[\frac{1+uv^2-2u(u^2+(\epsilon/m)^2)^{-1}}{1+\lambda^2(1-v^2)}-\frac{2v^2[u-u(u^2+(\epsilon/m)^2)^{-1}]}{[1+\lambda^2(1-v^2)]^2}\right]\right\}. \quad (2.41)$$

b. Vacuum Polarization Terms

This subsection is concerned with the contributions from the second line of (2.3):

$$(d\sigma/d\Omega)_v=4\pi^4e^3\,\mathrm{Re}\,\mathrm{tr}(m-\gamma p)\times(\mathbf{p}|\gamma AG_0\gamma A|\mathbf{q})(m-\gamma q)(\mathbf{q}|\gamma A'|\mathbf{p})+4\pi^4e^3\,\mathrm{Re}\,\mathrm{tr}(m-\gamma p)(\mathbf{p}|\gamma A'G_0\gamma A+\gamma AG_0\gamma A'|\mathbf{q})(m-\gamma q)(\mathbf{q}|\gamma A|\mathbf{p}). \quad (2.42)$$

The lowest-order vacuum polarization potential is, after charge normalization is carried out,[25]

$$A_\mu'(\mathbf{k})=A_\mu(\mathbf{k})f(k), \quad (2.43)$$

$$f(k)=\frac{\alpha}{4\pi}\int_0^1dvv^2(1-\tfrac{1}{3}v^2)[m^2+\tfrac{1}{4}k^2(1-v^2)]^{-1}. \quad (2.44)$$

Equation (2.8) then shows that the contribution due to the first line of (2.42) is simply $f(-2m\lambda)$ times the second BA:

$$f(-2m\lambda)=(\alpha/\pi)\lambda^2(F_1-\tfrac{1}{3}F_2),$$

where the functions F_n are those used by Schwinger[19]:

$$F_n=\int_0^1dv\frac{v^{2n}}{1+\lambda^2(1-v^2)}, \quad (2.45)$$

$$F_0=\lambda^{-1}(1+\lambda^2)^{-\frac{1}{2}}\log[(1+\lambda^2)^{\frac{1}{2}}+\lambda],\quad F_1=(1+\lambda^{-2})F_0-\lambda^{-2},\quad\text{etc.} \quad (2.46)$$

The second line of (2.42) is easily seen to contribute

$$f(m\lambda k_1)+f(m\lambda k_2),$$

operated on by the $\mathbf{k}$-integral that yields the second BA. Appendix B is devoted to the simple calculation of that and we may take the result from there. It

[25] J. Schwinger, Phys. Rev. **82**, 678 (1951); the renormalization is accomplished as indicated in I (6.10).

follows that

$$\Gamma_v=4\lambda^2\frac{(\mathbf{Q}\cdot\mathbf{k}-4\gamma)}{\eta}\left[F_1-\tfrac{1}{3}F_2+\tfrac{1}{2}\int_0^1 dv v^2(1-\tfrac{1}{3}v^2)\frac{k_1^2}{1+\lambda^2k_1^2(1-v^2)/4}\right]. \quad (2.47)$$

3. INELASTIC CROSS SECTION

As in the first-order calculation, the radiative correction to the elastic cross section contains an infrared divergence which cancels a corresponding one in the bremsstrahlung cross section. We therefore again have to add the "slightly inelastic" cross section, i.e., the low energy limit of the one-quantum bremsstrahlung cross section. It was shown in I, Sec. V, that this is, to all orders in the external field, a multiple of the elastic cross section, the factor being

$$\frac{\alpha}{2\pi^2}\int_{k_0=0}^{\Delta E}(dk)^4\delta(k^2)\left(\frac{p}{pk}-\frac{q}{qk}\right)^2. \quad (3.1)$$

As shown in I, Schwinger's evaluation of the integral (2.48) is modified in the presence of a small photon mass ϵ to yield

$$\int_{k_0=0}^{k_0=\Delta E}(dk)^4\delta(k^2)\left(\frac{p}{pk}-\frac{q}{qk}\right)^2=4\pi(|\mathbf{p}|/m)^2\sin^2\tfrac{1}{2}\vartheta \times[(F_0+F_1)\log(\Delta Em/\epsilon p_0)+F_1+\tfrac{1}{2}G+H], \quad (3.2)$$

[Eq. (5.11) in I contains two typographical errors which are corrected by (3.2)], with the functions H and G defined in reference 19.

The result of Appendix B for the second Born approximation therefore yields for the inelastic contribution to Γ:

$$\Gamma_h=8\lambda^2[(F_0+F_1)\log(\Delta Em/\epsilon p_0)+F_1+\tfrac{1}{2}G+H](\mathbf{Q}\cdot\mathbf{k}-4\gamma)\eta^{-1}. \quad (3.3)$$

4. FIRST INTEGRATION, AND TREATMENT OF THE DIVERGENCIES

Two kinds of divergencies appear in Γ. One is the infrared divergence which also appears in the first-order calculation and for the sake of which the slightly inelastic cross section was introduced. In Γ, as in the first-order terms, it appears as a divergence at $u=0$ and is treated as discussed in I, Sec. IV A. The introduction of a finite photon mass, ϵ, enables us to replace every u^{-1} by $u[u^2+(\epsilon/m)^2]^{-1}$. The u-integration can then be carried out without divergence difficulties.

The other divergence is also a familiar one due to the slow decrease of the Coulomb field at infinity. Since $\mathbf{k}$ is essentially the Fourier conjugate variable to the distance $\mathbf{r}$, this divergence appears at small $\mathbf{k}$; more precisely, at $\mathbf{k}_1$ or $\mathbf{k}_2$ equal to zero, since these variables belong separately to Coulomb fields. The customary way of dealing with such divergencies is to "screen" the Coulomb field, i.e., to replace k^{-2} by $(k^2+\mu^2)^{-1}$. This means that the Coulomb potential is replaced by one of the Yukawa type. The latter is very nearly like the Coulomb potential near the origin, where most of the scattering occurs, but it falls off exponentially at large distances. The physical justification for this procedure is, of course, the fact that the nuclear scattering centers are surrounded by electrons and that, therefore, outside the atom no field exists at all. The Coulomb field is thus physically screened, and the screening parameter, μ, is related to the size of the atom. Most of the scattering occurs deep inside the atom, where a small μ has little influence; the screening parameter is therefore retained only in terms which do not vanish in the limit as μ tends to zero. Since the mathematical reason for the introduction of μ was the occurrence of divergencies, there will be terms left which tend to infinity (logarithmically) as μ tends to zero. In these, one would give μ the value physically determined by the atomic size.

However, it will be shown that all terms in Γ which would tend to infinity as μ tends to zero cancel each other. There is therefore no reason why the divergent integrals should be cut off in such a relatively complicated, though physically realistic manner. In the case of cancellation the only purpose of the screening process is the proper "fitting" of various divergent integrals. This fitting can be accomplished in a much simpler manner in the present case.

The Coulomb divergencies arise in terms of the following nature[26]:

$$\int(dk)k_1^{-2}k_2^{-2}\eta^{-1}.$$

It is easily seen (for example, by a shift of $\mathbf{k}$ by $2\mathbf{n}$) that this diverges when either $\eta=\mathbf{k}_1=0$, or $\eta=\mathbf{k}_2=0$, and nowhere else. If η is therefore prevented from having zeros coinciding with $\mathbf{k}_1$ or $\mathbf{k}_2$, the integral will be finite. Now the origin of η^{-1} was $(m^2+p^2)^{-1}$ before matrix elements were taken, that is, from a zero-order Green's function. The definition of the outgoing wave Green's function, which is the proper one to use here, includes an addition of $-i\epsilon$ to the mass. Therefore η should properly be replaced by $\eta-i\epsilon$. As long as this ϵ remains different from zero, η cannot have any real roots and therefore the above integral will be finite. The Green's function ϵ is therefore a sufficient means of cutting off the Coulomb divergencies without screening until the terms diverging as ϵ tends to zero are found to cancel. Then, of course, ϵ will be allowed to vanish. That ϵ does not have the same physical reality as a screening parameter is of no consequence, since the latter would not have remained in the work anyway.

[26] The notation is that of (2.12) and (2.26).

The outlined procedure is substantially simpler than the customary screening.

Let us examine the above outlined situation somewhat more precisely. After the replacement $\eta\to\eta-i\epsilon$ has been made, and the **k**-integration carried out, only the imaginary part of the result, which is of no interest to us anyway, will diverge. The real part, i.e., the *principal part* of the integral, has a definite finite value. The real divergences occur in products such as $\eta^{-1}\eta_3^{-1}$, $\eta^{-1}\eta_4^{-1}$, and η_1^{-2}. Since $\eta_3=\eta_4=\eta_1=\eta$ when $u=0$, the real infinities will occur in the form of divergencies at $u=0$, just as the infrared ones. Thus, *the Coulomb field produces divergence difficulties only in the no-radiation limit.*

The term Γ_a requires special preparation. That part which contains η_1^{-1} is convergent. Equation (2.15) shows that near $v=0$, η_1 is of the form $au+bv$, and

$$\int_0^1 du(au+bv)^{-1}=\frac{1}{a}\log\left(\frac{a+bv}{bv}\right).$$

This, although tending to infinity as v tends to zero, will not diverge when integrated over v down to zero. The part containing η_1^{-2}, however, diverges logarithmically at $u=0$. In order to separate the divergence, η_1^{-2} is replaced by $(\eta_1^{-2}-\eta_1'^{-2})+\eta_1'^{-2}$, where

$$\eta_1'=\lambda^2 v(1-u)\eta+u[1+\lambda^2(1-w^2)]. \qquad (4.1)$$

The difference will no longer diverge, since $\eta_1\to\eta_1'$ as $v\to 0$. The v-integration over the last term is carried out immediately:

$$\begin{aligned}\lambda^2(1-u)&\int_0^1 dv\eta'^{-2}\\ &=-\eta^{-1}\{[u+\lambda^2u(1-w^2)+\lambda^2(1-u)\eta]^{-1}\\ &\quad-[u+\lambda^2u(1-w^2)]^{-1}\}\to\eta^{-1}[u(1-w^2)\lambda^2+u\\ &\quad+(\epsilon/m)^2u^{-1}]^{-1}+\lambda^2u^{-1}[1+\lambda^2(1-w^2)]^{-1}\\ &\quad\times\left(\frac{1-u}{(1-u)\eta\lambda^2+u[1+\lambda^2(1-w^2)]}-\frac{\lambda^{-2}}{\eta}\right).\end{aligned}$$

The w-integral over the first part is then easily carried out and the result is that

$$\begin{aligned}(1-u)\eta_1^{-2}\to&(1-u)(\eta_1^{-2}-\eta_1'^{-2})+u^{-1}[1+\lambda^2(1-w^2)]^{-1}\\ &\times\left(\frac{1-u}{\lambda^2(1-u)\eta+u+u\lambda^2(1-w^2)}-\frac{\lambda^{-2}}{\eta}\right)\\ &+\eta^{-1}\left[\frac{1}{2}\frac{G+2F_1}{1+2\lambda^2}+\frac{F_0}{\lambda^2}\log(m/\epsilon)\right]. \qquad (4.2)\end{aligned}$$

This replacement is, of course, necessary only in terms without a factor of u or v.

Equation (4.2) results in the following split up of Γ_a, (2.34):

$$\Gamma_k=4\lambda^2\int_0^1 dv\int_{-1}^1\frac{dw}{2}\left(\frac{1-v}{\eta_1^2}-\frac{1}{\eta_1'^2}\right)(1-u)\xi, \qquad (4.3)$$

$$\Gamma_l=4\lambda^2\int_0^1\frac{dw}{u}[1+\lambda^2(1-w^2)]^{-1}\times\left(\frac{1-u}{\lambda^2(1-u)\eta+u+u\lambda^2(1-w^2)}-\frac{\lambda^{-2}}{\eta}\right)\xi, \qquad (4.4)$$

$$\Gamma_m=4[F_0\log(m/\epsilon)+\lambda^2(1+2\lambda^2)^{-1}(F_1+\tfrac{1}{2}G)]\xi\eta^{-1}, \qquad (4.5)$$

$$\xi=1+4\gamma-\tfrac{1}{4}k^2+2\lambda^2(4\gamma-\mathbf{Q}\cdot\mathbf{k}), \qquad (4.6)$$

$$\Gamma_n=\lambda^4\int_0^1 dv(1-v)\int_{-1}^1\tfrac{1}{2}dwu\eta_1^{-2}\mathbf{Q}\cdot\mathbf{k}[2v(3-2u)^2-4-\tfrac{1}{2}vk^2+8(1-u)^2v\gamma], \qquad (4.7)$$

$$\Gamma_o=2\lambda^2\int_0^1 dv(1-v)\int_{-1}^1\tfrac{1}{2}dwu\eta_1^{-2}[4(3-4u+2u^2)-k^2-8(1-u)\lambda^2\gamma-8u(1-u)\gamma], \qquad (4.8)$$

$$\Gamma_p=4\int_0^1 dv(1-v)\int_{-1}^1\tfrac{1}{2}dwu\eta_1^{-1}[4\lambda^2\gamma(1-u)-2(1+u)-uv\lambda^2\mathbf{Q}\cdot\mathbf{k}]. \qquad (4.9)$$

Γ_m contains an infrared divergency and Γ_l, a Coulomb divergency. The latter, which is not apparent on the surface, arises from the fact that as $u\to 0$,

$$\lim\int[dk][u+u\lambda^2(1-w^2)+\lambda^2(1-u)(\eta-i\epsilon)]^{-1}\neq\int[dk](\eta-i\epsilon)^{-1}\lambda^{-2}.$$

We are now ready to collect the infrared divergencies occurring in Γ:

$$\Gamma_b^I=4\int_0^1 dv\eta_2, \qquad (4.10)$$

$$\Gamma_f^I=-4\frac{u}{u^2+(\epsilon/m)^2}[1+(1-\tfrac{1}{4}k^2+4\gamma)\eta^{-1}], \qquad (4.11)$$

$$\Gamma_g^I=-8\int_0^1 dv\lambda^2(\mathbf{Q}\cdot\mathbf{k}-4\gamma)\eta^{-1}\frac{u}{u^2+(\epsilon/m)^2}\times[(1+\lambda^2(1-v^2))^{-1}-v^2(1+\lambda^2(1-v^2))^{-2}], \qquad (4.12)$$

$$\Gamma_h^I=8\lambda^2(\mathbf{Q}\cdot\mathbf{k}-4\gamma)\eta^{-1}(F_0+F_1)\log(\Delta Em/\epsilon p_0), \qquad (4.13)$$

$$\Gamma_m^I=4F_0\log(m/\epsilon)[(1+2\lambda^2)(1+4\gamma-\tfrac{1}{4}k^2)\eta^{-1}+2\lambda^2]. \qquad (4.14)$$

The u- and v-integrations are readily carried out and all terms containing $\log\epsilon$ are seen to cancel each other.

We may now let the photon mass ϵ vanish with impunity, and the result remains unchanged.

Before the Coulomb divergencies can be cancelled, the **k**-integration has to be carried out. According to Appendix C ($\mathbf{B}\cdot\mathbf{n}=0$):

$$\int[dk](A+\mathbf{B}\cdot\mathbf{k}+C\mathbf{n}\cdot\mathbf{k}+\tfrac{1}{4}Dk^2)^{-1}=2[4B^2+(A-D)^2]^{-\frac{1}{2}}\times\cos^{-1}\left(1-2\frac{(A-D)^2+4B^2}{(A+D)^2-4C^2}\right) \quad (4.15)$$

for $AD>(B^2+C^2)$. In the case of η, $C=0$, $D=1$, $B^2=Q^2$, $A=-1+a$, where a is to tend to zero via $-i\epsilon$. For $a>1+Q^2\equiv\theta^2$, (4.15) holds directly. As $a\rightarrow<\theta^2$ the $\cos^{-1}$ becomes

$$\pi+i\log 16\theta^2a^{-2}.$$

If $a=\epsilon e^{i\varphi}$ and φ is allowed to change from zero to $-\pi/2$, then

$$\pi+i\log(16\theta^2\epsilon^{-2}e^{-2i\varphi})=\pi+2\varphi+i\log 16\theta^2\epsilon^{-2}\rightarrow i\log 16\theta^2\epsilon^{-2}.$$

Therefore, the real part of $\int[dk]\eta^{-1}$, which is the only one we are interested in, vanishes. (Notice that the sign of the imaginary part is fixed to be positive by the positive sign of ϵ.)

By the same kind of analytic continuation, the other integrals involving η are obtained from the results of Appendix C. [Notation: (2.22)]:

$$\int[dk]\eta^{-1}=0, \quad (4.16)$$

$$\int[dk]\tfrac{1}{4}k^2\eta^{-1}=2\pi(1+Q^2)^{-\frac{1}{2}}, \quad (4.17)$$

$$\int[dk]\frac{1}{\eta}\frac{1}{1+bk_1^2}=-\frac{4b}{(1+4b)(1+Q^2)^{\frac{1}{2}}}\cot^{-1}2[b(1+Q^2)]^{\frac{1}{2}}, \quad (4.18)$$

$$\int[dk]\frac{1}{\eta}\frac{k^2/4}{1+bk_1^2}=\frac{4}{(1+Q^2)^{\frac{1}{2}}}\frac{1+3b}{1+4b}\cot^{-1}2[b(1+Q^2)]^{\frac{1}{2}}. \quad (4.19)$$

By the use of Appendix C it is found that

$$\int[dk]\Gamma_l=16\int_0^1dw\int_0^\infty\frac{dU}{U}\frac{1}{\omega+\lambda^2U}\times\left[\frac{(1+2\lambda^2)(1+2\gamma)-\lambda^2U}{(4\theta^2+U^2-4U)^{\frac{1}{2}}}\cos^{-1}\left(8\frac{U-i\epsilon-\theta^2}{(U-i\epsilon)^2}-1\right)-\frac{1+2\lambda^2}{\theta}\cos^{-1}\left(1-2\frac{\theta^2}{U-i\epsilon}\right)-(\text{value at }U=0)\right], \quad (4.20)$$

$$\theta^2\equiv1+Q^2=\csc^2\tfrac{1}{2}\vartheta, \quad (4.21)$$

where $U=\omega\lambda^{-2}U(1-U)^{-1}$, and $\omega=1+\lambda^2(1-w^2)$. The U-integration is split up from zero to θ^2 and from θ^2 to ∞. In the first part,

$$\cos^{-1}\left(8\frac{U-i\epsilon-\theta^2}{(U-i\epsilon)^2}-1\right)=2\sin^{-1}\frac{(U^2-4U+4\theta^2)^{\frac{1}{2}}}{(U-i\epsilon)}=\pi+2i\log\frac{(U^2-4U+4\theta^2)^{\frac{1}{2}}+2(\theta^2-U)^{\frac{1}{2}}}{U-i\epsilon}\rightarrow\pi-2\tan^{-1}(\epsilon/U)=2\tan^{-1}(U/\epsilon), \quad (4.22)$$

whose value at $U=0$ is 0. Also

$$\cos^{-1}\left(1-2\frac{\theta}{U-i\epsilon}\right)=2\sin^{-1}\frac{\theta}{(U-i\epsilon)^{\frac{1}{2}}}=\pi+2i\log\frac{\theta+(\theta^2-U)^{\frac{1}{2}}}{(U-i\epsilon)^{\frac{1}{2}}}\rightarrow\pi-\tan^{-1}(\epsilon/U)=\tfrac{1}{2}\pi+\tan^{-1}(U/\epsilon), \quad (4.23)$$

whose value at $U=0$ is $\pi/2$. Therefore

$$\Gamma_{l1}=16\lambda^{-2}\int_0^1dw\int_0^{\theta^2}\frac{dU}{U}(\omega+U)^{-1}\times\left\{\frac{(1+2\lambda^2)(1+2\gamma)-\lambda^2U}{(4\theta^2+U^2-4U)^{\frac{1}{2}}}2\tan^{-1}\frac{U}{\epsilon}-\frac{1+2\lambda^2}{\theta}\tan^{-1}\frac{U}{\epsilon}\right\}. \quad (4.24)$$

The divergent part of this is

$$32(1+2\lambda^2)\gamma\theta^{-1}\int_0^1dw\omega^{-1}\int_0^{\theta^2}U^{-1}dU\tan^{-1}(U/\epsilon)=32\gamma(1+2\lambda^2)\theta^{-1}F_0\int_0^{\theta^2}U^{-1}dU\tan^{-1}(U/\epsilon). \quad (4.25)$$

In the second part of Γ_l, U is conveniently replaced by x^{-1}.

In similar fashion, the divergent parts of Γ_d and Γ_e are isolated and found to be

$$-32\gamma(2+\lambda^2F_0+\lambda^2F_1)\theta^{-1}\int_0^{\theta^2}U^{-1}dU\tan^{-1}(U/\epsilon) \quad (4.26)$$

and

$$32\gamma\theta^{-1}\int_0^{\theta^2}U^{-1}dU\tan^{-1}(U/\epsilon). \quad (4.27)$$

The sum of (4.25), (4.26), and (4.27) vanishes. Thus all the Coulomb divergencies cancel. The divergent parts (4.25) to (4.27) are subtracted from the terms from which they arose and the remainders possess a finite limit as ϵ tends to naught.

After the cancellation of both the infrared and the Coulomb divergencies has been accomplished, nothing but a number of convergent integrations remains. The **k**-integrations are given in Appendix C. Then there are the integrations over three auxiliary variables left. All of these can be reduced to at most one-dimensional integrals by extremely long and tedious procedures and the result is much too complicated to be amenable to either discussion or easy numerical evaluation.[27] We shall therefore immediately proceed to low-energy and high-energy approximations.

5. LOW-ENERGY LIMIT

The nonrelativistic limit means, in the present notation, $\lambda\to 0$. As we want to avoid carrying out explicitly all the integrations involved in Γ, the limit has to be taken under the integral sign and a certain amount of caution is mandatory. Since, nevertheless, the procedure is relatively simple, if slightly tedious, only the results will be furnished. It will be noticed that Γ_l, Γ_d, and Γ_e contain parts proportional to λ^{-2} so that the remaining terms in these contributions have to be extracted more carefully. Variables were changed on the whole as indicated in Appendix C. The arrow denotes the limit as $B\to 0$ (B is defined by (C.18) and, at low energies, equals β^2) and hence $\lambda\to 0$ (after both infrared and Coulomb divergencies have been cancelled out as described in Sec. 4).

$$\int[dk]\Gamma_b\to 8\pi, \tag{5.1}$$

$$\int[dk]\Gamma_c\to -8\pi, \tag{5.2}$$

$$\int[dk]\Gamma_d\to[-64\lambda^{-2}I_2-32\pi\log B+64I_1] -(16/3)\pi\theta^{-1}\log(16B) +8[4+(35/9)\theta^{-1}]\pi-32(2\theta^2-1)I_2, \tag{5.3}$$

$$\int[dk]\Gamma_e\to[32\lambda^{-2}I_2+16\pi\log B-32I_1] -8\pi\theta^{-1}\log(16B) +16(2\theta^2-1)I_2+4\pi(\theta^{-1}-4), \tag{5.4}$$

$$\int[dk]\Gamma_f\to 4\pi(1-\theta^{-1}), \tag{5.5}$$

$$\int[dk]\Gamma_k\to[-16\pi\log B+32I_1]-(8\pi/3)(3+\theta^{-1}), \tag{5.6}$$

$$\int[dk]\Gamma_l\to[32\lambda^{-2}I_2+32\pi\log B-64I_1] -8\pi\theta^{-1}\log B+(16/3)(5+6\theta^2)I_2, \tag{5.7}$$

$$\int[dk]\Gamma_o\to -8\pi, \tag{5.8}$$

$$\int[dk]\Gamma_p\to -4\pi, \tag{5.9}$$

$$\int[dk]\Gamma_v\to -(64/15)\pi\theta^{-1}, \tag{5.10}$$

while Γ_g, Γ_h, Γ_m, and Γ_n tend to zero.

The functions I_1 and I_2 are defined as follows:

$$I_1\equiv\int_0^{2\theta^{-2}}dyy^{-1}(Y^{-\frac{1}{2}}\sin^{-1}Y^{\frac{1}{2}}-\tfrac{1}{2}\pi) +\tfrac{1}{2}\pi\int_{2\theta^{-2}}^{\infty}dyy^{-1}Y^{-\frac{1}{2}}, \tag{5.11}$$

$$I_2\equiv\int_0^{2\theta^{-2}}dyY^{-\frac{1}{2}}\sin^{-1}Y^{\frac{1}{2}} +\tfrac{1}{2}\pi\int_{2\theta^{-2}}^{\infty}dy(Y^{-\frac{1}{2}}-\theta^{-1}y^{-1}), \tag{5.12}$$

and Y is given by (C.15). The brackets in (5.1) to (5.10) indicate parts that cancel out. Since I_1 cancels everywhere, the integrals in (5.11) need not be evaluated. The second integral in (5.12) is immediately found to be $-\frac{1}{2}\pi\theta^{-1}\log[\frac{1}{4}(\theta+1)]$. The first one is conveniently done by the substitution $y=\theta^{-2}(z+1)$ and subsequent application of the operator

$$-\theta^{-1}\int_\theta^\infty d\theta(\partial/\partial\theta)\theta,$$

whereupon its value is found to be $\pi\theta^{-1}\log(1+\theta^{-1})$. Hence

$$I_2=\tfrac{1}{2}\pi\theta^{-1}\log[4\theta^{-1}(1+\theta^{-1})]. \tag{5.13}$$

The sum of (5.1) to (5.10) yields, according to (2.21a), the low-energy limit of the radiative correction [in the notation of (2.16)]: For $\beta\ll 1$,

$$\delta_2=-(8\alpha/3\pi)\tan^2\tfrac{1}{2}\vartheta(1+\csc\tfrac{1}{2}\vartheta)\{(121/120) +\log[\tfrac{1}{2}\sqrt{2}\sin\tfrac{1}{2}\vartheta(1+\sin\tfrac{1}{2}\vartheta)\beta^{-2}]\}. \tag{5.14}$$

In this low-energy approximation the form factors S_1 and S_2 are, of course, unity.

Equation (5.14) in conjunction with (2.18) and Schwinger's result[19] yield the following at low energies (where $\mathcal{R}$ is the Rutherford cross section):

$$(d\sigma/d\Omega)/\mathcal{R}=1-\beta^2\sin^2(\tfrac{1}{2}\vartheta)\left\{1-\frac{8\alpha}{3\pi}\left(\frac{19}{30}+\log\frac{m}{2\Delta E}\right) +\frac{\pi Z\alpha}{\beta}\left[\csc\tfrac{1}{2}\vartheta-1 -\frac{8\alpha}{3\pi}\left(\frac{121}{120}+\log\frac{\sin\frac{1}{2}\vartheta(1+\sin\frac{1}{2}\vartheta)}{\sqrt{2}\beta^2}\right)\right]\right\}. \tag{5.15}$$

[27] This result occupies fourteen pages in the author's doctoral thesis at Harvard. It is not recommended for inspection for the possible purpose of numerical evaluation.

This equation shows that not only is $|\delta_2\beta_2| \gg |\delta_1|$ in the nonrelativistic region, but also $|\beta\delta_2 B_2| \gg |\delta_1|$ by a factor of $\log\beta$. It is well known that at low energies the Born series becomes unreliable, because it is essentially an expansion in $Z\alpha/\beta$ rather than $Z\alpha$ alone. By that reasoning one could, however, under the condition that $Z\alpha \ll \beta \ll 1$, expect the low-energy limit of the second BA to be a small correction. Equation (5.14) shows that that is not so, but that, instead, the necessary assumption is that

$$Z\alpha\beta^{-1}\log\beta^{-1} \ll 1 \quad \text{and} \quad \beta \ll 1. \tag{5.16}$$

Equation (5.15) for slow electrons, with the assumption (5.16) is in conflict with the conclusion of Mittleman,[28] who performed a nonrelativistic calculation of the radiative correction to all orders of $Z\alpha$ (and first order in the number of photons) and obtained a multiple of the uncorrected cross section. The ratio of the two cross sections in his work contains, however, a dependence on an unknown parameter needed for an ultraviolet cutoff. There follows the statement that this parameter is to be fixed by comparison with the first BA and it is concluded that the relative radiative correction for slow electrons is the same to all orders in $Z\alpha$; which contradicts (5.15).

The conflict is resolved by the realization that the ultraviolet cutoff parameter needed in the nonrelativistic calculation is a function of $Z\alpha$ and β[29]; comparison with the first BA yields only its zero-approximation in $Z\alpha$. The conclusion that the relative radiative correction to the over-all cross section equals that to the first BA is therefore incorrect. Furthermore, the fact that (5.14) contains a $\log\beta$ means that the low velocity limit of the relative radiative correction (even if $Z\alpha/\beta$ is kept constant in that process) does not exist, although that of the cutoff parameter does; not a very surprising result in view of the logarithmic dependence of one upon the other.

6. HIGH-ENERGY LIMIT

In order to simplify things to any appreciable extent, the extreme relativistic limit has to be combined with the assumption of not too small a scattering angle, so that $\lambda \gg 1$ (just as in the case of reference 19). The second sentence of Sec. 5 is even more applicable in the present case and the extraction of the limit as $\lambda \to \infty$ under the double integrals is rather lengthy and tedious. We shall list only the results obtained after the Coulomb divergencies (4.25) to (4.27) have been subtracted from their respective terms. The leading terms at high energies tend to infinity as $(\log 2\lambda)^2$. Therefore both the factors of $(\log 2\lambda)^2$ and $(\log 2\lambda)$ were kept. The constant terms contain a large number of integrals and their evaluation would involve an amount of work disproportionate to their expected importance.[30] In that sense, then, here are the asymptotic values in the pure Coulomb case:

$$\Gamma_c \sim 4\pi\theta^{-1}(\log 2\lambda)\{(\theta-1)(12\theta^{-3}+6\theta^{-2}-10\theta^{-1}-5-2\theta) +2(3-\theta-10\theta^{-2}+6\theta^{-4})\log[\tfrac{1}{2}(\theta+1)]+2(\theta-1) \times[1+2(\theta+1)(2\theta^{-2}-3\theta^{-4})]\log 2\}, \tag{6.1}$$

$$\Gamma_d \sim 48\pi(\theta-\theta^{-1})(\log 2\lambda)^2 +16\pi(\theta^{-1}-1)(\log 2\lambda)[(\theta+1)(3\theta^{-2}-2)\log(\theta+1) +4(\theta+1)\log 2-2(2\theta^2-1)(\theta-1)^{-1}\log\theta -(7/2)(\theta+1)-(5/2)\theta^{-1}+(3/2)\theta^{-2}+3\theta^{-3}], \tag{6.2}$$

$$\Gamma_e \sim -32\pi(\theta-\theta^{-1})\log 2\lambda, \tag{6.3}$$

$$\Gamma_k \sim -16\pi(\theta-\theta^{-1})(\log 2\lambda)^2-32\pi\theta^{-1}\log\theta\log 2\lambda, \tag{6.4}$$

$$\Gamma_l \sim -32\pi(\theta-\theta^{-1})[(\log 2\lambda)^2+2\log\tfrac{1}{2}\theta\log 2\lambda], \tag{6.5}$$

$$\Gamma_n \sim 8\pi(\log 2\lambda)[2\log 2-(1-\theta^{-1})\log(\theta+1)], \tag{6.6}$$

$$\Gamma_o \sim \text{const}, \quad \Gamma_p \sim \text{const}, \tag{6.7}$$

$$\Gamma_m+\Gamma_b+\Gamma_f+\Gamma_g+\Gamma_h \sim -4\pi(1-\theta^{-1})[(\log 2\lambda)^2+3(\log 2\lambda) +4(2\log 2\lambda-1)\log(\Delta E/E)], \tag{6.8}$$

$$\Gamma_v \sim -16\pi(1-\theta^{-1})\log 2\lambda. \tag{6.9}$$

The sum of (6.1) to (6.9) and the use of (2.21a) yields the following correction to the second BA in the case of pure Coulomb scattering and $(p_0/m)\sin\vartheta/2 \gg 1$:

$$\delta_2 \sim -(\alpha/\pi)\{\{\log[2(p_0/m)\sin\tfrac{1}{2}\vartheta]\}^2-2[\csc\tfrac{1}{2}\vartheta-\tfrac{3}{2} -\sec^2\tfrac{1}{2}\vartheta(1+\sin\tfrac{1}{2}\vartheta)\log\tfrac{1}{4}(1+\csc\tfrac{1}{2}\vartheta) +\tfrac{1}{4}(1+\csc\tfrac{1}{2}\vartheta)(1+3\cos\vartheta)(3-\cos\vartheta)\log(1+\csc\tfrac{1}{2}\vartheta) -2\log(\Delta E/E)]\log[2(p_0/m)\sin\tfrac{1}{2}\vartheta]+f(\vartheta)\}, \tag{6.10}$$

where $f(\vartheta)$ is not known and is assumed to be relatively small.

At an energy of 100 Mev and for right angle scattering, the size of the correction is $\delta_2 \approx 0.27$ if the energy resolution is $\Delta E/E = 0.01$; and $\delta_2 \approx 0.14$ if $\Delta E/E = 0.1$ ($f(\vartheta)$ was neglected). It depends thus rather strongly on the allowed energy loss. It is remarkable that at this energy and angle the contribution of the logarithmic term is about 6 times as large (for $\Delta E/E = 0.01$, about 3 times for $\Delta E/E = 0.1$) as that of the $(\log)^2$ term, and with the opposite sign. The values for δ_2 given above are to be judged in conjunction with those of the second BA, $B_2 \approx 0.5\times10^{-2}\times Z$ for this energy and angle, and the value $\frac{1}{2}$ for the relativistic correction $(1-\beta^2\sin^2\frac{1}{2}\vartheta)$.

[28] M. H. Mittleman, Phys. Rev. **93**, 453 (1954).

[29] This is equivalent to saying the following about details of reference 28: The integral in q_0^2 ought really to be extended to ∞, and this is done with impunity if the full relativistic theory is used. In the work of reference 28 the full $I(k)$ is replaced by its nonrelativistic approximation and a cutoff replaces the taking of the nonrelativistic limit after the integration is carried out. The size of this equivalent cutoff will depend on the order (in $Z\alpha$) where it is performed, since the integrals involved differ from order to order. Moreover, only if at a given order the low-energy limit of the cutoff exists and is not zero can one replace it by that limit alone.

[30] The author is in possession of these integrals and will gladly furnish them to anyone who considers it worth his while to evaluate them.

As the energy increases, the value of the correction decreases and becomes negative for very large energies. The relative importance of the energy resolution decreases with increasing impact energy. Furthermore, at extremely high energies, δ_2/δ_1 becomes arbitrarily large. Hence the Born series of one-photon radiative corrections becomes increasingly *unreliable* at very high energies. It is, of course, a moot question whether the nth term in the series contains $[\log(p_0/m)]^n$ as a leading term. Since parts[31] of the first term in the series (the Schwinger correction) also contain $[\log(p_0/m)]^2$, which however happen to cancel out, the increase in the power of the logarithm may be accidental. To the author, this seems unlikely. If the nth term in the series does have an asymptotic value proportional to $[\log(p_0/m)]^n$, then not only are the first terms quite insufficient as an approximation at very high energies, but the series can be expected to diverge beyond a certain value of the energy[32] (as well as below a certain small energy).

The entire discussion in the last paragraph of the behavior at high energies is, of course, restricted to pure Coulomb scattering, and therefore somewhat academic. In reality, the nucleus is not a point and no matter how small a non-point-charge distribution is, it leads to the form factors indicated in (2.16), (2.19) and (2.21). The nature of these modifications is such that in each term of the Born series of one-photon radiative corrections as the energy increases, the form factor tends to zero relative to that of the previous term and at least linearly[33] with (m/p_0). Any possible factor $[\log(p_0/m)]^n$ is therefore more than compensated and the series will, if it ever converges, tend to its first term at extremely high energies.

A general modification of the result (6.10) for an arbitrary extended source has not been accomplished. The following can, however, be stated. The terms proportional to $(\log 2\lambda)^2$ in (6.2), (6.4), and (6.5) cancel and the entire contribution to δ_2 in $(\log 2\lambda)^2$ comes from (6.8), where it stems from Γ_m and Γ_h only. Both Γ_m and Γ_h have precisely the same $\mathbf{k}$-dependence as does B_2, (B.5). Their shape dependence is therefore the same as that of the second BA, i.e., S_2, (B.7). At extremely high energies Eq. (2.6) therefore reads

$$(d\sigma/d\Omega)_2 = \mathcal{R} S_1[S_1 \cos^2\tfrac{1}{2}\vartheta(1-\delta_1)+B_2S_2(1-\delta_2')], \quad (6.11)$$

where now

$$\delta_2' = -(\alpha/\pi)\{\log[2(p_0/m)\sin\tfrac{1}{2}\vartheta]\}^2. \quad (6.12)$$

Equations (6.11) and (6.12) hold for arbitrary charge distributions when $\log[2(p_0/m)\sin\frac{1}{2}\vartheta] \gg 1$.

An important question, of course, is whether at the energies experimentally used today, the radiative correction will be substantially altered by the form factor. The value of the relevant parameter K, defined by (2.20), is approximately $4\times10^{-3}\times A^{\frac{1}{3}}$ (where A is the number of nucleons in the scattering nucleus); at 100 Mev and right angle scattering this leads to a value of $K\lambda \approx \frac{1}{2}A^{\frac{1}{3}}$ (≈ 1.5 for Al, ≈ 2 for Cu, ≈ 3 for Au). Since there is no evidence that the charge distribution in the nucleus is extremely peaked at the center, the factor $\sin(v\lambda Kk_1)/(v\lambda Kk_1)$ will in effect be appreciably smaller than unity for $k_1 > 4A^{-\frac{1}{3}}$. It is hard to estimate the effect of this upon the value of the shape integrals, especially to compare the effects on S_1 and S_2, or those in δ_2. Presumably, when the first BA leads to a real zero of the cross section (a situation where comparison with a phase-shift analysis shows that the first BA is very misleading[12–14]) the form factors of the second BA and its radiative correction do not vanish; and hence (2.16) shows that the second BA (with correction) will predominate over the first (with correction). However, (2.16) also shows that both still have one factor S_1 in common and hence all of $(d\sigma/d\Omega)_2$ still vanishes when the first BA does. It is not until parts of the *third* BA (and its radiative correction) that the shape factor S_1 will disappear altogether. (See Appendix E for an elaboration of this matter.) The present calculation can therefore not contribute to the "filling in" of the fictitious dips of the first Born approximation.[34]

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APPENDIX A. TRACES

The following equation, which is used for the evaluation of the traces in (2.10), is easily derived:

$$\tfrac{1}{4}\,\mathrm{tr}(m-\gamma q)\gamma a(m-\gamma p)(1,\gamma_\mu,\gamma_\mu\gamma_\nu) = 2m(ap,\, m\lambda^2 a_\mu + ap\lambda Q_\mu,\, m(a_\mu\lambda_\nu - a_\nu\lambda_\mu) - \delta_{\mu\nu}ap), \quad (A.1)$$

[where $p_0 = q_0$, and the notation (2.6) and (2.27) was used, with an obvious extention to the zero-component; a is an arbitrary four-vector]. The substitutions indicated in (2.5), the notations (2.12), (2.26), and the convention

$$kk \equiv \mathbf{k}_1\cdot\mathbf{k}_2 = 1-\tfrac{1}{4}k^2 \quad (A.2)$$

then yield from (A.1) the following traces. The arrow

[31] The function $G(\lambda)$, in the notation of reference 19.

[32] Unless it is such that it converges for arbitrarily large $Z\alpha$. Hardly anyone would be likely to argue for that.

[33] Possibly more than linearly due to the increasing number of factors of the kind $\sin(k\lambda K)$ and their rapid oscillation.

[34] Appendix D will give a brief discussion of the nature of the integrals involved in S_2 and S_3.

stands for

$$-\tfrac{1}{4}\,\mathrm{tr}\gamma_0(m-\gamma p)\cdots(m-\gamma q)/p_0m^2A_0(m\lambda k_1)A_0(m\lambda k_2).$$

$$FF\to -4m\lambda^2kk,$$
$$F(m+\gamma p)F\to 2m^2\lambda^4\mathbf{Q}\cdot\mathbf{k},$$
$$\gamma FF\gamma\to -4m\lambda^2kk,$$
$$\gamma F(m-\gamma p)F\gamma\to -m^2\lambda^4(4\gamma kk+(\mathbf{Q}\cdot\mathbf{k})^2),$$
$$\gamma F(m-\gamma p)^2F\gamma\to 2m^3\lambda^4(2\eta-4\gamma-(\mathbf{Q}\cdot\mathbf{k})^2),$$
$$\sigma F\sigma F\to -8m\lambda^2kk,$$
$$\sigma F(m+\gamma p)\sigma F\to 16m^2\lambda^4[\tfrac{1}{4}\mathbf{Q}\cdot\mathbf{k}+\gamma]\eta+16m^2\lambda^4(\gamma+1),$$
$$\gamma Fk\sigma F\to -2im^2\lambda^4kk\mathbf{Q}\cdot\mathbf{k},$$
$$\sigma FkF\gamma\to 2im^2\lambda^4kk\mathbf{Q}\cdot\mathbf{k},$$
$$\gamma Fk(m-\gamma p)\sigma F\to -4im^3\lambda^4kkk_2^2,$$
$$\sigma F(m-\gamma p)kF\gamma\to 4im^3\lambda^4kkk_1^2,$$
$$\gamma J\sigma F\to 2m^2\lambda^4\mathbf{Q}\cdot\mathbf{k}k_2^2,$$
$$\sigma F\gamma J\to 2m^2\lambda^4\mathbf{Q}\cdot\mathbf{k}k_1^2,$$
$$\gamma J(m-\gamma p)\sigma F\to 4m^3\lambda^4k_1^2k_2^2,$$
$$\sigma F(m-\gamma p)\gamma J\to 4m^3\lambda^4k_1^2k_2^2,$$
$$JF\gamma\to im^2\lambda^4\mathbf{Q}\cdot\mathbf{k}k_1^2,$$
$$\gamma FJ\to im^2\lambda^4\mathbf{Q}\cdot\mathbf{k}k_2^2,$$
$$J(m-\gamma p)F\gamma\to -2im^3\lambda^4\eta k_1^2,$$
$$\gamma F(m-\gamma p)J\to -2im^3\lambda^4\eta k_2^2,$$
$$\gamma J(m-\gamma p)^2kF\gamma\to -2im^5\lambda^6kk(kk+4\gamma)k_1^2,$$
$$\gamma Fk(m-\gamma p)^2\gamma J\to 2im^5\lambda^6kk(kk+4\gamma)k_2^2,$$
$$\gamma J(m-\gamma p)kF\gamma\to im^4\lambda^6kk(\mathbf{Q}\cdot\mathbf{k}-4\gamma)k_1^2,$$
$$\gamma Fk(m-\gamma p)\gamma J\to -im^4\lambda^6kk(\mathbf{Q}\cdot\mathbf{k}-4\gamma)k_2^2,$$
$$\gamma J(m-\gamma p)\gamma J\to m^4\lambda^6(\mathbf{Q}\cdot\mathbf{k}-4\gamma)k_1^2k_2^2,$$
$$\gamma J(m-\gamma p)^2\gamma J\to -2m^5\lambda^6(kk+4\gamma)k_1^2k_2^2.$$

APPENDIX B. SECOND BORN APPROXIMATION

Equations (1.1) and (2.2) yield for the ordinary second BA

$$(d\sigma/d\Omega)_2=4\pi^4\,\mathrm{Re\,tr}(m-\gamma p)(\mathbf{p}|\gamma AG_0\gamma A|\mathbf{q})\times(m-\gamma q)(\mathbf{q}|\gamma A|\mathbf{p}). \quad \text{(B.1)}$$

Equations (2.8), the equivalent of (2.5) for the simple present case, and the substitutions indicated between (2.5) and (2.6), yield for this [with the notation of (2.7), (2.12), and (2.26)]

$$(d\sigma/d\Omega)_2=\frac{e^3m^2}{16(2\pi)^5}\lambda A_0(-2m\lambda)\int\frac{(dk)^3}{\eta}\times A_0(m\lambda k_1)A_0(m\lambda k_2)\tfrac{1}{4}\mathrm{tr}(m-\gamma p)\times(\gamma k\lambda-4\gamma_0p_0/m)(m-\gamma q)\gamma_0, \quad \text{(B.2)}$$

where convention (2.11) is used again for the Fourier transforms. The trace is easily found to be

$$\tfrac{1}{4}\,\mathrm{tr}\cdots=2mp_0\lambda^2(\mathbf{Q}\cdot\mathbf{k}-4\gamma), \quad \text{(B.3)}$$

in the notation of (2.27), and (2.40). Equations (2.14), (2.15), and (2.17), and the notations (2.20) and (2.22) then yield [with the same notation as in (2.16)]:

$$B_2S_2=\tfrac{1}{2}Z\alpha\beta\sin\tfrac{1}{2}\vartheta\int_0^1dv_1v_1^2\rho(v_1)\int_0^1dv_2v_2^2\rho(v_2)\times\int[dk]\left(\frac{\sin v_1K\lambda k_1}{v_1K\lambda k_1}\right)\left(\frac{\sin v_2K\lambda k_2}{v_2K\lambda k_2}\right)\frac{4\gamma-\mathbf{Q}\cdot\mathbf{k}}{\eta}. \quad \text{(B.4)}$$

For pure Coulomb scattering, $K=0$ and thus by (2.14):

$$B_2=\tfrac{1}{2}Z\alpha\beta\sin\tfrac{1}{2}\vartheta\int[dk](4\gamma-\mathbf{Q}\cdot\mathbf{k})/\eta. \quad \text{(B.5)}$$

Finally, Eqs. (2.26), (4.16), (4.17), and (C.10) allow us to carry out the integral, and we obtain

$$B_2=\pi Z\alpha\beta\sin\tfrac{1}{2}\vartheta(1-\sin\tfrac{1}{2}\vartheta), \quad \text{(B.6)}$$

in agreement with McKinley and Feshbach.[9]

We may now also write down the form factor of the second Born approximation:

$$S_2=\tfrac{1}{2}\pi^{-1}\sec^2\tfrac{1}{2}\vartheta(1+\sin\tfrac{1}{2}\vartheta)\int_0^1dv_1v_1^2\rho(v_1)\int_0^1dv_2v_2^2\rho(v_2)\times\int[dk]\left(\frac{\sin v_1\lambda Kk_1}{v_1\lambda Kk_1}\right)\left(\frac{\sin v_2\lambda Kk_2}{v_2\lambda Kk_2}\right)\frac{4\gamma-\mathbf{Q}\cdot\mathbf{k}}{\eta}. \quad \text{(B.7)}$$

APPENDIX C. k-INTEGRATIONS

The principal integral occurring in the **k**-integration of Γ is of the following type:

$$\mathcal{I}=\tfrac{1}{2}\pi^{-2}\int(dk)^3\mathbf{k}_1^{-2}\mathbf{k}_2^{-2}\mathfrak{H}^{-1}\equiv\int[dk]\mathfrak{H}^{-1},$$
$$\mathbf{k}_1=\mathbf{n}+\tfrac{1}{2}\mathbf{k},\quad \mathbf{k}_2=\mathbf{n}-\tfrac{1}{2}\mathbf{k}, \quad \text{(C.1)}$$
$$\mathfrak{H}\equiv A+\mathbf{B}\cdot\mathbf{k}+C\mathbf{n}\cdot\mathbf{k}+\tfrac{1}{4}Dk^2,$$

where $\mathbf{n}$ is a unit vector and $\mathbf{B}\cdot\mathbf{n}=0$. The split of the part of the denominator $\mathfrak{H}$ linear in $\mathbf{k}$ is, of course, no restriction and can always be made. It is particularly convenient here since $\mathbf{Q}\cdot\mathbf{n}=0$, in the notation (2.6) and (2.27).

A sufficient condition for convergence of $\mathcal{I}$ is

$$AD>B^2+C^2, \quad \text{(C.2)}$$

which we assume to be satisfied. Whenever (C.2) is not satisfied, $\mathcal{I}$ will be evaluated by analytic continuation from the case where it is.

The evaluation of $\mathcal{I}$ proceeds by means of the familiar identity

$$(abc)^{-1}=\int_{-1}^{1}\tfrac{1}{2}d\alpha\int_{1}^{1}\tfrac{1}{2}d\beta(1-\beta)\{\tfrac{1}{2}a(1+\beta)+[\tfrac{1}{2}b(1-\alpha)+\tfrac{1}{4}c(1+\alpha)(1-\beta)\}^{-3}. \quad \text{(C.3)}$$

Then the **k**-integration is carried out and the auxiliary integrations can be accomplished, first over β and then over α. The result is

$$\mathcal{J}=2[(A-D)^2+4B^2]^{-\frac{1}{2}}\times\cos^{-1}\left[1-2\frac{(A-D)^2+4B^2}{(A+D)^2-4C^2}\right]. \quad \text{(C.4)}$$

Since for $B=C=D=0$, $\mathcal{J}$ must have the sign of A and the result never vanishes so long as (C.2) stays satisfied, it follows that the sign of $\mathcal{J}$ must remain that of A if the convergence domain is never left.

Other integrals may be obtained from (C.4) by means of differentiation as, for example,

$$\int[dk]\tfrac{1}{4}k^2\mathfrak{H}^{-1}=\int_\infty^A dA\,(\partial/\partial D)\mathcal{J}.$$

It is thus found that

$$\tfrac{1}{2}\int[dk](1+\tfrac{1}{4}k^2)\mathfrak{H}^{-1}=[(D+C)^2+B^2]^{-\frac{1}{2}}\times\cos^{-1}\left(1-2\frac{(D+C)^2+B^2}{D(A+D+2C)}\right)+(C\to-C). \quad \text{(C.5)}$$

Similarly,

$$\int[dk]\mathfrak{H}^{-2}=-\partial\mathcal{J}/\partial A, \quad \text{(C.6)}$$

$$\int[dk](\tfrac{1}{4}k^2)\mathfrak{H}^{-2}=-\partial\mathcal{J}/\partial D, \quad \text{(C.7)}$$

$$\int[dk]\mathbf{B}\cdot\mathbf{k}\mathfrak{H}^{-2}=-2B\partial\mathcal{J}/\partial B^2. \quad \text{(C.8)}$$

Another, somewhat more complicated integral is evaluated in the same manner as $\mathcal{J}$:

$$\begin{aligned}&\int[dk](2g+k_1^2)^{-1}k_1^2\mathfrak{H}^{-1}\\&\quad=4F^{-1}\sin^{-1}[2FP^{-\frac{1}{2}}R^{-\frac{1}{2}}(g+2)^{-\frac{1}{2}}],\quad g>0,\\&F^2=[A-gC-D(1+g)]^2+B^2(2+g)^2,\\&P=A+D+2C,\\&R=A+D-2C+2Dg+2[2g(AD-B^2-C^2)]^{\frac{1}{2}}.\end{aligned} \quad \text{(C.9)}$$

We now apply the general results to the most important cases needed in the body of the work.

Equation (C.4) yields as a special case, $A=1$, $B=C=D=0$, a result easily obtained directly:

$$\int[dk]=2\pi. \quad \text{(C.10)}$$

In case $\mathfrak{H}=\eta$, as in most other cases in the paper it proves convenient to make the change of variables

$$U=u(1-u)^{-1}v^{-1}. \quad \text{(C.11)}$$

Every integral then falls into two distinct parts, separated by U_0, which is determined simply as the value of U for which the argument of the antitrigonometric function involved become unity. For $U>U_0$ the formulas (C.4) to (C.9) are directly applicable, since (C.2) is fulfilled. For $U<U_0$ one must exercise some care in analytically continuing the value of the integral. When the value at $U=0$ is needed, it is important to realize the universal presence of $-i\epsilon$ in the Green's function $(m^2+p^2-i\epsilon)^{-1}$ from which all denominators η_i originate. With this in mind, one easily obtains the following equations:

$$\begin{aligned}&\int_0^1du\int[dk]\eta_1^{-1}\\&\quad=4\lambda^{-2}\int_0^{y_0}\frac{dy}{yL+2v}Y^{-\frac{1}{2}}\sin^{-1}\left[\frac{YL^2}{(L+2l^+)(L+2l^-)}\right]^{\frac{1}{2}}\\&\qquad+2\pi\lambda^{-2}\int_{y_0}^\infty\frac{dy}{yL+2v}Y^{-\frac{1}{2}},\end{aligned} \quad \text{(C.12)}$$

where

$$L=\lambda^{-2}+(1-v)^2(1-w^2), \quad \text{(C.13)}$$

$$l^{\pm}=v(1-v)(1\pm w), \quad \text{(C.14)}$$

$$Y=\theta^2y^2-2y+1, \quad \text{(C.15)}$$

and $y=2U^{-1}L^{-1}$, while θ is defined by (4.21). Similarly, because of the symmetry in the w-integration,

$$\begin{aligned}&\int_0^1du\int[dk]k_1^2\eta_1^{-1}=\int_0^1du\int[dk]\eta_1^{-1}(1+\tfrac{1}{4}k^2)\\&\quad=8\lambda^{-2}\,\mathrm{Re}\int_0^\infty\frac{dy}{v+yl^+}Y_+^{-\frac{1}{2}}\\&\qquad\times\sin^{-1}\left[\frac{Y_+l^{+2}}{(L+2l^+)(l^+y+v(1-v))}\right]^{\frac{1}{2}}.\end{aligned} \quad \text{(C.16)}$$

The corresponding integrals involving η_1^{-2} are obtained from the above by applying the operator $-\lambda^{-4}(1+v^{-1}U^{-1})(\partial/\partial\lambda^{-2})\lambda^2$ to (C.12) and (C.16). For $w=1$ these results immediately yield the integrals in which η_1 is replaced by η_3.

Further integrals of η_3 needed are the following, where the ϵ of the Green's function has been kept and can be allowed to vanish only after Coulomb divergencies are cancelled:

$$\int_0^1 du\int[dk]\frac{\lambda^2 ak_1^2}{1+\lambda^2 ak_1^2}\frac{1}{\eta_3}$$

$$=\frac{4\theta^{-1}a}{1+4\lambda^2 a}\int_0^{U_0}\frac{dU}{1+Uv}\bigg[\tan^{-1}\frac{U(1+4\lambda^2 a)}{\epsilon\lambda^2}$$

$$+\tan^{-1}\frac{2Ua+1}{2a^{\frac{1}{2}}(B-U-U^2a)^{\frac{1}{2}}}$$

$$-2\tan^{-1}\frac{Ua^{\frac{1}{2}}}{B^{\frac{1}{2}}+(B-U-U^2a)^{\frac{1}{2}}}\bigg]$$

$$+\frac{8\theta^{-1}a}{1+4\lambda^2 a}\sin^{-1}[B^{-1}U(1+2Ua)$$

$$+2(U^2a+U-B)^{\frac{1}{2}}a^{\frac{1}{2}}]^{-\frac{1}{2}}, \quad (C.17)$$

where

$$B\equiv\lambda^2\theta^2=\beta^2(1-\beta^2)^{-1}. \quad (C.18)$$

A similar, but more complicated result, is obtained when the left hand side of (C.17) contains $(1+\frac{1}{4}k^2)$ as a factor. We shall not bother filling space with it here.

The following results are needed frequently in the course of the approximation:

$$\int[dk][\eta-i\epsilon+U]^{-1}$$

$$=\begin{cases}2yY^{-\frac{1}{2}}\sin^{-1}Y^{\frac{1}{2}}, & U>\theta^2\\ 2yY^{-\frac{1}{2}}\tan^{-1}(U/\epsilon), & U<\theta^2,\end{cases} \quad (C.19)$$

where $y=2U^{-1}$ and Y is defined by (C.15).

$$\int[dk](1+\tfrac{1}{4}k^2)[\eta-i\epsilon+U]^{-1}$$

$$=\begin{cases}8\theta^{-1}\sin^{-1}\theta U^{-\frac{1}{2}}, & U>\theta^2\\ 4\theta^{-1}(\frac{1}{2}\pi+\tan^{-1}(U/\epsilon)), & U<\theta^2.\end{cases} \quad (C.20)$$

APPENDIX D. FORM FACTOR INTEGRAL

This appendix will briefly deal with a transformation of the integral in S_2.

In the integral in (B.7) we introduce the new variables

$$2z_1=k_1,\quad 2z_2=k_2,\quad \xi=\mathbf{k}\cdot\mathbf{Q}k^{-1}Q^{-1}. \quad (D.1)$$

Then

$$\int[dk]f(k_1,k_2)\eta^{-1}=4\pi^{-2}\,\mathrm{Re}\int_0^\infty\frac{dz_1}{z_1}\int_{|1-z_1|}^{1+z_1}\frac{dz_2}{z_2}$$

$$\times\int_{-\infty}^{\infty}d\xi\left[\frac{2(z_1^2+z_2^2)-1}{(2(z_1^2+z_2^2)-1)(1-\xi^2)-(z_1^2-z_2^2)^2}\right]^{\frac{1}{2}}$$

$$\times f(2z_1,2z_2)\eta^{-1}, \quad (D.2)$$

and η becomes

$$\tfrac{1}{2}\eta=z_1^2+z_2^2-1-Q\xi[2(z_1^2+z_2^2)-1]^{\frac{1}{2}}. \quad (D.3)$$

The ξ integral is readily carried out and one obtains

$$I\equiv\int[dk]\eta^{-1}f(k_1)f(k_2)[(1+\tfrac{1}{4}k^2),1]$$

$$=4\pi\,\mathrm{Re}\int_0^\infty\frac{dz_1}{z_1}\int_{|1-z_1|}^{1+z_1}\frac{dz_2}{z_2}$$

$$\times\frac{f(2z_1)f(2z_2)[2(z_1^2+z_2^2),1]\,\mathrm{sgn}(z_1^2+z_2^2-1)}{[(z_1^2+z_2^2-1)^2+Q^2((z_1^2-z_2^2)^2+1-2(z_1^2+z_2^2))]^{\frac{1}{2}}} \quad (D.4)$$

The transformation $u=z_1^2+z_2^2$, $v=z_1^2-z_2^2$ changes this to

$$I=2\pi\int\int\frac{dudv}{u^2-v^2}$$

$$\times\frac{[2u,1]\,\mathrm{sgn}(u-1)f((2u+2v)^{\frac{1}{2}})f((2u-2v)^{\frac{1}{2}})}{[(u-1)^2-Q^2(2u-1-v^2)]^{\frac{1}{2}}}, \quad (D.5)$$

where the integral extends above the parabola $u=\frac{1}{2}(1+v^2)$, with the exception of the ellipse $(u-\theta^2)^2+(\theta^2-1)v^2=\theta^2(\theta^2-1)$, which osculates the former at $u=1$, $v=\pm1$. The entire part above the ellipse can be transferred into that below by the substitution $u=u'(2u'-1)^{-1}$, $v=v'(2u'-1)^{-1}$. This yields

$$I=2\pi\int\int\frac{dudv}{u^2-v^2}[(1-u)^2-(\theta^2-1)(2u-1-v^2)]^{-\frac{1}{2}}$$

$$\times\left\{\left[\frac{2u}{2u-1},1\right]f\left(\left[2\frac{u+v}{2u-1}\right]^{\frac{1}{2}}\right)f\left(\left[2\frac{u-v}{2u-1}\right]^{\frac{1}{2}}\right)\right.$$

$$\left.-[u,1]f((2u+2v)^{\frac{1}{2}})f((2u-2v)^{\frac{1}{2}})\right\}, \quad (D.6)$$

where the integral extends over the crescent between $u=\frac{1}{2}(1+v^2)$ and $u=\theta^2-(\theta^2-1)^{\frac{1}{2}}(\theta^2-v^2)^{\frac{1}{2}}$, whose vortices are at $u=1$, $v=\pm1$, and which cuts the u-axis at $u=\frac{1}{2}$ and $u=\theta^2-\theta(\theta^2-1)^{\frac{1}{2}}$. The function $f(x)$ in (D.6) refers to $\int_0^1 dv v^2\rho(v)(\sin v\lambda Kx)/(v\lambda Kx)$.[35]

The transformations indicated above may serve to facilitate the evaluation, numerically or otherwise, of S_2 and the form factor of the third BA.

APPENDIX E. THIRD BORN APPROXIMATION AT THE ZEROS OF THE FIRST

The zeros of the first Born approximation to the scattering of high-energy electrons by nuclei of finite size are due to the vanishing of the matrix element

[35] For $\lambda=0$, $f=1$, and (D.6) verifies (4.16) and (4.17).

$(\mathbf{p}|V|\mathbf{q})$ of the potential between initial and final momenta. Every term of the second Born approximation, as well as its radiative correction, still contains such a matrix element as a factor. The first term without it occurs in the *third* Born approximation, so that at the position of these zeros of the first Born approximation the leading term in the scattering cross section is[36]

$$(d\sigma/d\Omega)=2\pi^4e^4\,\mathrm{tr}(m-\gamma p)(\mathbf{p}|\gamma AG_0\gamma A|\mathbf{q})\times(m-\gamma q)(\mathbf{q}|\gamma AG_0\gamma A|\mathbf{p}).\quad (\mathrm{E}.1)$$

A consequence of this simple observation is that in the region where the Born series may be expected to be useful, i.e., for Z not too large, the cross section ought to vary as Z^4 at the position of the minima (in contrast to Z^2 variation elsewhere). In other words, the relative depth of the minima ought to vary as Z^{-2}. Since the Born approximation for the lighter elements such as Al and, say, even Cu is computationally very much simpler and also admits of easier insight into the relation between charge distribution and scattering cross section than the otherwise much more accurate method of phase shifts,[14] it may be worth while to use it appropriately corrected at the zeros by (E.1).[37]

The matrix elements and trace in (E.1) are straightforward to evaluate. The result is the following at high energies $[(p_0/m)\sin\vartheta/2\gg1]$:

$$d\sigma/d\Omega=\mathcal{R}(\tfrac{1}{4}Z\alpha)^2[\sin^2\tfrac{1}{2}\vartheta\,\mathfrak{F}_1^2+\mathfrak{F}_2^2],\quad (\mathrm{E}.2)$$

where the first Born approximation vanishes. Here $\mathcal{R}$ is the Rutherford cross section,

$$\mathfrak{F}_1\equiv\tfrac{1}{2}\pi^{-2}\int_0^1 dv_1v_1^2\rho(v_1)\int_0^1 dv_2v_2^2\rho(v_2)\times\int\frac{(dk)^3}{k_1^2k_2^2}\left(\frac{\sin Kk_1v_1}{Kk_1v_1}\right)\left(\frac{\sin Kk_2v_2}{Kk_2v_2}\right)\frac{\mathbf{m}\times\mathbf{n}\cdot\mathbf{k}}{\eta}$$

$$\equiv\int(dk)^3\mathfrak{f}(\mathbf{k})(\mathbf{m}\times\mathbf{n}\cdot\mathbf{k}),\quad (\mathrm{E}.3)$$

$$\mathfrak{F}_2\equiv\int(dk)^3\mathfrak{f}(\mathbf{k})(4\,\mathrm{ctn}\tfrac{1}{2}\vartheta-\mathbf{m}\cdot\mathbf{k}),\quad (\mathrm{E}.4)^{38}$$

η is defined by (2.26), k_1 and k_2 by (2.12),

$$\mathbf{Q}=\mathbf{m}\,\mathrm{ctn}\tfrac{1}{2}\vartheta,\quad (\mathrm{E}.5)$$

$$\mathbf{n}\cdot\mathbf{m}=0,\quad \mathbf{n}^2=\mathbf{m}^2=1,\quad (\mathrm{E}.6)$$

$$K\equiv(rp_0/\hbar c)\sin\tfrac{1}{2}\vartheta,\quad (\mathrm{E}.7)$$

and r is the nuclear radius; ρ is the nuclear charge distribution normalized by (2.14). The integrals $\mathfrak{F}_1$ and $\mathfrak{F}_2$ would have to be carried out numerically. The $\mathbf{k}$-integrations can be reduced to two dimensions by the introduction of $|\mathbf{k}_1|$, $|\mathbf{k}_2|$, and $|\mathbf{k}|^{-1}\mathbf{m}\cdot\mathbf{k}$ as new variables. The latter integral can then be carried out. It may be advisable to use $u=\frac{1}{4}(k_1^2+k_2^2)$, $v=\frac{1}{4}(k_1^2-k_2^2)$ as new variables and proceed as in Appendix D.

[36] See I, Sec. III.

[37] At higher energies than the ones experimentally used at the present such zeros will, of course, appear for any assumed charge distribution, not only for the uniform one, as in reference 14.

[38] The form factor of the second Born approximation can be expressed in terms of $\mathfrak{F}_2$: $S_2=\frac{1}{2}\pi^{-1}\sec^2\frac{1}{2}\vartheta(1+\sin\frac{1}{2}\vartheta)S_1\mathfrak{F}_2$, where S_1^2 is the form factor of the first Born approximation (all normalized to tend to unity at low energies).