which represent the omission of momenta greater than k_0 from the Fermi splitting itself. For $k_0=k_0^D$ equal to the π-meson mass, Eqs. (92) and (93) yield a value of (5.8×10^{-5}) for F ($F=0.9\times10^{-5}$ for $k_0=M$), which is larger than the value of δ' given in Eq. (104). This result merely emphasizes the prevalent feeling that a crude nonrelativistic "spreading" of the nucleon bears little relation to reality.

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Classical Field Theory in the Hamilton-Jacobi Formalism*

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A Hamilton-Jacobi formalism of classical relativistic field theory is developed. Both "time-independent" and "time-dependent" formulations are given, and the relation between them is discussed. In the former, the constants of the motion are identified with the "new" field variables, whereas in the latter they are the values of the fields on a suitable spacelike surface. The explicit introduction of a Hamiltonian density is avoided. As an illustration of the respective procedures, the classical Dirac and Klein-Gordon free fields are solved explicitly. A perturbation method is formulated for the case of fields in interaction. The metric tensor is not treated as a field quantity.

INTRODUCTION

THE purpose of the present paper is to serve as a starting point for the extension of Bohm's reinterpretation of particle quantum mechanics[1] to the theory of quantized fields. It will be shown in a subsequent paper that such an extension is indeed possible, and can be based on a Hamilton-Jacobi formulation of classical field theory. It was thought preferable to develop the necessary Hamilton-Jacobi formalism in a preliminary paper, so as not to break the continuity in the argument of the subsequent paper, and also because a Hamilton-Jacobi formalism for field theory may be of some interest in its own right. The formulation in the present paper actually goes beyond what is needed for a causal presentation of the theory of quantized fields.

The usual particle Hamilton-Jacobi formalism[2] is based on Hamiltonian mechanics. However, the essential features of Hamilton-Jacobi theory (i.e., the transformation to "appropriate" variables, which are essentially the constants of the motion,[3] transformation theory, and the reduction of the entire problem under consideration to the solution of a nonlinear first order partial differential equation) can be based as well on a Lagrangian formulation. In view of the greater adaptability of the purely Lagrangian approach to the requirements of covariance,[4] it is used throughout. A Hamiltonian density could be introduced explicitly, for instance by carrying out the differentiation in the right-hand side of (10). However, no useful purpose would be served, as it is not desired to develop a Hamiltonian formalism.[5]

In particle mechanics, the case of conservative systems can be treated by "time-independent" Hamilton-Jacobi theory $[H(q,\partial\bar{S}/\partial q)=E]$, whereas for non-conservative systems, "time-dependent" theory $[H+\partial S/\partial t=0$, which will be written $L=dS/dt]$ is required. The functions $\bar{S}$ and S are not identical, but for a conservative system, which can be treated by either method, S can be obtained from $\bar{S}$. From the point of view of generality, one might think that the time-dependent formalism should suffice. This is so for particle mechanics, but not for field theory. It might be impossible to express the Lagrangian L as a function of the field variables and conjugate momenta alone, but still possible to express suitable constants of the motion in terms of these variables (these constants of the motion playing a role analogous to that of H for particle mechanics), in which case time-independent theory is an indispensible tool. That is, in fact, what happens in the case of the Dirac field. Aside from this contingency, solutions of field Hamilton-Jacobi equations are usually quite difficult to obtain, and one or the other method might prove more convenient.

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[1] D. Bohm, Phys. Rev. **85**, 166 (1952).

[2] See, e.g., C. Lanczos, *The Variational Principles of Mechanics* (University of Toronto Press, Toronto, 1949).

[3] In some formulations of the theory, the momenta conjugate to the "appropriate" variables are the constants of the motion. Such formulations, while perfectly acceptable for particle dynamics, lead to difficulties in field theory.

[4] J. Schwinger, Phys. Rev. **82**, 914 (1951); P. G. Bergmann, Phys. Rev. **89**, 4 (1953).

[5] Such as that of R. H. Good, Jr., Phys. Rev. **93**, 239 (1954).

Particular attention is paid to the problem of perturbation of "free" fields by an "interaction," as it is in such interactions that quantum effects are observed. The perturbation method developed here for handling such classical interaction problems is simply the translation of well-known methods of celestial mechanics into a convenient relativistic field formalism.

The formulation given here is somewhat primitive. It is covariant under the general group of transformations in space-time, but it obviously does not include the theory of gravitation or its generalizations, both because of the assumed form of the Lagrangian density, and because the components of the metric tensor are not treated as field variables. Likewise, questions of constraint are not discussed. A more general treatment is under development.

NOTATION

Space-time is described by four coordinates, ξ^μ, which are not required to be Cartesian. In (at least locally) flat space-time and pseudo-Cartesian coordinates x^μ, $x^1=x$, $\cdots$, $x^4=ct$, and $g_{\mu\nu}=\mathrm{diag}(-1,-1,-1,1)$. The field variables are denoted by y^A. If F is a function of ξ^μ, and possibly also of y^A, then $\partial F/\partial\xi^\mu$ refers only to the explicit dependence upon ξ^μ, whereas $\partial_\mu F=\partial F/\partial\xi^\mu+\partial_\mu y^A(\partial F/\partial y^A)$. Likewise, $\partial F/\partial y^A$ refers only to the explicit dependence upon y^A. ∇_μ denotes the covariant derivative, identical to ∂_μ for scalars, and for tensors as well in pseudo-Cartesian coordinates.

The dynamical behavior of the fields is obtained, as usual, from a variational principle, $\delta W=0$, where $W=(1/c)\int_\Omega L d\omega$. The Lagrangian density $L(y^A,y^B{}_\mu,\xi^\nu)$ is a scalar function. $y^A{}_\mu$ means $\nabla_\mu y^A$. Ω is a volume of space-time, and $d\omega$ is a scalar element of integration in space-time: $d\omega=\sqrt{(-g)}\,(d\xi)^4$. δy^A vanishes on the boundary of Ω, the essential part of which consists of two non-intersecting space-like surfaces σ_1 and σ_2. This leads (provided $\pi^{A\mu}$ is contragredient to y^A, which is assumed) to the usual Eulerian field equations, $L^A\equiv\partial L/\partial y^A-\nabla_\alpha\pi^{A\alpha}=0$, where $\pi^{A\mu}=\partial L/\partial y^A{}_\mu$. The integrated field momenta are $p^A=\int_\sigma\pi^{A\alpha}d\sigma_\alpha$, where $d\sigma_\mu$ is a vector element of integration on the space-like surface σ. The summation convention is used between covariant and contravariant tensor indices, and also between field indices, the latter always being written as superscripts.

"TIME-INDEPENDENT" METHOD

Conservation laws associated with a field are expressed by the vanishing divergence of a number of density tensors (current-charge j^ν, stress energy $T_\mu{}^\nu$, angular momentum $G_{\lambda\mu}{}^\nu$) for which t^ν shall be a generic symbol. If one or more such tensors can be expressed in terms of y^A and $\pi^{A\mu}$ alone, then one may seek to solve the field equations by transforming to new ("appropriate," "ignorable") field variables Y^A and conjugate field momentum densities $P^{A\mu}$, in terms of which the integrated equations of field motion are

$$Y^A=\bar\alpha^A,\quad \nabla_\alpha P^{A\alpha}=1. \tag{1}$$

[Equations (1) are the relativistic field analog of $Q_i=\bar\alpha_i$, $P_i=t-\bar\beta_i$, which hold for ignorable coordinates in classical mechanics.] The conditions under which such variables Y^A exist are not examined here; what is given, rather, is a method of proceeding in cases in which they do exist, i.e., in which the Hamilton-Jacobi equation (6) has a solution. If such variables exist, then Eqs. (1) can be derived from a new Lagrangian density $\bar L$, which need not be identical with L;[6] but if the variational principle $\delta\bar W=0$ is to hold in terms of the new variables as well as the old, it is sufficient that

$$L-\bar L=c\nabla_\alpha\bar S^\alpha. \tag{2}$$

The action density $\bar S^\mu(y^A,Y^B)$ (a vector) does not explicitly depend upon the space-time coordinates ξ^μ.

Integrating (2) from an arbitrary space-like surface σ_i to the space-like surface σ under consideration, one obtains

$$W-\bar W=\bar S(\sigma)-\bar S(\sigma_i), \tag{3}$$

where $\bar S(\sigma)=\int_\sigma\bar S^\alpha d\sigma_\alpha$. If one now considers a variation of (3) from the physical history (i.e., from the history during which the field equations apply throughout), in which the field variables are varied on σ as well as throughout the history from σ_i to σ, but not on σ_i, the result

$$(1/c)\int_\sigma(\pi^{A\alpha}\delta y^A-P^{A\alpha}\delta Y^A)d\sigma_\alpha = \int_\sigma(\partial\bar S^\alpha/\partial Y^A\delta Y^A+\partial\bar S^\alpha/\partial y^A\delta y^A)d\sigma_\alpha \tag{4}$$

is obtained. Since the variations δy^A and δY^A on σ are arbitrary (it is assumed that the relation between y^A and Y^A does not prevent such arbitrary variations), it follows that

$$\pi^{A\mu}=c\partial\bar S^\mu/\partial y^A,\quad P^{A\mu}=-c\partial\bar S^\mu/\partial Y^A. \tag{5}$$

Substitution of the first of Eqs. (5) into the conservation law (or laws) yields the time-independent Hamilton-Jacobi equation,

$$\nabla_\alpha t^\alpha(y^A,c\partial\bar S^\mu/\partial y^B)=0. \tag{6}$$

The constants of separation $\bar\alpha^A$ which arise in the solution are identified with the new field variables Y^A, yielding $\bar S(y^A,\bar\alpha^B)$, and, by (1) and (5), the integrated field equations in terms of the old field variables,[7]

$$c\nabla_\alpha\partial\bar S^\alpha/\partial\bar\alpha^A=-1. \tag{7}$$

$\bar\alpha^A$ have the dimensions of energy per unit volume, and

[6] For instance, the Lagrangian density (in pseudo-Cartesian co-ordinates) $\bar L=\Sigma_A[Y^A+\frac{1}{4}Y^A{}_\alpha(x^\alpha-4\bar\beta^{A\alpha})]$ leading to $P^{A\mu}=\frac{1}{4}(x^\mu-4\bar\beta^{A\mu})$ is an example of a suitable new Lagrangian density. In general, such new Lagrangian densities, where they exist, will be of the form $\bar L=\Sigma_A\,Y^A+$terms in $Y^A{}_\mu$; the explicit space-time dependence of L is purely formal, as $Y^A{}_\mu=0$. $\bar\beta^{A\mu}$ is a set of constant vectors.

[7] In pseudo-Cartesian coordinates, $c\partial\bar S^\mu/\partial\bar\alpha^A=\frac{1}{4}(4\bar\beta^{A\mu}-x^\mu)$.

can be identified, once the solution $\bar{S}^\mu$ has been found, from the first of Eqs. (5).

"TIME-DEPENDENT" METHOD

It may be preferable not to view Hamilton-Jacobi theory as the theory of canonical transformations from a given system of field variables to the system appropriate to the problem under consideration, but from the configuration $y^A=\alpha^A$on some appropriate space-like surface σ_k to the resulting configuration y^A on σ, the space-like surface under consideration. In that case, the generator of the transformation, the time-dependent action density $S^\mu(y^A,\alpha^B,\xi^\nu)$, must depend upon the space-time coordinates characterizing σ. This procedure is usually required if L depends explicitly upon the ξ^μ, in which case there may not be any suitable constants of the motion to set up (6), (i.e., the physical system under consideration is not closed), or if L consists of two "separate" fields in interaction.

The fundamental variational principle $\delta W=0$ implies the existence of a functional of configurations

$$S(y^A,\alpha^B,\sigma)=(1/c)\int L d\omega,$$

the integration being performed from the configuration on σ_k to that on σ, following the physical history. It is assumed that S can be expressed as a surface integral,

$$S(y^A,\alpha^B,\sigma)=\int_\sigma S^\alpha(y^A,\alpha^B,\xi^\nu)d\sigma_\alpha, \tag{8}$$

from which it follows that $L=c\nabla_\alpha S^\alpha$. By a simple extension of the derivation of (5), one again obtains

$$\pi^{A\mu}=c\partial S^\mu/\partial y^A. \tag{9}$$

If the Lagrangian density can be expressed as a function of the field variables and conjugate momentum densities alone, the time-dependent Hamilton-Jacobi equation is

$$L(y^A,c\partial S^\mu/\partial y^B,\xi^\nu)=c\nabla_\alpha S^\alpha. \tag{10}$$

If this equation can be solved for $S^\mu(y^A,\alpha^B,\xi^\nu)$ (the α^B again appearing as constants of separation) the integrated field equations are

$$c\partial S^\mu/\partial\alpha^A=\beta^{A\mu}, \tag{11}$$

where, by a procedure analogous to that leading to (5), the $\beta^{A\mu}$ are identified as minus the (constant) values of the conjugate momentum densities on σ_k.

It was assumed throughout that in $S^\mu(y^A,\alpha^B,\xi^\nu)$, y^A and α^B are independent. The derivation of Eqs. (9) and (11) depends upon that assumption. Actually, this is true only in the case of second order field equations. In the case of first order field equations, the specification of a configuration on a space-like determines the configuration on all other space-like surfaces. Therefore y^A and α^A are not independent, and (9) and (11) cannot hold simultaneously. One approach would be to keep (11), abandoning (9). In that case, the left hand side of (10) cannot be written down. However, in the case of the Dirac field, the most interesting first order case, $L=0$, as can be seen from the fact that L contains the Dirac equation as a factor, or by eliminating $\partial_\mu\psi$ with the aid of the Dirac equation. In that case, a time-dependent Hamilton-Jacobi equation is obtained by setting the right hand side of (10) equal to zero.

RELATION BETWEEN "TIME-INDEPENDENT" AND "TIME-DEPENDENT" METHODS

If a problem has been solved by the time-independent method, the time-dependent action density $S^\mu(y^A,\alpha^B,\xi^\nu)$ may be found from the time-independent function $\bar{S}^\mu(y^A,\bar{\alpha}^B)$ by the relation

$$c\nabla_\alpha S^\alpha-\textstyle\sum_A \bar{\alpha}^A=c\nabla_\alpha\bar{S}^\alpha, \tag{12}$$

which follows from (2), (10), and footnote 6, provided it is integrable.[8] The constants $\bar{\alpha}^A$ thus introduced into S^μ may be identified with the α^A. Consistency of the two methods is further confirmed by differentiating (12) with respect to α^A and comparing

$$c\nabla_\alpha\partial S^\alpha/\partial\alpha^A-1=c\nabla_\alpha\partial\bar{S}^\alpha/\partial\alpha^A \tag{13}$$

with (7) and (11); the $\bar{\beta}^{A\mu}$ may likewise be identified with the $\beta^{A\mu}$.

The time-dependent action density S^μ for a system which can be treated by the time-independent method is not constant, but its change is trivial. The system is evolving, but without changing its "state," i.e., the set of appropriate field variables $\bar{\alpha}^A$ related to the constants of the motion.

PERTURBATION METHOD

Suppose that the Lagrangian density consists of two parts, called, respectively, the free part and the perturbation (one hopes that the latter is sufficiently weak to induce only small changes):

$$L=L_0+L_I. \tag{14}$$

This happens in two important cases which may be considered together:

(*a*) The system under consideration is not closed, and L_I represents the effect of the "outside";

(*b*) The system divides naturally into two or more component parts, with interaction between them.

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[8] In pseudo-Cartesian co-ordinates, the solution is

$$S^\mu=\Sigma_A(\bar{\alpha}^A/4c)x^\mu+\bar{S}^\mu.$$

The time-dependent action density of the free part, $S_0{}^\mu$, may be found by either of the preceding methods. It satisfies $c\nabla_\alpha S_0{}^\alpha = L_0$, which, if subtracted from (10), leaves

$$L_I(y^A, c\partial S^\mu/\partial y^B, \xi^\nu) = c\nabla_\alpha \Delta S^\alpha \tag{15}$$

(where $\Delta S^\mu = S^\mu - S_0{}^\mu$) as the Hamilton-Jacobi equation of the perturbation problem. It may be possible to solve a problem by first obtaining $S_0{}^\mu$, substituting the result into (15), and then solve for ΔS^μ (and thus for S^μ). The integrated field equations are then given by (11).

In addition to possible simplification of the solution, the perturbation method gives a better insight into the physical phenomenon: the perturbation induces a change in the "state" of the system.

EXAMPLE OF TIME INDEPENDENT METHOD: THE FREE DIRAC FIELD

This example, like the following, will be considered in pseudo-Cartesian coordinates.

$$L_D = -\hbar c \psi_{\text{adj}} (i\gamma^\mu \partial_\mu + \kappa)\psi.$$

The field variables, ψ and ψ_{adj}, are spinors. ψ_{adj} denotes the adjoint of ψ, $\psi_{\text{adj}} = \psi^\dagger \gamma^4$, where $\psi^\dagger$ is the Hermitian conjugate of ψ. γ^μ is a vector, the components of which satisfy $[\gamma_\lambda, \gamma^\mu]_+ = 2g_\lambda{}^\mu$. The conjugate momentum densities are $\pi^\mu = -i\hbar c \psi_{\text{adj}} \gamma^\mu$ and $\pi_{\text{adj}}{}^\mu = 0$. It is not possible to express the conventional stress-energy density tensor $T_\mu{}^\nu$ as a function of the field variables and conjugate momentum densities alone; but it is possible so to express the current charge density vector $j^\mu \sim \psi_{\text{adj}} \gamma^\mu \psi \sim \pi^\mu \psi$ yielding the Hamilton-Jacobi equation (6),

$$\partial_\alpha (c(\partial \bar{S}^\alpha/\partial\psi)\psi) = 0,$$

a special solution of which is

$$\bar{S}^\mu = (\hbar\kappa^2/4i) k^\mu \ln(\psi/u).$$

Here, $\bar{\alpha} = \hbar\kappa^4 c$, $k^\alpha k_\alpha = \kappa^2$, and u is a constant spinor. The integrated field equations are (by footnote 7)

$$\psi = u \exp[ik_\alpha(4\bar{\beta}^\alpha - x^\alpha)].$$

EXAMPLE OF TIME-DEPENDENT METHOD: THE FREE KLEIN-GORDON FIELD

$$L_{\text{KG}} = -\tfrac{1}{2}(c\hbar/\kappa)(\psi^{*\alpha}\psi_\alpha - \kappa^2\psi^*\psi).$$

The field variables are the complex scalars ψ and ψ^*. ψ_μ means $\partial_\mu\psi$. The conjugate momenta are

$$\pi^\mu = -\tfrac{1}{2}(c\hbar/\kappa)\psi^{*\mu}, \quad \pi^{*\mu} = -\tfrac{1}{2}(c\hbar/\kappa)\psi^\mu.$$

The Lagrangian density can be expressed in terms of the field variables and momentum densities alone, leading to the time-dependent Hamilton-Jacobi equation (10),

$$-(2\kappa c/\hbar)(\partial S^\alpha/\partial\psi)(\partial S_\alpha/\partial\psi^*) + (c\hbar\kappa/2)\psi^*\psi = c\partial_\alpha S^\alpha.$$

Carrying out the differentiation on the right hand side[9] and changing field variables by $\psi = \psi_1 + i\psi_2$, $\psi^* = \psi_1 - i\psi_2$, one is left with

$$\tfrac{1}{2}(c\kappa/\hbar)[(\partial S^\alpha/\partial\psi_1)(\partial S_\alpha/\partial\psi_1) + (\partial S^\alpha/\partial\psi_2)(\partial S_\alpha/\partial\psi_2)] + \tfrac{1}{2}c\hbar\kappa(\psi_1{}^2 + \psi_2{}^2) = c\partial S^\alpha/\partial x^\alpha,$$

in which the *Ansatz*

$$S^\mu(\psi_1, \psi_2, x^\nu) = S_1{}^\mu(\psi_1) + S_2{}^\mu(\psi^2) + S_x{}^\mu(x^\nu)$$

can be used to separate the variables:

$$\tfrac{1}{2}(c\kappa/\hbar)(\partial S_i{}^\alpha/\partial\psi_i)(\partial S_{i\alpha}/\partial\psi_i) + \tfrac{1}{2}c\hbar\kappa\psi_i{}^2 = \alpha_i, \quad (i = 1, 2),$$

$$c\partial S_x{}^\alpha/\partial x^\alpha = \alpha_x, \quad \alpha_1 + \alpha_2 = \alpha_x.$$

A special solution of these equations is

$$S_i{}^\mu = \tfrac{1}{8}\hbar v^\mu \{\psi_i[2\alpha_i/(\hbar c\kappa) - \psi_i{}^2]^{\frac{1}{2}} + [2\alpha_i/(\hbar c\kappa)] \sin^{-1}[\psi_i(2\alpha_i)^{-\frac{1}{2}}(\hbar c\kappa)^{\frac{1}{2}}]\},$$

$$S_x{}^\mu = \alpha_x x^\mu/(4c),$$

where $v^\mu = k^\mu/\kappa$ is an arbitrary direction cosine, and the integrated field equations are, by (11),

$$\psi_i = (2\alpha_i)^{\frac{1}{2}}(\hbar c\kappa)^{-\frac{1}{2}} \sin[k_\alpha(4\beta_i{}^\alpha - x^\alpha)].$$

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[9] This corresponds to the introduction of a Hamiltonian density