Mass Corrections to the Hyperfine Structure in Hydrogen

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In a previous paper, a covariant wave equation for bound states was used to calculate corrections to fine structure. The techniques developed in that paper are extended here to the study of hyperfine structure (hfs) in the hydrogen atom. An expression is derived for corrections of relative order $(\alpha m/M)$ to the hfs splitting of any S-state of hydrogen, arising from the finite mass of the nucleus. The proton is considered as a point particle with an anomalous magnetic moment of the Pauli type in addition to its Dirac moment. It is shown that the corrections obtained with an unmodified Pauli moment diverge logarithmically, and a cut-off in momentum space is introduced. Numerical results are given for the limit of large cut-off momentum. It is shown that the leading terms, which involve $\log(M/m)$ as a factor, can also be obtained from a modified form of three-dimensional perturbation theory.

1. INTRODUCTION

THE Fermi formula for the hyperfine structure splitting of S-states in hydrogen (hereafter designated by hfs) is

$$\text{hfs}=(2\pi\alpha\mu_p/3mM)(\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b)|\phi(0)|^2, \qquad (1)$$

where α is the fine structure constant, μ_p is the proton magnetic moment in nuclear magnetons, m and M are the electron and proton masses respectively, and $\phi(0)$ is the Schrödinger wave function evaluated at the nucleus. We are using a system of units in which $\hbar=c=1$. Since $|\phi(0)|^2$ is proportional to m_R^3, where m_R is the reduced mass $mM/(M+m)$, the hfs splitting contains a factor $(1-3m/M)$, the simple "reduced mass correction." This factor, derived by a nonrelativistic argument, accounts approximately for the recoil of the proton. Using a relativistic approach, we would expect further mass corrections of relative order $\alpha m/M$. In 1947, Breit and Meyerott[1,2] applied the approximately covariant Breit equation to this problem. They found that the Breit equation gives the reduced mass correction, but no terms of order $\alpha m/M$.

Bethe and Salpeter[3] have derived a completely covariant procedure for handling such problems, which was applied by Salpeter[4] to finding corrections of relative order $\alpha m/M$ to the fine structure (fs). Karplus, Klein, and Schwinger[5,6] have also found a method, based on the Schwinger-Tomonaga formulation of quantum electrodynamics, for solving bound-state problems, and have used it to obtain mass corrections to the hfs in positronium.[7] This procedure has also been applied by Arnowitt[8] to finding mass corrections to the hfs in hydrogen. This paper presents another calculation of the hydrogen hfs for S-states, by a method analogous to that used in S. Since hfs is of order $(m/M)\times$(fs), the present calculation consists essentially of carrying the expansion in absolute powers of (m/M) of S one order higher, retaining only spin-dependent terms, and including the effect of a Pauli-type magnetic moment. Wherever possible we shall use the notation of S.

As in S, we find it convenient to separate the instantaneous Coulomb interaction from the effect of the transverse photons in accordance with the equation[9]

$$\gamma_\mu{}^a\gamma_\mu{}^b/k_\mu{}^2=-\gamma_4{}^a\gamma_4{}^b[1/k^2+\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b/k_\mu{}^2], \qquad (2)$$

where $\boldsymbol{\alpha}_\perp$ is the component of $\boldsymbol{\alpha}$ perpendicular to $\mathbf{k}$. A similar separation will be used for the Pauli terms.

The mass corrections are all produced by processes in which the proton and electron interact twice, either through the exchange of two transverse photons (double photon terms) or through one transverse photon and one instantaneous interaction. Processes involving two instantaneous interactions contribute to the fs corrections, but, since they are not spin-dependent, they do not contribute to the hfs.

Let $\mathbf{k}'$ and $\mathbf{k}$ be the momenta absorbed by the proton during the first and second interactions respectively. It will be shown that the main contributions to the hfs mass corrections come from values of k between m and M. Since these values are large compared with the Bohr momentum of the atom, which is of order αm, it will be possible to neglect the internal momentum of the atom and to approximate $\mathbf{k}'$ by $(-\mathbf{k})$. In this approximation, the mass correction terms depend on the atomic wave function only through the factor $|\phi(0)|^2$.

Most of these terms may be approximated by ex-

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[1] G. Breit and R. E. Meyerott, Phys. Rev. **72**, 1023 (1947).

[2] Breit, Brown, and Arfken, Phys. Rev. **76**, 1299 (1949).

[3] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

[4] E. E. Salpeter, Phys. Rev. **87**, 328 (1952), hereafter referred to as S.

[5] R. Karplus and A. Klein, Phys. Rev. **85**, 972 (1952).

[6] Karplus, Klein, and Schwinger, Phys. Rev. **86**, 288 (1952).

[7] R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952).

[8] R. Arnowitt, Phys. Rev. **92**, 1002 (1953).

[9] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

pressions of the form

$$\Delta E=C(\alpha m/M)(\text{hfs})\int_m^M dk/k$$
$$=C(\alpha m/M)\log(M/m)(\text{hfs}), \quad (3)$$

where C is a constant factor of order unity, and (hfs) is given by Eq. (1). If we evaluate the integrals more carefully, we also find terms which do not contain the large factor $\log(M/m)$, but none with a factor $\log\alpha$. All such terms, whether or not they contain $\log(M/m)$, are said to be of relative order $\alpha m/M$.

We treat the proton as a point particle with a Dirac moment of unity and a Pauli moment of (μ_p-1). Since the two moments behave in entirely different ways, it is necessary to treat them separately. The terms involving two Pauli interactions turn out to be logarithmically divergent. This divergence would presumably be removed in a consistent theory for the anomalous proton moment. In the absence of such a theory, we shall cut off the Pauli terms at a finite momentum k_0, corresponding to a crude spreading-out of the Pauli moment over a radius k_0^{-1}.

The four-dimensional calculations will be presented in Secs. 2 to 4. To show the relation of this four-dimensional method to conventional, noncovariant theory and to check the results, the mass corrections will also be evaluated by three-dimensional perturbation theory. This procedure yields only the terms containing $\log(M/m)$. It is carried through for single-photon terms in Sec. 5 and four double-photon terms in Sec. 6.[10-12]

2. SINGLE-PHOTON TERMS WITH DIRAC INTERACTIONS ONLY

The hfs correction arising from the exchange of a single transverse photon is given by

$$\Delta E_{CD}=\Delta E_{CD}{}^0+\Delta E_{CD}{}^\times-\Delta E_B. \quad (4)$$

$\Delta E_{CD}{}^0$ and $\Delta E_{CD}{}^\times$ represent the first order corrections arising respectively from diagram (1b) and from the crossed diagrams (2b) and (2c). The subscript CD indicates that these terms involve a Coulomb interaction and a Dirac photon.[13] In Fig. (1b), no Coulomb interaction is shown. The Coulomb interaction in $\Delta E_{CD}{}^0$ arises instead from the iteration of the initial or final wave function according to Eq. (29) of S. As in S, it is convenient to subtract ΔE_B, the single photon

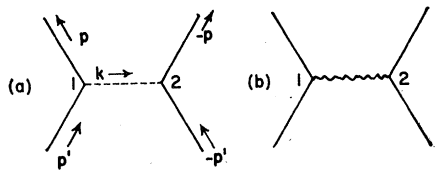

FIG. 1. Second order Feynman diagrams: Fermions are denoted by solid lines, Coulomb interactions by dotted lines, and transverse photons by wavy lines. Electrons are on the left, protons on the right.

correction arising from a solution of the Breit equation in which the transverse photon term is evaluated by means of first order perturbation theory. Since such a solution of the Breit equation gives the ordinary hfs with reduced mass correction and nothing else,[1,2] this subtraction procedure will remove these large effects, leaving only the corrections of order $\alpha m/M$.

The Breit equation in momentum space with a transverse photon term is

$$[E-H_a(\mathbf{p})-H_b(\mathbf{p})]\phi(\mathbf{p})$$
$$=-(e^2/2\pi^2)\int(d^3k/k^2)[1-\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b]\phi(\mathbf{p}+\mathbf{k}), \quad (5)$$

where H_a and H_b are the free-particle Dirac Hamiltonians for the electron and proton respectively:

$$H_a(\mathbf{p})=\boldsymbol{\alpha}^a\cdot\mathbf{p}+\beta^a m, \quad (6a)$$
$$H_b(\mathbf{p})=-\boldsymbol{\alpha}^b\cdot\mathbf{p}+\beta^b m. \quad (6b)$$

The transverse photon term produces a perturbation energy of

$$\Delta E_B=(e^2/2\pi^2)\int(d^3pd^3k/k^2)\phi^*(\mathbf{p})\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\phi(\mathbf{p}+\mathbf{k}), \quad (7)$$

where ϕ satisfies

$$[E-H_a(\mathbf{p})-H_b(\mathbf{p})]\phi(\mathbf{p})$$
$$=-(e^2/2\pi^2)\int(d^3k/k^2)\phi(\mathbf{p}+\mathbf{k}). \quad (8)$$

It is convenient to express ϕ in terms of positive and negative energy components, using the Casimir projection operators as in S:

$$\phi_{\pm\pm}(\mathbf{p})=\Lambda_\pm{}^a(\mathbf{p})\Lambda_\pm{}^b(\mathbf{p})\phi(\mathbf{p}), \quad (9)$$
$$\Lambda_\pm(\mathbf{p})=[E(\mathbf{p})\pm H(\mathbf{p})]/2E(\mathbf{p}), \quad (10)$$

where $E(\mathbf{p})=(m^2+p^2)^{\frac{1}{2}}$. ϕ_{++} is larger by a factor of α^{-1} than the largest of the other three components of ϕ.

With ϕ and ϕ^* each expressed as the sum of four terms, ΔE_B breaks up into sixteen terms, of which only the terms containing either ϕ_{++} or $\phi_{++}{}^*$ or both contribute to the required order of magnitude. The largest

[10] The results of the three-dimensional calculation for hydrogen and of a very crude calculation for deuterium have previously been published by the authors (see reference 11). These results were incorrect, since the instantaneous part of the Pauli interaction was erroneously neglected. A more adequate treatment for deuterium will be given in a future publication.[12]

[11] E. E. Salpeter and W. A. Newcomb, Phys. Rev. 87, 150 (1952).

[12] Greifinger, Newcomb, and Salpeter (to be published).

[13] A Dirac (Pauli) photon is a photon which interacts with the Dirac (Pauli) moment of the proton.

term is

$$\Delta E_B{}^{++}=(e^2/2\pi^2)\int(d^3pd^3k/k^2)\times\phi_{++}{}^*(\mathbf{p})\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\phi_{++}(\mathbf{p}+\mathbf{k}). \quad (11)$$

The two terms containing either ϕ_{-+} or $\phi_{-+}{}^*$ can be combined to form the following approximate expression, after expressing ϕ_{-+} in terms of ϕ_{++} by means of the $(-+)$ component of Eq. (8) and dropping higher orders in α,

$$\Delta E_B{}^{-+}=-2\left(\frac{e^2}{2\pi^2}\right)^2\int\frac{d^3pd^3p'd^3p''\phi_{++}{}^*(\mathbf{p})\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\Lambda_-{}^a(\mathbf{p}')\Lambda_+{}^b(\mathbf{p}')\phi_{++}(\mathbf{p}'')}{k^2k'^2(E+E_a'-E_b')}, \quad (12)$$

where $\mathbf{k}=\mathbf{p}'-\mathbf{p}$, $\mathbf{k}'=\mathbf{p}''-\mathbf{p}'$, $E_a'=E_a(\mathbf{p}')$, and $E_b'=E_b(\mathbf{p}')$. Two further terms $\Delta E_B{}^{+-}$ and $\Delta E_B{}^{--}$ are obtained by analogy with Eq. (12).

The next term to be considered is $\Delta E_{CD}{}^0$, the first order contribution from diagram (1b). By using Eqs. (23a) and (38) of S,

$$\Delta E_{CD}{}^0=-(e^2/2\pi^2)\int d^4pd^4p'\psi^*(p_\mu)\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\times\psi(p_\mu')(\omega^2-k^2+i\Delta)^{-1}, \quad (13)$$

where ψ is the four-dimensional wave function discussed in S, and where ω is the fourth component of $k_\mu=p_\mu'-p_\mu$. If we split ψ and ψ^* into positive and negative energy components, the main term will be the one involving ψ_{++} and $\psi_{++}{}^*$. By using Eqs. (10) and (15) of S, ψ_{++} may be expressed in terms of the positive energy component ϕ_{++} of a three-dimensional wave function. After carrying out the integration over ϵ and ϵ', the fourth components of p_μ and p_μ', we find that

$$\Delta E_{CD}{}^{0++}=(e^2/2\pi^2)\int d^3pd^3p'\phi_{++}{}^*(\mathbf{p})\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\phi_{++}(\mathbf{p}')\times\left[\frac{1}{k^2-(E_b'-E_b)^2}+\frac{E-E_a-E_b}{2k(E-E_a-E_b'-k)(E_b-E_b'-k)}+\frac{E-E_a'-E_b'}{2k(E-E_a'-E_b-k)(E_b'-E_b-k)}\right]. \quad (14)$$

ϕ_{++} is the $(++)$ component of the solution of the zero-order three-dimensional Bethe-Salpeter equation,

$$[E-H_a(\mathbf{p})-H_b(\mathbf{p})]\phi(\mathbf{p})=-(e^2/2\pi^2)[\Lambda_+{}^a(\mathbf{p})\Lambda_+{}^b(\mathbf{p})-\Lambda_-{}^a(\mathbf{p})\Lambda_-{}^b(\mathbf{p})]\times\int(d^3k/k^2)\phi(\mathbf{p}+\mathbf{k}). \quad (15)$$

To the order of accuracy required, the $(++)$ components of the solutions of Eqs. (8) and (15) agree.

The three terms in brackets in Eq. (14) will hereafter be designated by the letters a, b, and c. The first term $\Delta E_{CD}{}^{0++a}$ is much larger than the effects we are studying. In fact, it gives rise to the ordinary hfs, as we may readily see by evaluating it to lowest order in α. The matrix element in Eq. (14) is, for $k\ll m$,

$$\phi_{++}{}^*(\mathbf{p})\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\phi_{++}(\mathbf{p}')=(4mM)^{-1}\phi_{++}{}^{++*}(\mathbf{p})[-4p_\perp{}^2+2i\mathbf{p}\cdot(\boldsymbol{\sigma}^a\times\mathbf{k})+2i\mathbf{p}\cdot(\boldsymbol{\sigma}^b\times\mathbf{k})+(\boldsymbol{\sigma}^a\times\mathbf{k})\cdot(\boldsymbol{\sigma}^b\times\mathbf{k})]\phi_{++}{}^{++}(\mathbf{p}'), \quad (16)$$

where $\phi_{++}{}^{++}$ designates the upper four components of the sixteen component wave function ϕ_{++} in the notation of S.[14]

The first two terms of (16) are independent of the proton spin, and therefore do not contribute to the hfs. The main contribution to $\Delta E_{CD}{}^{0++a}$ is from the region where p and p' are both of order αm. In this region, Eq. (16) is valid, $\phi_{++}{}^{++}$ may be approximated to lowest order in α by the Schrödinger wave function ϕ_0, and $(E_b'-E_b)^2$ is negligible compared with k^2. The third term in (16) then gives the orbital hfs, which vanishes for S states. The fourth term gives, after averaging over angles,

$$\Delta E=(e^2/12\pi^2mM)\int d^3pd^3p'\phi_0{}^*(\mathbf{p})\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b\phi_0(\mathbf{p}'), \quad (17)$$

which immediately reduces to $\mu_p{}^{-1}$ times the Fermi expression, Eq. (1), i.e., to that part of the ordinary spin-spin hfs which is due to the Dirac moment of the proton.

To get the mass correction, we subtract $\Delta E_B{}^{++}$ from $\Delta E_{CD}{}^{0++a}$. The result is

$$(e^2/2\pi^2)\int d^3pd^3p'\phi_{++}{}^*(\mathbf{p})\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\phi_{++}(\mathbf{p}')\times(E_b'-E_b)^2k^{-2}[k^2-(E_b'-E_b)^2]^{-1}. \quad (18)$$

The main contributions to this integral come from the regions $p\sim\alpha m$, $p'\sim M$, and $p\sim M$, $p'\sim\alpha m$. The contributions are equal to each other and of order $\alpha(m/M)$(hfs). We therefore evaluate this term only to lowest order in α. If $p\sim\alpha m$, $\phi_{++}{}^*(\mathbf{p})$ may be replaced by $\phi_0{}^*(\mathbf{p})$, the conjugate of the Schrödinger wave function. However, since p' will then be large, $\phi_{++}(\mathbf{p}')$ must not be approximated by ϕ_0. To get an approximation to $\phi_{++}(\mathbf{p}')$ which is good to lowest order in α for

[14] Throughout this paper, as discussed in the Appendix of S, $+$ or $-$ signs as subscripts to wave functions refer to eigenstates of the Casimir projection operators for the relevant momentum; $+$ or $-$ signs as superscripts refer to the eigenstates of the Dirac operator β.

any value of $\mathbf{p}'$, we may use the iterated wave function ϕ_1, defined by Eq. (29) of S.

By using ϕ_1, we shall eventually be able to express any contribution $\Delta E_\alpha{}^\beta$ to the hfs mass correction in the form

$$\Delta E_\alpha{}^\beta = (e^2/2\pi^2)^2 \int d^3p d^3p' d^3p'' \phi_{++}{}^*(\mathbf{p}) \times \mathcal{F}_\alpha{}^\beta \phi_{++}(\mathbf{p}'')/k^2 k'^2, \quad (19)$$

where $\mathbf{k}$ and $\mathbf{k}'$ are defined as in Eq. (12) and α, β stand for arbitrary subscripts and superscripts. In all such expressions we shall replace ϕ_{++} and $\phi_{++}{}^*$ by ϕ_0 and $\phi_0{}^*$.

With this notation, we find

$$\mathcal{F}_{CD}{}^{0++a} - \mathcal{F}_B{}^{++} = -2 \frac{\boldsymbol{\alpha}_\perp{}^a \cdot \boldsymbol{\alpha}_\perp{}^b \Lambda_+{}^a(\mathbf{p}') \Lambda_+{}^b(\mathbf{p}') (E_b' - E_b)^2}{(E - E_a' - E_b')[k^2 - (E_b' - E_b)^2]}. \quad (20)$$

The terms b and c in (14) are equal, as may be seen by interchanging $\mathbf{p}$ and $\mathbf{p}'$, and taking the complex conjugate. Term c has no contributions of the required order from values of p large compared to αm. To get an approximation to lowest order in α, we must therefore iterate $\phi_{++}(\mathbf{p}')$, but not $\phi_{++}{}^*(\mathbf{p})$. The result is

$$\mathcal{F}_{CD}{}^{0++b} + \mathcal{F}_{CD}{}^{0++c} = 2\mathcal{F}_{CD}{}^{0++c} = -\frac{k \boldsymbol{\alpha}_\perp{}^a \cdot \boldsymbol{\alpha}_\perp{}^b \Lambda_+{}^a(\mathbf{p}') \Lambda_+{}^b(\mathbf{p}')}{(E - E_a' - E_b - k)(E_b' - E_b - k)}. \quad (21)$$

The iteration procedure used to derive Eqs. (20) and (21) has the effect of bringing in a Coulomb interaction which is not crossed with the photon. The same thing will happen with all the other components of $\Delta E_{CD}{}^0$.

The contributions to the required order of these other components consist of terms containing $\psi_{++}{}^*$ and one of the functions ψ_{-+}, ψ_{+-}, and ψ_{--}, and of the complex conjugates of such terms. The four-dimensional wave functions are then expressed in terms of ϕ_{++}, using the uniterated expression for ψ_{++} and the iterated form for the negative energy components, in accordance with Eqs. (10), (15), (15a), and (15b) of S. After carrying out the integration over the fourth components of the momenta, we find

$$\mathcal{F}_{CD}{}^{0-+} = \frac{k \boldsymbol{\alpha}_\perp{}^a \cdot \boldsymbol{\alpha}_\perp{}^b \Lambda_-{}^a(\mathbf{p}') \Lambda_+{}^b(\mathbf{p}')}{(E_a' + E_a + k)(E - E_a - E_b' - k)}, \quad (22)$$

$$\mathcal{F}_{CD}{}^{0+-} = \frac{k \boldsymbol{\alpha}_\perp{}^a \cdot \boldsymbol{\alpha}_\perp{}^b \Lambda_+{}^a(\mathbf{p}') \Lambda_-{}^b(\mathbf{p}')}{(E_b' + E_b + k)(E - E_a' - E_b - k)}, \quad (23)$$

$$\mathcal{F}_{CD}{}^{0--} = \boldsymbol{\alpha}_\perp{}^a \cdot \boldsymbol{\alpha}_\perp{}^b \Lambda_-{}^a(\mathbf{p}') \Lambda_-{}^b(\mathbf{p}') \times [k(E_b' + E_b + k)^{-1}(E + E_a' - E_b - k)^{-1} - 2k^2(E + E_a' + E_b')^{-1}\{(E + E_a' - E_b)^2 - k^2\}]. \quad (24)$$

We now turn to the crossed terms. Using Eq. (56) of S, and observing that diagrams (2b) and (2c) give

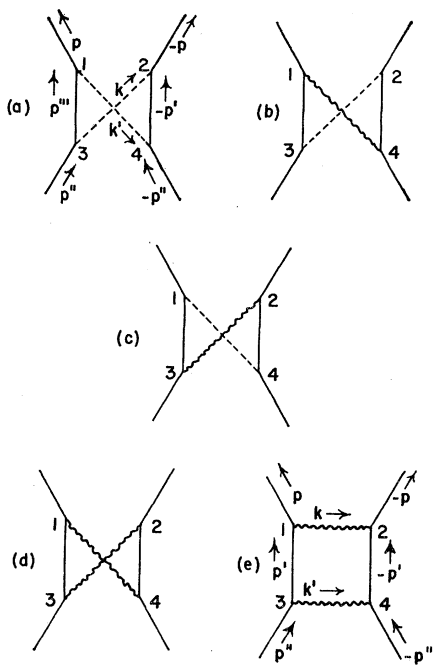

FIG. 2. Fourth order Feynman diagrams: p, p', etc. denote the momentum four-vector of a particle minus (m/m_a) times the four-vector of the whole atom, where m and m_a are the masses of the particle and the atom respectively. In all crossed diagrams, the momenta are as in Fig. 2(a).

equal contributions to $\Delta E_{CD}{}^\times$, we may write

$$\Delta E_{CD}{}^\times = (i/\pi)(e^2/2\pi^2)^2 \int d^4p d^4p' d^4p'' \times k'^{-2}(\omega^2 - k^2 + i\Delta)^{-1} \psi^*(p_\mu) \times \sum_{i=1}^{2} \alpha_i{}^b [\eta_a E - H_a(\mathbf{p}''') - \omega + \epsilon'' + i\delta\beta^a]^{-1} \times [\eta_b E - H_b(\mathbf{p}') - \omega - \epsilon + i\delta\beta^b]^{-1} \alpha_i{}^a \psi(p_\mu''), \quad (25)$$

where $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{p}''$, $\mathbf{k}$, and $\mathbf{k}'$ are the same as in the uncrossed terms, and where $p_\mu''' = p_\mu'' - k_\mu$. $\sum$ represents a summation over the two directions perpendicular to $\mathbf{k}$. η_a and η_b (denoted by μ_a and μ_b in S) are given by

$$\eta_a = m(M+m), \quad (26a)$$

$$\eta_b = M/(M+m). \quad (26b)$$

To resolve $\Delta E_{CD}{}^\times$ into positive and negative energy components, we introduce the operator

$$[\Lambda_+{}^a(\mathbf{p}''') + \Lambda_-{}^a(\mathbf{p}''')][\Lambda_+{}^b(\mathbf{p}') + \Lambda_-{}^b(\mathbf{p}')] = 1$$

into the integrand of (25). This eliminates the Dirac matrices in the denominator. Since the integrand is appreciable only when p and p'' are both of order αm, $\psi^*(p_\mu)$ and $\psi(p_\mu'')$ may be replaced by $\psi_{++}{}^*$ and ψ_{++}, which are expressible in terms of $\phi_{++}{}^*$ and ϕ_{++} as before. After carrying out the integrations over the

fourth components of the momenta, we obtain expressions for $\mathfrak{F}_{CD}{}^{\times++}$, $\mathfrak{F}_{CD}{}^{\times-+}$, etc. For example,

$$\mathfrak{F}_{CD}{}^{\times++}=\frac{k\sum_i \alpha_i{}^b\Lambda_+{}^a(\mathbf{p}''')\Lambda_+{}^b(\mathbf{p}')\alpha_i{}^a}{(E-E_a'''-E_b''-k)(E-E_a-E_b'-k)}. \quad (27)$$

Aside from the main term $\Delta E_{CD}{}^{0++a}$, which is cancelled by $-\Delta E_B{}^{++}$ in the region $k\sim\alpha m$, $\Delta E_{CD}{}^{\times++}$ and $\Delta E_{CD}{}^{0++bc}$ are the only terms which give appreciable contributions from this region. It is easy to show that these two terms cancel to lowest order in α for $k\sim\alpha m$, except for terms which are independent of the electron spin. This means that the region $k\gtrsim m\gg\alpha m$ is the only important one. On the other hand, the only important values of p and p'' are of order αm, the Bohr momentum, since the Schrödinger wave functions decrease rapidly for larger momenta. Since $p,p''\ll k$, we may, to the required order of accuracy, replace $\mathbf{p}$ and $\mathbf{p}''$ by zero, $\mathbf{p}'$ by $\mathbf{k}$, and $\mathbf{p}'''$ by $-\mathbf{k}$ in each expression $\mathfrak{F}_\alpha{}^\beta$. With these approximations, the integral over $\mathbf{p}$ and $\mathbf{p}''$ in Eq. (19) reduces to

$$\int d^3pd^3p''\phi_0{}^*(\mathbf{p})\phi_0(\mathbf{p}'')=(2\pi)^3|\phi_0(0)|^2. \quad (28)$$

Any $\Delta E_\alpha{}^\beta$ is then expressible in the form

$$\Delta E_\alpha{}^\beta=(\alpha m/\pi\mu_p M)(\text{hfs})T_\alpha{}^\beta, \quad (29)$$

where

$$T_\alpha{}^\beta=12M^2(\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b)^{-1}\int_0^\infty dkk^{-2}(\mathfrak{F}_\alpha{}^\beta). \quad (30)$$

$(\mathfrak{F}_\alpha{}^\beta)$ is the expectation value of $\mathfrak{F}_\alpha{}^\beta$ for a state with zero momentum.

To illustrate the procedure, we will carry out the evaluation of $\Delta E_{CD}{}^{0++bc}+\Delta E_{CD}{}^{\times++}$. The expectation values of the numerators of (21) and (27) are given by

$$\begin{aligned}k(\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\Lambda_+{}^a(\mathbf{k})\Lambda_+{}^b(\mathbf{k}))&\\ =k(\textstyle\sum_i\alpha_i{}^b\Lambda_+{}^a(-\mathbf{k})\Lambda_+{}^b(\mathbf{k})\alpha_i{}^a)&\\ =k^3(\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b)/6E_a{}^kE_b{}^k,&\quad (31)\end{aligned}$$

after averaging over angles. $E_a{}^k$ and $E_b{}^k$ are abbreviations for $E_a(k)$ and $E_b(k)$.

We now have, using Eqs. (21), (27), (30), and (31)

$$\begin{aligned}T_{CD}{}^{0++bc}+T_{CD}{}^{\times++}&\\ =-2M\int_0^\infty(kdk/E_a{}^kE_b{}^k)(E_a{}^k-m+k)^{-1}.&\quad (32)\end{aligned}$$

To simplify the evaluation of the $T_\alpha{}^\beta$, we split the integral into two ranges, $k<A$ and $k>A$, where $m\ll A\ll M$. For $k<A$ we expand in powers of k/M and m/M, and for $k>A$ in powers of m/k and m/M, keeping only the lowest order term. After carrying out the integration, we neglect (m/A) and (A/M) compared to unity. These approximations, which introduce errors of relative order $(m/M)^{\frac{1}{2}}$, finally give

$$T_{CD}{}^{0++bc}+T_{CD}{}^{\times++}=-\log(M/m)-2\log 2-1. \quad (33)$$

The other single-photon terms are treated in precisely the same way. The results, with crossed, uncrossed, and Breit terms combined, but with positive and negative energy contributions separated, are

$$T_{CD}{}^{++}=-\log(M/m)-1, \quad (34a)$$

$$T_{CD}{}^{-+}=-5\log(M/m)+9, \quad (34b)$$

$$T_{CD}{}^{+-}=-\log(M/m)-3, \quad (34c)$$

$$T_{CD}{}^{--}=3\log(M/m)-5. \quad (34d)$$

The total single-photon contribution with Dirac interactions alone, the sum of Eqs. (34), is then

$$T_{CD}=-4\log(M/m). \quad (35)$$

The integrals for the various $T_\alpha{}^\beta$ can also be evaluated in an elementary manner if a cut-off is introduced, i.e., if the upper limit of the k-integration is put equal to a constant, $k_0{}^D$. This cut-off can be introduced into all integrals except in those for the Breit terms. These must be integrated to infinity to be equivalent to the results of Breit and Meyerott,[1,2] which we have used. With such a cut-off, the expression for T_{CD} is

$$\begin{aligned}T_{CD}=-4\log(M/m)-8(E_0{}^D-k_0{}^D)/k_0{}^D&\\ +4\log[(E_0{}^D+k_0{}^D)/2k_0{}^D],&\quad (35a)\end{aligned}$$

where $E_0{}^D=(M^2+k_0{}^{D2})^{\frac{1}{2}}$.

3. SINGLE-PHOTON TERMS WITH PAULI INTERACTIONS

To take account of the anomalous magnetic moment of the proton, we replace $\gamma_\mu{}^b$ in the Dirac interaction terms by

$$\Gamma_\mu{}^b(q_\rho)=(\mu/4M)q_\rho(\gamma_\rho{}^b\gamma_\mu{}^b-\gamma_\mu{}^b\gamma_\rho{}^b), \quad (36)$$

where $\mu=\mu_p-1$ is the anomalous moment in nuclear magnetons, and where q_ρ is the four-momentum absorbed by the proton. q_ρ is equal to k_ρ' or k_ρ according as the interaction in question is the first or the second along the world line of the proton, counting the bottom of a diagram as the earlier part.

Using the new "vertex part" Γ_μ, we may eliminate longitudinal waves just as before to obtain

$$\begin{aligned}\gamma_\mu{}^a\Gamma_\mu{}^b(q_\rho)/(q_4{}^2-q^2+i\Delta)=-\gamma_4{}^a\gamma_4{}^b[A_4{}^b(q_\rho)/q^2&\\ +\sum_{i=1,2}\alpha_i{}^aA_i{}^b(q_\rho)/(q_4{}^2-q^2+i\Delta)^{-1}],&\quad (37)\end{aligned}$$

where $A_\mu{}^b(q_\rho)=\gamma_4{}^b\Gamma_\mu{}^b(q_\rho)$. It should be pointed out that Eq. (37) is a strict identity of matrices, whereas the corresponding relation for a Dirac moment asserts only that the relevant matrix elements are equal.

The first term on the right-hand side of Eq. (37) represents an instantaneous interaction, hereafter designated as a Q-interaction. It arises from the fact that the motion of the anomalous magnetic moment induces an electric moment which interacts with the Coulomb field of the electron. To find the effect of the Q-interaction on the single-photon terms, we must replace the Coulomb interaction operator q^{-2} by the operator

$$q^{-2}[1+(\mu/2M)\beta^b\boldsymbol{\alpha}^b\cdot\mathbf{q}]. \tag{38}$$

This is the operator which should now be used in iterating the Schrödinger wave function to obtain ϕ_1. This will affect the value of $\Delta E_{CD}{}^0$, the contribution from diagram (1b). The additional term, $\Delta E_{QD}{}^0$, is gotten by substituting Q-interactions for Coulomb interactions in Eqs. (21) to (24), and in the part of Eq. (20) corresponding to $\Delta E_{CD}{}^{0++a}$. These terms represent a Q-interaction and a Dirac photon, the two interactions being uncrossed. Finally, we will have to add terms $\Delta E_{QD}{}^\times$ representing new diagrams resulting from diagrams (2b) and (2c) by the substitution of a Q-interation for the Coulomb interaction. Of course, we no longer subtract Breit terms.

For positive energy intermediate states of the proton, the integrands are smaller by a factor of order $(k/M)^2$ if we use a Q- instead of a Coulomb interaction. For negative energy states, they are not changed in order of magnitude. We may, therefore, still neglect the region $k\sim\alpha m$, since only the positive energy terms gave contributions of the required order to ΔE_{CD} from this region. These contributions, one of which was of order (hfs), are now reduced to $\alpha^2(m/M)^2$ (hfs) or less.

For $k\gg\alpha m$, it is easy to show that introducing the matrix $(\mu/2M)\beta^b\boldsymbol{\alpha}^b\cdot\mathbf{q}$ simply multiplies the integrand by a factor f^Q, given by

$$f_+{}^Q=-(\mu/2M)(E_b{}^k-M), \tag{39a}$$

$$f_-{}^Q=(\mu/2M)(E_b{}^k+M), \tag{39b}$$

where $f_+{}^Q$ or $f_-{}^Q$ is used according as the intermediate proton energy is positive or negative. Such multipliers, used to convert Dirac terms to Pauli terms, will hereafter be called f factors.

To compute $\Delta E_{QD}=\Delta E_{QD}{}^0+\Delta E_{QD}{}^\times$, we simply multiply the integrands in the CD^0 and the $CD^\times$ terms by f^Q and re-evaluate the integrals.[15] Since some of the Pauli terms diverge logarithmically, we introduce a cut-off k_0 into the k integrals for all Pauli terms. The result for the QD terms is

$$T_{QD}=2\mu\log(M/m)-2\mu\log[(E_0+k_0)/2k_0], \tag{40}$$

where $E_0=(M^2+k_0{}^2)^{\frac{1}{2}}$. The positive and negative energy components of T_{QD} converge individually even if k_0 tends to infinity. In this limit they are

$$T_{QD}{}^{++}=T_{QD}{}^{-+}=\mu(-\tfrac{1}{2}-\log 2), \tag{41a}$$

$$T_{QD}{}^{+-}=\mu[\log(M/m)+\tfrac{3}{2}+\log 2], \tag{41b}$$

$$T_{QD}{}^{--}=\mu[\log(M/m)-\tfrac{1}{2}+\log 2]. \tag{41c}$$

We will now examine the terms ΔE_{CP}, involving a Coulomb interaction and a Pauli photon. These terms are obtained from the corresponding Dirac terms by replacing $\alpha_i{}^b$ by

$$\begin{aligned}A_i{}^b(q_\rho)&=(\mu/4M)\beta^b q_\rho(\gamma_\rho{}^b\gamma_i{}^b-\gamma_i{}^b\gamma_\rho{}^b)\\&=(i\mu/2M)\beta^b(\boldsymbol{\sigma}^b\times\mathbf{q})_i+(\mu q_4/2M)\beta^b\alpha_i{}^b.\end{aligned} \tag{42}$$

The first term of (42) represents the interaction of the magnetic moment of the proton with the magnetic vector of the photon wave, while the second term represents the interaction of the electric moment with the electric vector.

It can be shown that, to the required order, the CP integrands are equal to the Dirac integrands multiplied by μ for $k\sim\alpha m$. We may, therefore, cancel the Pauli terms in this region by multiplying the Breit terms by μ and subtracting them. As before, the Breit terms will give the ordinary hfs produced by the Pauli moment with the reduced mass factor $(1-3m/M)$, while our deviation terms will give additional mass corrections of relative order $\alpha m/M$.

For $k\gg\alpha m$, replacement of $\alpha_i{}^b$ by $A_i{}^b(q_\rho)$ is equivalent to multiplying the integrand by one of the following f-factors:

$$f_+=(\mu/2M)(E_b{}^k+M+k_4), \tag{43a}$$

$$f_+{}'=(\mu/2M)(E_b{}^k+M-k_4'), \tag{43b}$$

$$f_-=-(\mu/2M)(E_b{}^k-M-k_4), \tag{43c}$$

$$f_-{}'=-(\mu/2M)(E_b{}^k-M+k_4'). \tag{43d}$$

f_+ is the appropriate one to use when the intermediate proton energy is positive and when k_μ rather than k_μ' is the four-momentum absorbed by the proton from the photon, i.e., when the photon is the second interaction along the world line of the proton. The other three cases are specified analogously.

To find k_4 (or k_4') for a given term, we examine the integration over ω and ω', the fourth components of the photon momenta, and put $k_4(k_4')$ equal to the value of $\omega(\omega')$ at the pole. If the term in question arises from several poles, it must be split up into partial terms, each arising from a single pole. This is the case with $\Delta E_{CD}{}^{0--}$, for example.

After the appropriate f-factors have been introduced into the CD terms, and the integrals re-evaluated up to the cut-off k_0, we find

$$\begin{aligned}T_{CP}=&-4\mu\log(M/m)-(8\mu/k_0)(E_0-k_0)\\&+4\mu\log[(E_0+k_0)/2k_0].\end{aligned} \tag{44}$$

[15] ΔE_{QD} and ΔE_{QP} are the terms which were neglected erroneously in reference 11.

If k_0 tends to infinity, the positive and negative energy components of T_{CP} again converge. Their limiting values are

$$T_{CP}{}^{++}=2u\log 2, \tag{45a}$$

$$T_{CP}{}^{-+}=-4\mu\log(M/m)-6\mu\log 2+8\mu, \tag{45b}$$

$$T_{CP}{}^{+-}=-2\mu\log(M/m)-2\mu\log 2-4\mu, \tag{45c}$$

$$T_{CP}{}^{--}=2\mu\log(M/m)+6u\log 2-4\mu. \tag{45d}$$

Finally, we may replace the Coulomb interaction in the CP terms by a Q interaction. For this we use the same procedure as in the conversion of T_{CD} into T_{QD}. The results, using the finite cut-off k_0, are

$$T_{QP}{}^{++}=T_{QP}{}^{-+}$$
$$=\mu^2\{(E_0-M-k_0)/2M-\tfrac{1}{2}\log[(E_0+k_0)/M]\}, \tag{46a}$$

$$T_{QP}{}^{+-}=T_{QP}{}^{--}=\mu^2\{-(E_0-M-k_0)/2M$$
$$-\tfrac{1}{2}\log[(E_0+k_0)/M]\}, \tag{46b}$$

with a total of

$$T_{QP}=-2\mu^2\log[(E_0+k_0)/M], \tag{47}$$

which is logarithmically divergent if k_0 approaches infinity.

4. DOUBLE-PHOTON TERMS

The Dirac double-photon terms include the first order energy $\Delta E_{DD}{}^{\times}$ from diagram (2d) and the second order energy $\Delta E_{DD}{}^{0}$ from diagram (1b). The latter, as shown by S, is gotten by computing a first order energy for the uncrossed diagram (2e), as if this diagram were irreducible. These two terms can be written in the form

$$\Delta E_{DD}{}^{\alpha}=(i/2\pi)(e^2/2\pi^2)^2\int d^4p d^4p' d^4p''\psi^*(p_\mu)$$
$$\times\mathcal{K}^{\alpha}\psi(p_\mu'')(\omega^2-k^2+i\Delta)^{-1}(\omega'^2-k'^2+i\Delta')^{-1}, \tag{48}$$

where α stands for 0 or $\times$, and where ω and ω' are the fourth components of $k_\mu=p_\mu'-p_\mu$ and $k_\mu'=p_\mu''-p_\mu'$. The expressions for $\mathcal{K}^{\alpha}$ are

$$\mathcal{K}^{\times}=\sum_{i=1}^{2}\sum_{j=1}^{2}{}'\,\alpha_j{}^a\alpha_i{}^b[\eta_a E-H_a(\mathbf{p}''')+\epsilon''-\omega+i\delta\beta^a]^{-1}$$
$$\times[\eta_b E-H_b(\mathbf{p}')-\epsilon-\omega+i\delta\beta^b]^{-1}\alpha_i{}^a\alpha_j{}^b, \tag{49a}$$

where $p_\mu'''=p_\mu''-k_\mu$, where ϵ and ϵ'' are the fourth components of p_μ and p_μ'', and where $\sum$ and $\sum'$ represent summations over the two directions perpendicular to $\mathbf{k}$ and to $\mathbf{k}'$ respectively; and

$$\mathcal{K}^{0}=\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b[\eta_a E-H_a(\mathbf{p}')+\epsilon+\omega+i\delta\beta^a]^{-1}$$
$$\times[\eta_b E-H_b(\mathbf{p}')-\epsilon-\omega+i\delta\beta^b]^{-1}\boldsymbol{\alpha}_{\perp'}{}^a\cdot\boldsymbol{\alpha}_{\perp'}{}^b. \tag{49b}$$

As in the case of $\Delta E_{CD}{}^{0}$, these are evaluated by setting ψ,ψ^* equal to $\psi_{++},\psi_{++}{}^*$, expressing these in terms of $\phi_{++},\phi_{++}{}^*$, and interposing projection operators to remove the Dirac matrices from the denominator.

The terms arising from negative energy proton states give contributions of order α(hfs) from the range $k\sim\alpha m$. As in the single photon terms, however, these contributions cancel each other to lowest order in α, except for terms independent of spin. We may therefore restrict outselves to values of k much greater than αm and, after expressing the DD terms in the form (19), replace $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{p}''$, and $\mathbf{p}'''$ in $\mathcal{F}_\alpha{}^\beta$ by 0, $\mathbf{k}$, 0, and $-\mathbf{k}$ respectively, as in Sec. 2. The integrands will contain the expectation values

$$\sum_{i=1}^{2}\sum_{j=1}^{2}{}'(\alpha_i{}^a\alpha_j{}^b\Lambda_{\epsilon 1}{}^a(-\mathbf{k})\Lambda_{\epsilon 2}{}^b(\mathbf{k})\alpha_j{}^a\alpha_i{}^b) \tag{50a}$$

and

$$-(\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\Lambda_{\epsilon 1}{}^a(\mathbf{k})\Lambda_{\epsilon 2}{}^b(\mathbf{k})\boldsymbol{\alpha}_{\perp'}{}^a\cdot\boldsymbol{\alpha}_{\perp'}{}^b) \tag{50b}$$

for a state with zero momentum, where $\epsilon 1$ and $\epsilon 2$ are equal to ± 1. After averaging over angles, these are each equal to

$$(E_a{}^k-\epsilon 1 m)(E_b{}^k-\epsilon 2 m)(\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b)/6E_a{}^kE_b{}^k. \tag{50c}$$

Using (50a, b, c), the DD terms may be expressed as one-dimensional integrals $T_\alpha{}^\beta$ in accordance with Eqs. (29) and (30). If, as in the case of the CD terms, we introduce a cut-off $k_0{}^D$, we find

$$T_{DD}=\log(M/m)-\log[(E_0{}^D+k_0{}^D)/2k_0{}^D]. \tag{51}$$

The positive and negative energy components, in the limit as $k_0{}^D$ approaches infinity, are

$$T_{DD}{}^{++}=\tfrac{3}{8}\log(M/m)+\tfrac{1}{4}\log 2+\tfrac{1}{16}, \tag{52a}$$

$$T_{DD}{}^{-+}=\tfrac{3}{8}\log(M/m)+\tfrac{1}{4}\log 2-\tfrac{7}{16}, \tag{52b}$$

$$T_{DD}{}^{+-}=T_{DD}{}^{--}=\tfrac{1}{8}\log(M/m)-\tfrac{1}{4}\log 2+\tfrac{3}{16}. \tag{52c}$$

Any term in which either of the two photons interacts with the Pauli moment, the other photon interacting with the Dirac moment, is obtained by multiplying the integrand by the sum of the two appropriate f factors, as given by Eqs. (43). The sum of these terms, up to the cut-off k_0, is

$$T_{PD}=2\mu\log(M/m)-2\mu\log[(E_0+k_0)/2k_0]. \tag{53}$$

The positive and negative energy components, for infinite k_0, are

$$T_{PD}{}^{++}=\tfrac{3}{4}\mu\log(M/m)+\tfrac{1}{2}\mu\log 2+\tfrac{1}{8}\mu, \tag{54a}$$

$$T_{PD}{}^{-+}=\tfrac{3}{4}\mu\log(M/m)+\tfrac{1}{2}\mu\log 2-\tfrac{7}{8}\mu, \tag{54b}$$

$$T_{PD}{}^{+-}=T_{PD}{}^{--}=\tfrac{1}{4}\mu\log(M/m)-\tfrac{1}{2}\mu\log 2+\tfrac{3}{8}\mu. \tag{54c}$$

The PD terms are exactly 2μ times the DD terms.

Finally, the terms in which both photons interact with the Pauli moment are derived from the DD terms by multiplying the integrands by the products of the

two appropriate f factors. With a finite k_0, they give

$$T_{PP}^{++}=\mu^2\{\tfrac{3}{8}\log(M/m)+\tfrac{1}{16}+(E_0-k_0)/4M +\tfrac{1}{16}\log[512Mk_0^6/(E_0+k_0)^4(E_0+M)^3]\}, \quad (55a)$$

$$T_{PP}^{-+}=T_{PP}^{++}-\tfrac{1}{2}\mu^2, \quad (55b)$$

$$T_{PP}^{+-}=T_{PP}^{--}=\mu^2\{\tfrac{1}{4}-(E_0-k_0)/4M -\tfrac{1}{16}\log[8(E_0+k_0)^4/M(E_0+M)^3]\}. \quad (55c)$$

The total is

$$T_{PP}=\mu^2\{\tfrac{3}{4}\log(M/m)+\tfrac{1}{8}-\tfrac{3}{4}\log[(E_0+k_0)/2k_0] -\tfrac{1}{4}\log[(E_0+k_0)/M]\}, \quad (56)$$

which diverges logarithmically as k_0 approaches infinity. Notice that the only divergent terms are ΔE_{QP} and ΔE_{PP}, i.e., the terms involving two Pauli interactions.

5. THREE-DIMENSIONAL METHOD, SINGLE PHOTON TERMS

As an alternative to the covariant procedure, we shall now describe an approximate, but simpler, method for calculating the mass corrections to hfs, based on orthodox three-dimensional perturbation theory. This method, like the four-dimensional one, involves an integration over a photon momentum, k. We shall expand in powers of k/M as well as m/M, treating the proton velocity as non-relativistic throughout, and arbitrarily impose an upper limit of order M on the k integration. Unlike the three-dimensional treatment of the fs, described in S, the hfs calculation gives no contribution of the required order from the region $\alpha m \lesssim k \ll m$, so that no terms involving $\log\alpha$ are obtained. It will therefore be possible to expand in powers of m/k also, and to introduce a lower limit of order m for the k integration. These approximations will give only the mass corrections involving $\log(M/m)$, neglecting terms of order unity relative to this logarithm.

For the interaction of the electron and of the Dirac part of the proton moment with the transverse electromagnetic field, we use the customary perturbation Hamiltonian:

$$H'=H^a+H^b, \quad (57)$$

where

$$H^a=e\sum_{\mathbf{k}}\sum_{i=1}^{2}\alpha_i^a(q_{\mathbf{k},i}^{\mathrm{op}}e^{i\mathbf{k}\cdot\mathbf{r}_a}+\bar{q}_{\mathbf{k},i}^{\mathrm{op}}e^{-i\mathbf{k}\cdot\mathbf{r}_a}), \quad (58)$$

in the notation of S. The expression for H^b is obtained from (58) by replacing $\mathbf{r}_a$ and $e\alpha_i^a$ by $\mathbf{r}_b$ and $(-e\alpha_i^b)$. $q_{\mathbf{k},i}^{\mathrm{op}}$ and $\bar{q}_{\mathbf{k},i}^{\mathrm{op}}$ are the usual absorption and emission operators for a photon of momentum $\mathbf{k}$ and polarization i.

In the usual lowest-order perturbation treatment of hfs, the wave equation is first solved for a Hamiltonian containing the Coulomb interaction, but not containing H'. Let $|0\rangle$ denote the eigenstate of such a Hamiltonian corresponding to the initial state of the hydrogen atom and $|m,\mathbf{k},i\rangle$ the eigenstate of the same Hamiltonian containing the hydrogen atom in state $|m\rangle$ plus a photon of momentum $\mathbf{k}$ and polarization i. The Fermi expression is then contained in a term of the form

$$\sum_{m,\mathbf{k},i}\langle 0|H^b|m,\mathbf{k},i\rangle\langle m,\mathbf{k},i|H^a|0\rangle(E_0-E_{mk})^{-1}, \quad (59)$$

plus a similar term with H^a and H^b interchanged. This procedure is not well adapted to the present problem, in which we are interested mainly in evaluating accurately the small contributions to the hfs of momenta $\mathbf{k}$ large compared with the Bohr momentum αm (in fact, larger than m). If we consider the atomic wave functions in momentum space, the main part of the ground-state wave function lies in the region of small momentum for both the electron and the proton. The momentum spread is of order αm, with only a small "high-momentum tail" depending mainly on the high-frequency Fourier components of the Coulomb field. The main part of $H^a|0\rangle$ has a momentum spread of the same order around an electron of momentum $\mathbf{k}$ and a proton at rest, whereas $H^b|0\rangle$ is similarly centered around a proton of momentum $-\mathbf{k}$ and an electron at rest. The high-frequency components of the Coulomb field again add small "high-momentum tails" to these wave functions, extending over large distances from the peak of the momentum distribution. Hence, if we use a representation in which the unbound states of the atom consist of plane waves perturbed by the Coulomb potential, it is impossible for the main part of any state $|m,\mathbf{k},i\rangle$ to overlap[16] the main parts of both $H^a|0\rangle$ and $H^b|0\rangle$ simultaneously, and the matrix element in Eq. (59) depends critically on the "high-momentum tail" of at least one of the wave functions involved. In Eq. (59), relativistic wave functions accurate over a large range of internal momenta would therefore be required, and it would no longer be correct to replace $|0\rangle$ and $|m\rangle$ by a Schrödinger wave function and a plane wave respectively.

Because of this difficulty, we shall, in the case of the single-photon terms, use a somewhat modified perturbation procedure, which is made possible by the fact that we are interested only in terms for which $k \gg \alpha m$. We consider the Coulomb interaction V in momentum space and split it into two terms, V_0 containing momenta less than B, and V' containing momenta greater than B, where $\alpha m \ll B \ll m$. As unperturbed wave functions we take the eigenfunctions of a Hamiltonian containing only the part V_0 of the Coulomb potential, and treat V' as an additional perturbation. These unperturbed wave functions decrease rapidly outside regions of diameter B at the most, so that the state vectors $|m,\mathbf{k},i\rangle$, $H^a|0\rangle$, and $H^b|0\rangle$ no longer overlap appreciably. Term (59) is now negligible for $k \gg B$, and

[16] In other representations, e.g., the angular momentum representation, the main parts of certain states $|m,\mathbf{k},i\rangle$ would overlap simultaneously with $H^a|0\rangle$ and $H^b|0\rangle$, but the large contribution from such a state would be cancelled by the contributions of other such states.

the high-frequency contributions to the hfs are given by third order perturbation theory as a sum of six terms, ΔE_1 to ΔE_6, where

$$\Delta E_1=\sum_{m,n,\mathbf{k},i}\frac{\langle 0|H^b|m,\mathbf{k},i\rangle\langle m,\mathbf{k},i|H^a|n\rangle\langle n|V'|0\rangle}{(E_0-E_{mk})(E_0-E_n)}, \quad (60)$$

and the other five terms differ from ΔE_1 by having a different order for the operators H^a, H^b, and V'.

In (60), the main parts of the various state vectors overlap, so that: (1) the initial and intermediate atomic wave functions may be replaced by a Schrödinger wave function and a plane wave, and (2) momentum is approximately conserved in all the transitions. Hence, for $k\gg\alpha m$, the state $|m\rangle$ consists approximately of a free electron at rest and a free proton of momentum $(-\mathbf{k})$, while state $|n\rangle$ is like $|m\rangle$ except that the electron has momentum $\mathbf{k}$. This will enable us to replace the energy denominators by simpler expressions. Finally, the third matrix element in Eq. (60) corresponds to the transfer of a momentum approximately equal to $\mathbf{k}$ from the proton to the electron by means of the Coulomb interaction V'. An expression similar to Eq. (60) for ΔE_1 could have been obtained by substituting the iterated wave function ϕ_1 given by Eq. (29) of S, which is a good approximation everywhere in momentum space, for $|0\rangle$ in the second matrix element of (59). The term similar to (60), but having the operators in the order $H^bV'H^a$, corresponds to taking the Coulomb interaction in the intermediate state of (59) into account.

To evaluate ΔE_1, we first split it up into four parts by inserting the sum of the projection operators $\Lambda_\pm{}^a(\mathbf{k})$ after H^a and $\Lambda_\pm{}^b(\mathbf{k})$ after H^b. Consider first the part $\Delta E_1{}^{++}$ involving $\Lambda_+{}^a\Lambda_+{}^b$. From the arguments of the last paragraph, and from the presence of the positive energy projection operators, it follows that we may substitute $(k+k^2/2M)$ and $(E_a{}^k-m+k^2/2M)$ for the energy denominators in Eq. (60). Since the energy denominators are now independent of the indices m and n, we can apply a sum rule to eliminate the intermediate states altogether. Introducing the explicit expressions for H^a and H^b, we finally get

$$\Delta E_1{}^{++}=-\frac{e^2}{4\pi^2}\int\frac{d^3k}{k}\cdot\frac{\langle 0|\boldsymbol{\alpha}_\perp{}^a\cdot\boldsymbol{\alpha}_\perp{}^b\Lambda_+{}^a(\mathbf{k})\Lambda_+{}^b(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{r}}V'|0\rangle}{(k+k^2/2M)(E_a{}^k-m+k^2/2M)}, \quad (61)$$

where $\mathbf{r}=\mathbf{r}^a-\mathbf{r}^b$ and V' is the high-frequency part of $(-e^2/r)$. Expressing V' and $|0\rangle$ in the momentum representation, and neglecting the initial and final momenta p'' and p in comparison to k,

$$\Delta E_1{}^{++}=\frac{e^4}{8\pi^4}\int\frac{d^3kd^3pd^3p''\,\mathfrak{M}}{k^3(k+k^2/2M)(E_a{}^k-m+k^2/2M)}, \quad (62)$$

where

$$\mathfrak{M}=(k^2/6ME_a{}^k)\phi_0{}^*(\mathbf{p})\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b\phi_0(\mathbf{p}''), \quad (63)$$

after averaging over angles.

Since the energy denominators in Eq. (62) do not contain $\mathbf{p}$ and $\mathbf{p}''$, the integration over these two variables can be carried out immediately, and an expression of the form (29) is obtained for $\Delta E_1{}^{++}$, in which

$$T_1{}^{++}=\int_0^\infty dk(Mk/E_a{}^k)(k+k^2/2M)^{-1}\times(E_a{}^k-m+k^2/2M)^{-1}. \quad (64)$$

To obtain the contribution of any other ordering of the operators, we replace $(k+k^2/2M)$ and $(E_a{}^k-m+k^2/2M)$ in (64) by the energy denominators appropriate to the process in question. The total positive-energy contribution is then given by

$$T^{++}=2M\int_0^\infty(kdk/E_a{}^k)[(ab)^{-1}+(ac)^{-1}+(bc)^{-1}], \quad (65)$$

where

$$a=k+k^2/2M, \quad (66a)$$

$$b=E_a{}^k-m+k, \quad (66b)$$

$$c=E_a{}^k-m+k^2/2M, \quad (66c)$$

To find the contributions of negative-energy electron states, we must use hole theory to get the correct sign. It is easy to show that the negative-energy electron terms can be derived from the known positive-energy terms by replacing $(E_a{}^k-m)$ in Eqs. (66) by $(E_a{}^k+m)$. Similarly, negative-energy proton states are accounted for by replacing $k^2/2M$ in Eqs. (66) by $(2M+k^2/2M)$, which may be set equal to $2M$.

We will first examine the range $m\ll k\ll M$. Expanding in powers of m/k, we find

$$T^{++}=\int_m^M dk[(4M/k^2)+(4Mm/k^3)-(3/k)]. \quad (67)$$

Since we are interested only in mass corrections, we subtract the part of ΔE^{++} which would be given by an infinitely heavy proton and an electron with the reduced mass $m_R=mM/(M+m)$. Referring to Eq. (29), the part of T^{++} which remains is, in our approximation

$$T_{CD}{}^{++}=\int_m^M dk(-3/k)=-3\log(M/m). \quad (68)$$

The negative-energy electron term T^{-+} differs from T^{++} only by a change in sign of m in the energy denominators which, to the present approximation, does not affect the mass correction term. Hence, $T_{CD}{}^{-+}$ and $T_{CD}{}^{++}$ are equal. For negative-energy proton states, we substitute $2M$, $2k$, and $2M$ for a, b, and c respectively, so that

$$T_{CD}{}^{+-}=T_{CD}{}^{--}=\log(M/m). \quad (69)$$

Nothing has to be subtracted from Eq. (69), since negative proton energy terms would not arise for an infinitely heavy proton.

Next, we consider the range $\alpha m<k<m$ to see whether there are any terms containing $\log\alpha$. For $k\ll m$, we expand the integrand of T^{++} as a power series in k, carrying only the first two terms. We find that

$$T^{++}=\int dk(8Mm_R/m)[k^{-2}+(1/2m_R)k^{-1}+\cdots]. \quad (70)$$

Equation (70) is identical with the equivalent term for an infinitely heavy proton and an electron of mass m_R. Hence, there are no mass corrections containing $\log\alpha$ from the positive-energy components. Essentially the same argument holds for the $(-+)$ terms, and the negative-energy proton terms are obviously negligible in this tange. The total contribution from Dirac terms for all values of $k\gg\alpha m$ is therefore

$$T_{CD}=-4\log(M/m), \quad (71)$$

in agreement with Eq. (35).[17]

We shall now examine the effect of the Pauli moment. First of all, if the Coulomb interaction is replaced by a Q-interaction, the positive and negative proton energy integrands are multiplied by factors of $(-\mu k^2/4M^2)$ and μ respectively. The result is obviously

$$T_{QD}{}^{++}=T_{QD}{}^{-+}=0, \quad (72a)$$

$$T_{QD}{}^{+-}=T_{QD}{}^{--}=\mu\log(M/m), \quad (72b)$$

with a total of

$$T_{QD}=2\mu\log(M/m), \quad (73)$$

in agreement with Eqs. (40) and (41).

To evaluate T_{CP}, we approximate the f-factors in Eqs. (43) by

$$f_+=\mu(1+k_4/2M), \quad (74a)$$

$$f_+'=\mu(1-k_4'/2M), \quad (74b)$$

$$f_-=\mu k_4/2M, \quad (74c)$$

$$f_-'=-\mu k_4'/2M. \quad (74d)$$

In Eqs. (74), $k_4(k_4')$ is again the fourth component of the momentum absorbed by the proton from the photon when the photon represents the second (first) interaction along the world line of the proton. We shall set it equal to $\pm k$ according as the photon is absorbed or emitted by the proton. In converting from positive to negative proton energies, we must interchange k_μ and k_μ', since the order of interactions along the proton world line is the reverse of the temporal order for negative-energy states.

The negative proton energy terms clearly give zero, since they all gave contributions no larger than the required order to the Dirac terms, and since they are now multiplied by small f-factors. The positive energy terms, however, are not simply multiplied by μ, since they contained large terms which were subtracted off. For the Pauli terms, the subtraction should not be carried out until after the integrands have been multiplied by the f-factors (74a) or (74b). The resulting mass correction is

$$T_{CP}{}^{++}=T_{CP}{}^{-+}=-2\mu\log(M/m), \quad (75)$$

so that the total is

$$T_{CP}=-4\mu\log(M/m), \quad (76)$$

in agreement[17] with Eq. (44).

The term T_{QP} is obviously zero, in agreement with Eq. (47), since only positive proton energy terms contribute to T_{CP}, and these are reduced below the required order by the f-factor $(-\mu k^2/4M^2)$ in T_{QP}.

6. THREE-DIMENSIONAL METHOD, DOUBLE-PHOTON TERMS

We consider now the exchange between electron and proton of two photons, transferring momenta $\mathbf{k}$ and $\mathbf{k}'$ respectively from the electron to the proton. $\mathbf{k}'$ again represents the first of the two momenta absorbed by the proton along its own world line. We treat these processes by means of orthodox fourth order perturbation theory, as in Sec. 5 of S. The operators H^a and H^b (or their Pauli equivalents) each occur twice, and the atom starts and finishes in the same state. There are twelve such terms, differing from each other in the temporal order of the emissions and absorptions. Since we are interested only in the range $k,k'\gg\alpha m$, we can again neglect the momentum spread in the initial and intermediate states and break each term into its positive and negative energy components by inserting projection operators. The intermediate states are then given approximately by free-particle states of positive or negative energy, depending on which projection operators occur. Neglecting the difference between k and k', which is of order αm, the three energy denominators can be approximated by expressions depending only on k, but not on the indices of the intermediate atomic states. We then apply sum rules to eliminate these intermediate states.

Restricting our attention first to the Dirac part of the proton moment, we can write each fourth-order term $\Delta E_\alpha{}^\beta$ in the form

$$\Delta E_\alpha{}^\beta=-\alpha^2(2\pi)^{-4}\int d^3kd^3pd^3p''k^{-2}\phi_0{}^*(\mathbf{p})\times\mathfrak{N}_\alpha{}^\beta\phi_0(\mathbf{p}'')\mathcal{P}_\alpha{}^\beta, \quad (77)$$

where ϕ_0 is the momentum-space wave function for the

[17] There is an apparent discrepancy between the logarithmic terms of Eqs. (34) and of Eqs. (68) and (69). This discrepancy arises from the fact that the Breit terms are subtracted in Eqs. (34), but not in Eqs. (68) and (69). Each of these terms individually gives a logarithmic contribution, but their sum is zero. The same situation holds for the CP terms.

initial state, $\mathfrak{N}_\alpha{}^\beta$ is a product of four Dirac matrices and two projection operators, and $\mathcal{P}_\alpha{}^\beta$ is the product of the reciprocals of the three energy denominators $(E_{\text{inter}}-E_{\text{init}})$. After averaging over angles and neglecting p and p'' relative to k, the expectation value for proton and electron at rest of each positive energy $\mathfrak{N}_\alpha$ reduces to

$$(\mathfrak{N}_\alpha{}^{++})=\epsilon_\alpha\tfrac{1}{6}(k^2/2M)(1-m/E_a{}^k)(\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b), \quad (78)$$

where ϵ_α is ±1 according as the electron interacts first with the $\mathbf{k}$ or the $\mathbf{k}'$ photon. Again neglecting p and p'' relative to k, $\mathcal{P}_\alpha{}^\beta$ is independent of p and p'', so that the integration over these two variables can be carried out immediately. This reduces $\Delta E_\alpha{}^{++}$ to an expression of form (29), where

$$T_\alpha{}^{++}=(M^2/2)\int_0^\infty dk(k^2/2M)(1-m/E_a{}^k)\epsilon_\alpha\mathcal{P}_\alpha{}^{++}. \quad (79)$$

Summing over all orders of the emissions and absorptions,

$$\textstyle\sum_\alpha \epsilon_\alpha\mathcal{P}_\alpha{}^{++}=[(d-c)/cd][(a+b)^2/a^2b^2], \quad (80)$$

where a, b, and c are given by Eqs. (66), and where

$$d=E_a{}^k-m+k^2/2M+2k. \quad (81)$$

It is easy to verify that the range $\alpha m\ll k\ll m$ does not contribute to the required order; hence no $\log\alpha$ terms can arise. For $m\ll k\ll M$, we can replace a, b, c, and d by k, $2k$, k, and $3k$ respectively, reducing $\sum\epsilon_\alpha\mathcal{P}_\alpha{}^{++}$ to $(3/2k^3)$. Using this approximation in Eq. (79), and integrating only from m to M, we get

$$T_{DD}{}^{++}=\tfrac{3}{8}\log(M/m). \quad (82)$$

The terms contributing to $T_{DD}{}^{++}$ are all of the same order, and there is no cancellation of large terms.

Taking account of hole theory, negative energy states are treated by replacing $(E_a{}^k-m)$ by $(E_a{}^k+m)$ for the electron and $k^2/2M$ by $2M$ for the proton in Eq. (78) and in the energy denominators a, b, c, and d. Also, $\mathbf{k}$ and $\mathbf{k}'$ are interchanged for negative proton energy terms. Since we neglect m compared with k, the expressions for $(\epsilon\mathcal{P})$ and T are the same for positive and negative electron energies.

For negative energy proton states, some of the processes give contributions of a larger order of magnitude than the total, which cancel to lowest order.

The negative proton energy totals are

$$T_{DD}{}^{+-}=T_{DD}{}^{--}=\tfrac{1}{8}\log(M/m), \quad (83)$$

so that

$$T_{DD}=\log(M/m). \quad (84)$$

These results are in agreement with Eqs. (51) and (52).

If only one of the Dirac interactions is replaced by a Pauli interaction, we must multiply the integrand by twice the appropriate f-factor. Since there is no cancellation of large terms for positive-energy proton states, the positive-energy f-factor is needed only to lowest order. We therefore use $f=\mu$, yielding

$$T_{PD}{}^{++}=T_{PD}{}^{-+}=\tfrac{3}{4}\mu\log(M/m). \quad (85)$$

For negative-energy proton states, we use the f-factors (74c) and (74d). Although f is small compared to one, the large terms which previously cancelled now have the same sign, and give a contribution of the required order when multiplied by f. The result is

$$T_{PD}{}^{+-}=T_{PD}{}^{--}=\tfrac{1}{4}\mu\log(M/m), \quad (86)$$

giving a total of

$$T_{PD}=2\mu\log(M/m), \quad (87)$$

in agreement with Eqs. (53) and (54).

If both photons interact with the Pauli moment, the positive-proton energy terms are multiplied by μ^2, whereas the negative proton energy terms are multiplied by a factor of order μ^2k^2/M^2, which makes them negligibly small. We have, therefore

$$T_{PP}{}^{++}=T_{PP}{}^{-+}=\tfrac{3}{8}\mu^2\log(M/m), \quad (88)$$

$$T_{PP}{}^{+-}=T_{PP}{}^{--}=0, \quad (89)$$

$$T_{PP}=\tfrac{3}{4}\mu^2\log(M/m), \quad (90)$$

in agreement with Eqs. (55) and (56).

We have now succeeded in deriving all the $\log(M/m)$ terms by three-dimensional perturbation theory, and in showing that they agree in detail with our previous results from the covariant method.

7. NUMERICAL RESULTS AND DISCUSSION

In Secs. 2 to 4 we have calculated the total proton recoil correction ΔE_{tot} of relative order $\alpha m/M$ to the Fermi formula (with the simple reduced-mass factor) for the hfs splitting of S-states of the hydrogen atom. We denote the ratio of this correction to the Fermi splitting itself by $(-F)$. Using $\mu_P=2.79$, it follows from Eq. (29) that

$$F=-(\alpha/\pi\mu_P)(m/M)T_{\text{tot}}=-(4.53\times10^{-7})T_{\text{tot}}. \quad (91)$$

Writing T_{tot} as the sum of contributions T_s and T_d from single and double photon terms respectively, and collecting all terms given in Eqs. (35a), (40), (44), (47), (51), (53), and (56), we have

$$\begin{aligned}T_s&=T_{CD}+T_{QD}+T_{CP}+T_{QP}\\&=-2(\mu_P+1)\log(M/m)-8(E_0{}^D-k_0{}^D)/k_0{}^D\\&\quad-8(\mu_P-1)(E_0-k_0)/k_0\\&\quad+4\log[(E_0{}^D+k_0{}^D)/2k_0{}^D]\\&\quad+2(\mu_p-1)\log[(E_0+k_0)/2k_0]\\&\quad-2(\mu_p-1)^2\log[(E_0+k_0)/M], \quad (92)\end{aligned}$$

and

$$T_d=T_{DD}+T_{PD}+T_{PP}$$
$$=\tfrac{1}{4}(\mu_P+1)(3\mu_p-1)\log(M/m)$$
$$+\tfrac{1}{8}(\mu_p-1)^2-\log[(E_0{}^D+k_0{}^D)/2k_0{}^D]$$
$$-\tfrac{1}{4}(\mu_p-1)(3\mu_p+5)\log[(E_0+k_0)/2k_0]$$
$$-\tfrac{1}{4}(\mu_p-1)^2\log[(E_0+k_0)/M]. \quad (93)$$

In Eqs. (92) and (93) we have cut-offs $k_0{}^D$ for terms not containing any Pauli interaction and k_0 for terms containing at least one Pauli interaction. E_0 stands for $(M^2+k_0{}^2)^{\frac{1}{2}}$, with a similar expression for $E_0{}^D$.

As was pointed out before, both T_s and T_d diverge logarithmically if the Pauli cut-off approaches infinity, and the divergences in the two terms do not cancel each other. If we let $k_0{}^D$ equal k_0, and take the limit as (k_0/M) tends to infinity, we get

$$T_s=-2(\mu_p+1)\log(M/m)-2(\mu_p-1)^2\log(2k_0/M) \quad (94)$$
$$=-57.0-(6.4)\log(2k_0/M), \quad (95)$$
$$T_d=\tfrac{1}{4}(\mu_p+1)(3\mu_p-1)\log(M/m)+\tfrac{1}{8}(\mu_p-1)^2$$
$$-\tfrac{1}{4}(\mu_p-1)^2\log(2k_0/M)$$
$$=52.9-(0.8)\log(2k_0/M), \quad (96)$$

and

$$T_{\rm tot}=\tfrac{3}{4}(\mu_p-3)(\mu_p+1)\log(M/m)+\tfrac{1}{8}(\mu_p-1)^2$$
$$-(9/4)(\mu_p-1)^2\log(2k_0/M)$$
$$=-4.1-(7.2)\log(2k_0/M), \quad (97)$$

so that

$$F=(1.86\times10^{-6})+(3.26\times10^{-6})\log(2k_0/M). \quad (98)$$

Note that there is an almost complete, but quite fortuitous, cancellation between the finite parts of T_s and T_d, making the finite part of F much smaller than $(\alpha m/M)\log(M/m)=(3.0\times10^{-5})$, which is the order of magnitude one might have expected.

In this paper we have attempted to treat the proton as a point particle with no internal structure and with an anomalous magnetic moment exactly of the Pauli type. In addition to the fractional mass correction $(-F)$ which we have calculated above, a consistent field-theoretic treatment taking the meson field into account would give an additional correction arising from the internal structure of the proton. Such a consistent treatment would presumably modify the behavior of the proton at short distances, or large momenta, and remove the divergence obtained in Eq. (98) with an unmodified Pauli moment.

The divergence makes an unambiguous separation of the correction terms into mass corrections and structure corrections impossible. But, when a consistent theory is available, one might be able to calculate the fractional difference δ between the values obtained for the hfs splitting with a point proton and with the correct proton structure, using a cut-off $k_0\gg M$ in the same manner as in this paper.[18] If we call δ' the sum of δ and the term in Eq. (98) involving k_0, one would hope that δ' would approach a finite limit as k_0 approaches infinity, and we would have

$$F+\delta=(1.86\times10^{-6})+\delta'. \quad (99)$$

The structure corrections will presumably be negligible for photon momenta less than some value A, where $m\ll A\ll M$. For this reason, we give our results integrated up to such a cut-off A:[19]

$$T_s(k<A)=-2(\mu_p+1)\log(2A/m)+8\mu_p, \quad (100)$$
$$T_d(k<A)=\tfrac{1}{4}(3\mu_p-1)(\mu_p+1)\log(2A/m)$$
$$+\tfrac{1}{8}(\mu_p-1)^2. \quad (101)$$

If the structure corrections are really negligible for $k<A$, the consistent theory may be able to use Eqs. (100) and (101) for this range, while making an expansion in powers of (m/k) for $k>A$.

At least a rough experimental determination of the combined correction factors $(F+\delta)$ is now available. Fine-structure measurements,[20] whose interpretation is virtually independent of nucleon structure effects, give a value for the fine-structure constant of

$$\alpha^{-1}=(137.0365\pm0.0012). \quad (102)$$

Measurements of the hfs splitting in the hydrogen ground state,[21] after applying the known radiative corrections[22,23] of orders α and α^2 and the simple reduced-mass correction to the Fermi formula, yield the relation:[24,25]

$$\alpha^{-1}=(137.0365\pm0.0006)[1+\tfrac{1}{2}(F+\delta)]. \quad (103)$$

A comparison of Eqs. (99), (102), and (103) therefore yields

$$\delta'=(F+\delta)-(1.86\times10^{-6})=(-0.2\pm2.0)\times10^{-5}, \quad (104)$$

where the estimated error is the rms of the errors in Eqs. (102) and (103).

The mean of this experimental value for δ' is much smaller than one might have expected. Taking a finite cut-off $k_0=k_0{}^D$ in the calculations of this paper is equivalent to including in a naive way some non-relativistic structure effects, i.e., a crude smearing of the total moment of the proton over distances of the order of $k_0{}^{-1}$. In fact, for $k_0\lesssim M$, most of the cut-off dependence of T, and hence of F, comes from terms

[18] Of course, this might not be an unambiguous procedure, since the cut-off was not introduced in a covariant way in this paper.

[19] Equations (100) and (101) cannot be derived by setting k_0 and $k_0{}^D$ both equal to A in Eqs. (92) and (93), since the latter were derived by neglecting A in comparison with k_0 and $k_0{}^D$.

[20] Dayhoff, Triebwasser, and Lamb, Phys. Rev. **89**, 106 (1953).

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[22] N. M. Kroll and F. Pollock, Phys. Rev. **84**, 594 (1951).

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[24] N. M. Kroll and F. Pollock, Phys. Rev. **86**, 876 (1952).

[25] The stated error includes an estimate of the unknown radiative corrections of order α^3.

which represent the omission of momenta greater than k_0 from the Fermi splitting itself. For $k_0=k_0^D$ equal to the π-meson mass, Eqs. (92) and (93) yield a value of (5.8×10^{-5}) for F ($F=0.9\times10^{-5}$ for $k_0=M$), which is larger than the value of δ' given in Eq. (104). This result merely emphasizes the prevalent feeling that a crude nonrelativistic "spreading" of the nucleon bears little relation to reality.

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Classical Field Theory in the Hamilton-Jacobi Formalism*

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A Hamilton-Jacobi formalism of classical relativistic field theory is developed. Both "time-independent" and "time-dependent" formulations are given, and the relation between them is discussed. In the former, the constants of the motion are identified with the "new" field variables, whereas in the latter they are the values of the fields on a suitable spacelike surface. The explicit introduction of a Hamiltonian density is avoided. As an illustration of the respective procedures, the classical Dirac and Klein-Gordon free fields are solved explicitly. A perturbation method is formulated for the case of fields in interaction. The metric tensor is not treated as a field quantity.

INTRODUCTION

THE purpose of the present paper is to serve as a starting point for the extension of Bohm's reinterpretation of particle quantum mechanics[1] to the theory of quantized fields. It will be shown in a subsequent paper that such an extension is indeed possible, and can be based on a Hamilton-Jacobi formulation of classical field theory. It was thought preferable to develop the necessary Hamilton-Jacobi formalism in a preliminary paper, so as not to break the continuity in the argument of the subsequent paper, and also because a Hamilton-Jacobi formalism for field theory may be of some interest in its own right. The formulation in the present paper actually goes beyond what is needed for a causal presentation of the theory of quantized fields.

The usual particle Hamilton-Jacobi formalism[2] is based on Hamiltonian mechanics. However, the essential features of Hamilton-Jacobi theory (i.e., the transformation to "appropriate" variables, which are essentially the constants of the motion,[3] transformation theory, and the reduction of the entire problem under consideration to the solution of a nonlinear first order partial differential equation) can be based as well on a Lagrangian formulation. In view of the greater adaptability of the purely Lagrangian approach to the requirements of covariance,[4] it is used throughout. A Hamiltonian density could be introduced explicitly, for instance by carrying out the differentiation in the right-hand side of (10). However, no useful purpose would be served, as it is not desired to develop a Hamiltonian formalism.[5]

In particle mechanics, the case of conservative systems can be treated by "time-independent" Hamilton-Jacobi theory $[H(q,\partial\bar{S}/\partial q)=E]$, whereas for non-conservative systems, "time-dependent" theory $[H+\partial S/\partial t=0$, which will be written $L=dS/dt]$ is required. The functions $\bar{S}$ and S are not identical, but for a conservative system, which can be treated by either method, S can be obtained from $\bar{S}$. From the point of view of generality, one might think that the time-dependent formalism should suffice. This is so for particle mechanics, but not for field theory. It might be impossible to express the Lagrangian L as a function of the field variables and conjugate momenta alone, but still possible to express suitable constants of the motion in terms of these variables (these constants of the motion playing a role analogous to that of H for particle mechanics), in which case time-independent theory is an indispensible tool. That is, in fact, what happens in the case of the Dirac field. Aside from this contingency, solutions of field Hamilton-Jacobi equations are usually quite difficult to obtain, and one or the other method might prove more convenient.

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[1] D. Bohm, Phys. Rev. **85**, 166 (1952).

[2] See, e.g., C. Lanczos, *The Variational Principles of Mechanics* (University of Toronto Press, Toronto, 1949).

[3] In some formulations of the theory, the momenta conjugate to the "appropriate" variables are the constants of the motion. Such formulations, while perfectly acceptable for particle dynamics, lead to difficulties in field theory.

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[5] Such as that of R. H. Good, Jr., Phys. Rev. **93**, 239 (1954).