instead of $\Lambda^{(2}{}^{\alpha}$, the angular momentum operator in his ω space, $m_3=i(\omega_1\partial/\partial\omega_2-\omega_2\partial/\partial\omega_1)$, etc. The only difference is that his theory allows transitions between states with different k's. Here there is only one k, which is given. In both cases, an M is found which is invariant for rotations in spin-space. This seems unnecessary, but if it is desired, this is a possible M.

In order that $[J,M]=0$, for a conservation equation, we must take

$$J=e(\Lambda^{(1)3}+\Lambda^{(2)3}).$$

There is only one conserved quantity which must be interpreted as electric charge. The eigenvalues of J are

$$\pm(k+\tfrac{1}{2}),\quad \pm(k-\tfrac{1}{2}),\quad \cdots.$$

There are thus inevitably multiply-charged particles.

12. CHARGE CONJUGATION

$j_{\beta\alpha}$ is imaginary as has been shown. Thus, with our previous notation,

$$j_{\beta\alpha}\phi_\alpha{}^\dagger=-j\phi_\alpha{}^\dagger.$$

This expresses the familiar fact that by interchanging the roles of creation and annihilation operators in ϕ, the boson current is reversed.

The Λ^α and the $-\Lambda^{T\alpha}$ have the same commutation relations. Thus, as usual,

$$-\Lambda^{T\alpha}=T\Lambda^\alpha T^{-1},$$

where

$$TT^\dagger=1$$

$$T=\pm T^T.$$

It follows that

$$Jv(J_r)=J_r v(J_r)$$

implies

$$J(T^{-1}v^\dagger)=-J_r(T^{-1}v^\dagger).$$

The eigenvalues of J occur in pairs $\pm J_r$ with eigenvectors v and $T^{-1}v^\dagger$. This is just what is needed for fermion charge conjugation.

Usually,

$$\psi'=C^{-1}\bar{\psi}.$$

Here we have

$$\psi'=T^{-1}C^{-1}\bar{\psi}.$$

13. CONCLUSION

A possible framework for dealing with families of apparently different particles has been formulated. Perhaps it may prove possible to describe in this way what is actually observed to occur.

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Proper-Time Electron Formalism*

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A classical and first quantized formalism is presented which gives a complete description of electrons in a given electromagnetic field, including real and virtual pair processes. The Feynman viewpoint of electrons propagating through space-time is adopted throughout. Interactions between electrons are considered only in the classical theory, and a nonlocal interaction is assumed to make all effects finite. Consideration of interactions in the quantized theory is reserved to field quantization, which will be presented in a following paper. The calculation of transition probabilities gives the results of hole theory as interpreted by Feynman.

I. INTRODUCTION

USE has been made of the proper time[1] in quantum electrodynamics as a means of rewriting the Dirac equation.[2–4] It has provided covariant methods of calculation and prescriptions for the evaluation of divergent terms.

There are several reasons for investigating the possibilities of a more extensive use of this parameter. Feynman graphs[5] in field-theoretical calculations suggest the interpretation of the electron motion as evolving in four-space in the course of proper time. Also, the introduction of a covariant nonlocal interaction between the electron and electromagnetic field[6–8] suggests a shape to the electron in four-space at each value of the proper time. As a consequence, a quantum electrodynamics has been formulated in which the proper time plays an essential role throughout.

The concepts to be employed are first introduced

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[1] "Proper time" is used in the general sense of an invariant parameter describing the motion of the electron.

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into the classical theory. The state of the electron is specified by its four-dimensional position and momentum at a value of the independent, invariant, parameter, τ. In the course of τ, the electron draws out its world line in four-space in accordance with the equations of motion and the initial (in τ) conditions. The forces acting on the electron result from a nonlocal interaction between the electron and the electromagnetic field. In the corresponding quantized theory, the state is specified by the wave function, $\psi(q,\tau)$, where q is the coordinate vector for the particle in four-space. The development of ψ in τ is obtained from an operator Hamiltonian corresponding to the classical one.

The nonlocal interaction spreads the current density, j, over a small region about the world line of the electron. The field equations themselves are unchanged. The effect, however, is that of introducing a factor $\exp[-f(\Box)]$ into the field equations, in the sense that $\exp[-f(\Box)]\Box A_\mu$ is equal to the point charge distribution. Pais and Uhlenbeck[9] have shown that this does not alter the positive definiteness of the field energy, but does destroy strict causality. In the classical theory, the nonlocal interaction is shown to result in a finite self-mass and a modified and acausal interaction for distances comparable to the electron radius. In the first quantized theory, only a given potential is considered and the nonlocal interaction results in an averaging of the potential over distances comparable to the electron radius.

We will use units for mass and time such that $\hbar$ and $c=1$. Electric charge will be expressed in "rational" units, with $e^2/4\pi=1/137=\alpha$. The components of a four-vector, x, will be written as x_μ, with $\mu=1$, 2, 3, and 4. x_4 is imaginary and equal to ix_0. The three-vector part of x will be written as $\mathbf{x}$; the magnitude of $\mathbf{x}$ is $|\mathbf{x}|$. The matrix $\gamma_\mu x_\mu$ will be written as $\boldsymbol{x}$. Complex and Hermitian conjugates will be denoted by an asterisk; the adjoint by a dagger.

II. CLASSICAL THEORY

1. Current Density

The field produced by a classical electron is determined by the current density, $j(x)$, corresponding to its world line. For a point electron, j is given by[10]

$$j(x)=e\int_{\infty}^{+\infty} ds(dq/ds)\delta^4(x-q). \tag{1.1}$$

s is the proper time in the conventional sense, with $ds^2=-dq^2$. Equation (1.1) will be generalized in two ways.

In the first place, s can be replaced by an arbitrary independent parameter, τ, to give

$$j(x)=e\int_{-\infty}^{+\infty} d\tau\dot{q}\delta^4(x-q). \tag{1.2}$$

(Dot denotes τ differentiation.) $q(\tau)$ describes the entire world line of the electron. If $\dot{q}$ is a positive time-like vector for all τ, then (1.2) is equal to (1.1). However, if for any part of the path, $\dot{q}$ is a negative time-like vector, then over these portions, (1.1) and (1.2) differ by a sign. In these regions, the particle would be observed as a positron.[5,11]

The second generalization of (1.1) is to replace the delta function distribution by a shape function, $F(x-q)$. $eF(x-q)$ can be interpreted as the charge distribution in four-space at a τ when the particle is at q, since it satisfies the continuity equation

$$-\frac{\partial}{\partial\tau}eF(x-q)=\frac{\partial}{\partial x_\mu}e\dot{q}_\mu F(x-q). \tag{1.3}$$

In terms of the independent parameter, τ, and the shape function, $F(x)$, the current density is

$$j(x)=e\int d\tau\dot{q}F(x-q). \tag{1.4}$$

Equation (1.4) expresses the observable quantity, $j(x)$, in terms of entire motion of the electron in the course of τ. References to the position of the electron, its state, or its charge distribution at a τ are to be taken in a formal sense, since measurements at one value of τ have no direct physical significance. If at a τ, the electron is localized to a macroscopic time, then measurements at this time can be interpreted as made at the corresponding τ.

2. The Shape Function

$F(x)$ must be a real, invariant, and odd function of x^2.[6] $F(x)$ is expressed in terms of a universal length a, and $F(x)$ vanishes for $|x^2|\gg a^2$. The length a is of the order of magnitude of the classical electron radius, $(137m)^{-1}$.

Expressing the dependence of F on x^2 by a Fourier integral, we write

$$F(x)=\frac{1}{2\pi}\int d\beta g(\beta)\exp(i\beta x^2). \tag{2.1}$$

Equation (2.1) can be written in a form which is considerably more convenient for calculation, as well as for interpretation, by making the substitutions

$$\begin{aligned}\Delta(x,u)&=\frac{-i\epsilon(u)}{(2\pi u)^2}\exp(ix^2/2u),\\ f(u)&=i\pi g(1/2u)\epsilon(u),\end{aligned} \tag{2.2}$$

where $\epsilon(u)$ is equal to the sign of u. With this substitution, (2.1) becomes

$$F(x)=\int duf(u)\Delta(x,u). \tag{2.3}$$

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In the limit of $u\to 0$ (from either direction), $\Delta(x,u)$ has the properties of the four-dimensional Dirac delta function. For a finite u, Δ oscillates with increasing rapidity for $|x^2|\gg u$. When Δ is a product in an integrand, contributions from $|x^2|$ less than u will predominate. Such integrations are to be performed in a way which combines different x^2 before integrating over the infinite surfaces of constant x^2. The integral of $\Delta(x,u)$ over all x is equal to one for all u.

As $F(x)$ is a distribution over distances of the order a, the range of u in $f(u)$ is of the order of a^2. Furthermore, for F real and an odd function of x^2, $f(u)=f(-u)=f^*(u)$. $F(x)$ is to be normalized to one $\int d^4xF(x)=\int du f(u)=1$.

The function $\Delta(x,u)$ is of particular value in later calculations. It is used in determining the Green's function for the wave equation, for it satisfies the homogeneous wave equation

$$-\tfrac{1}{2}\partial^2\Delta/\partial x_\mu{}^2=i\partial\Delta/\partial\tau. \tag{2.4}$$

Its derivatives are

$$\begin{aligned}\partial\Delta/\partial x_\mu&=i(x_\mu/\tau)\Delta,\\ \partial\Delta/\partial\tau&=-[(2/\tau)+ix^2/2\tau^2]\Delta;\end{aligned} \tag{2.5}$$

and some four-dimensional integrals involving it are

$$\int d^4x'\Delta(x-x',\tau)\Delta(x'-x'',\tau')=\Delta(x-x'',\tau+\tau'),$$

$$\int d^4x'x_\mu'\Delta(x-x',\tau)\Delta(x'-x'',\tau')=\bar{x}_\mu\Delta(x-x'',\tau+\tau'),$$

$$\begin{aligned}&\int d^4x'x_\mu'x_\nu'\Delta(x-x',\tau)\Delta(x'-x'',\tau')\\ &\quad=(\bar{x}_\mu\bar{x}_\nu+i\delta_{\mu\nu}\tau\tau'/\tau+\tau')\Delta(x-x'',\tau+\tau'),\\ &\int d^4x'e^{-ipx'}\Delta(x-x',\tau)\Delta(x'-x'',\tau')\\ &\quad=\{\exp-i[p\bar{x}+p^2\tau\tau'/2(\tau+\tau')]\}\Delta(x-x'',\tau+\tau').\end{aligned} \tag{2.6}$$

On the right in the equations of (2.6), $\bar{x}$ is the mean of x and x'' weighted by τ^{-1} and τ'^{-1} respectively:

$$\bar{x}_\mu=(x_\mu\tau'+x_\mu''\tau)/(\tau+\tau'). \tag{2.7}$$

With the use of the integrals of (2.6), the shape function, $F(x)$, and the Green's functions can be readily combined in coordinate space; the momentum representation is not required. However, the Fourier transform of $\Delta(x,\tau)$ is $\exp(-i\tau k^2/2)$.

In all interactions between electrons, the shape function will appear at least twice, in the emission and absorption of virtual photons. In these instances, integrals of the type $\int dudvf(u)f(v)M(u+v)$ appear. We define the weighting function $W(z)$ by

$$W(2z)=\int_{-\infty}^{+\infty}dyf(z+y)f(z-y), \tag{2.8}$$

so that

$$\int dudvf(u)f(v)M(u+v)=\int dzW(z)M(z). \tag{2.9}$$

The function $W(z)$ contains all the information about the shape function, $F(x)$, which affects electron interactions.

The limiting case of the point electron is included with $f(u)=\delta(u)$, and $W(z)=\delta(z)$. In general, the range of z in $W(z)$ is also of the order of a^2, the square of the electron radius.

3. Equations of Motion

The classical equations of motion are derived from the variational principle,

$$\delta\left[\int d\tau L_E(\dot{q})+\int d^4x\mathfrak{L}_I(x)+\int d^4x\mathfrak{L}_F(x)\right]=0. \tag{3.1}$$

L_E is the electron Lagrangian, and $\mathfrak{L}_I$ and $\mathfrak{L}_F$ are the interaction and field Lagrangian densities. They are:

$$L_E=\tfrac{1}{2}\dot{q}_\mu{}^2,\quad \mathfrak{L}_I=j_\mu A_\mu,\quad \mathfrak{L}_F=\tfrac{1}{4}f_{\mu\nu}f_{\mu\nu}, \tag{3.2}$$

$j(x)$ is (3.2) is the current density of the electron, (1.4), together with any external currents. The field strengths are defined in the usual way,

$$f_{\mu\nu}=\partial A_\nu/\partial x_\mu-\partial A_\mu/\partial x_\nu. \tag{3.3}$$

The nonlocal interaction does not alter the propagation character of the field itself, so that the fields produced by a given current distribution are uniquely determined by deriving them from the retarded potentials only. All the effects of the nonlocal interaction are included in $j(x)$.

We introduce a potential in coordinate space by

$$\bar{A}_\mu(q)=\int d^4xF(x-q)A_\mu(x). \tag{3.4}$$

For the point electron, the average potential, $\bar{A}_\mu$, is just A_μ. For a variation in the electron path,

$$\delta\left(\int d\tau L_E+\int d^4x\mathfrak{L}_I\right)=\delta\int d\tau L, \tag{3.5}$$

where L is defined by $L=\frac{1}{2}\dot{q}_\mu{}^2+e\dot{q}_\mu\bar{A}_\mu$. The equations of motion for the electron are

$$\ddot{q}_\mu=ef_{\mu\nu}(q)\dot{q}_\nu. \tag{3.6}$$

$\bar{f}_{\mu\nu}$ is defined analogously to $\bar{A}_\mu$ by

$$f_{\mu\nu}(q)=\partial\bar{A}_\nu/\partial q_\mu-\partial\bar{A}_\mu/\partial q_\nu=\int d^4xF(x-q)f_{\mu\nu}(x). \tag{3.7}$$

Equation (3.6) differs in two ways from the more usual equations of motion involving the proper time,

$$md^2q_\mu/ds^2=ef_{\mu\nu}(q)dq_\nu/ds. \tag{3.8}$$

In the first place, $\bar{f}_{\mu\nu}$ in (3.6) implies an averaging of $f_{\mu\nu}$ over the distribution, $F(x-q)$. This is the effect of extension of the electron on its motion in an external field. The second difference is that m does not appear in (3.6). However, as a consequence of (3.6), $\partial \dot{q}_\mu^2/\partial\tau=0$, and the length of the vector $\dot{q}$ is constant. To obtain the particle path corresponding to (3.8) we impose the initial condition in τ that $\dot{q}$ be time-like and with length m. Then, since its length is constant, it will be time-like with length m for all τ. The mass of the particle is simply the magnitude of its four-velocity, and gives the speed with which the particle moves through four-space in the course of τ. ds, the length of an element of the path, will then be equal to $md\tau$. With this relationship between s and τ, except for the effects of the extended charge distribution, (3.8) agrees with (3.6).

4. Interactions

The interactions among electrons, including self-actions, follow from the equations of motion of (3.6) if the field produced by the electrons is included in $\bar{f}_{\mu\nu}$. The field intensity, $f_{\mu\nu}(x)$, produced by the current distribution, $j_\mu(x)$ is obtained from the usual retarded Green's function, D_{ret}. However, the averaged field, $\bar{f}_{\mu\nu}$, produced by an electron whose world line is $q(\tau)$, is obtained from the averaging of D_{ret} over the electron charge distributions,

$$\bar{D}_{\text{ret}}(q-q')=\int d^4x d^4x' F(x-q)F(x'-q')D_{\text{ret}}(x-x'). \tag{4.1}$$

D_{ret} is equal to half the sum of the advanced and retarded Green's functions, plus half their difference. Their difference is a solution of the homogeneous wave equation, and is not affected by the averaging of (4.1). Their sum can be expressed in terms of the Δ function of (2.2) by

$$\tfrac{1}{2}(D_{\text{ret}}+D_{\text{adv}})=\tfrac{1}{4}i\int_{-\infty}^{+\infty} ds\epsilon(s)\Delta(x,s). \tag{4.2}$$

Then, using the first of the integrals of (2.6), the definition of $W(z)$ of (2.8), and the fact that $W(z)=W(-z)$, we obtain

$$\tfrac{1}{2}(\bar{D}_{\text{ret}}+\bar{D}_{\text{adv}})$$
$$=\tfrac{1}{4}i\int dzW(z)\left[\int_{|z|}^{\infty} ds\Delta(x,s)-\int_{-\infty}^{-|z|} ds\Delta(x,s)\right]. \tag{4.3}$$

In (4.3), the effect of the nonlocal interaction in removing the portion of the integration in the region about $s=0$ is shown. This is just the region where $\Delta(x,s)$ has its singularity. The nonlocal interaction will have essentially this same effect in quantized field theory.

The field derived from $\frac{1}{2}(D_{\text{ret}}-D_{\text{adv}})$ is responsible for the radiation-damping part of the electron self-action,[10] and is not affected by the nonlocal interaction. The field from $\frac{1}{2}(D_{\text{ret}}+D_{\text{adv}})$ at a point on the electron's path, to lowest order in the τ derivatives at that point, is

$$f_{\mu\nu}(q)=(\lambda/m^3)(\ddot{q}_\mu\dot{q}_\nu-\ddot{q}_\nu\dot{q}_\mu). \tag{4.4}$$

In (4.4), $m^2=-\dot{q}^2$ and λ is the integral over q_0 of $\frac{1}{4}e^2(\bar{D}_{\text{ret}}+\bar{D}_{\text{adv}})$ at $\mathbf{q}=0$,

$$\lambda=\frac{\alpha}{2\pi^{\frac{1}{2}}}\int dzW(z)|z|^{-\frac{1}{2}}. \tag{4.5}$$

In the presence of an additional external field, $F_{\mu\nu}(x)$, the equations of motion, (3.6), become

$$\ddot{q}_\mu(1+\lambda/m)=\bar{F}_{\mu\nu}(q)\dot{q}_\nu. \tag{4.6}$$

For a given external field, the curvature of the path will be reduced by this effect of the self-action. This gives the same resultant path as would be obtained if the self-action were disregarded, and the initial magnitude of the four velocity, m, increased by the coefficient of $\ddot{q}_\mu$ in (4.6), giving $m'=m+\lambda$. The λ of (4.5) is the classical self-mass of the electron in terms of the $W(z)$ of (2.8), and the well-known result that it is inversely proportional to the radius of the electron follows from the fact that the range of z in $W(z)$ is the square of the electron radius.

5. Hamiltonian Form

To facilitate quantization, the equations of motion (3.6) are put into Hamiltonian form. The Lagrangian of (3.5) is

$$L(q,\dot{q})=\tfrac{1}{2}\dot{q}_\mu^2+e\dot{q}_\mu\bar{A}_\mu(q). \tag{5.1}$$

We construct the momenta conjugate to q, and the Hamiltonian, by

$$p_\mu=\partial L/\partial\dot{q}_\mu=\dot{q}_\mu+e\bar{A}_\mu, \tag{5.2}$$

$$H=p_\mu\dot{q}_\mu-L=\tfrac{1}{2}(p-e\bar{A})_\mu^2. \tag{5.3}$$

Writing $\pi(p,q)=\dot{q}$, then from (5.2) we have for (5.3):

$$H=\tfrac{1}{2}\pi_\mu\pi_\mu. \tag{5.4}$$

Since for an actual path, $\dot{q}$ is time-like with magnitude m, the numerical value of the Hamiltonian of (5.4) is $-m^2/2$. $m^2/2$ will be called ω, and is analogous to the energy in the more usual canonical formulation. The Hamiltonian gives the τ dependence of q and p by

$$\dot{q}_\mu=\partial H/\partial p_\mu, \quad \dot{p}_\mu=-\partial H/\partial q_\mu. \tag{5.5}$$

These relationships can be conveniently expressed by introducing the Poisson Brackets of any two functions, u and v of p and q,

$$\{u,v\}=(\partial u/\partial q_\mu)(\partial v/\partial p_\mu)-(\partial v/\partial q_\mu)(\partial u/\partial p_\mu). \tag{5.6}$$

Then,

$$\{u,H\}=\dot{u}, \quad \{q_\mu,q_\nu\}=0, \quad \{q_\mu,\pi_\nu\}=\delta_{\mu\nu},$$
$$\{\pi_\mu,\pi_\nu\}=e\bar{f}_{\mu\nu}(q), \quad H=\tfrac{1}{2}\pi_\mu\pi_\mu. \tag{5.7}$$

The set of equations of (5.7) provides a simple, gauge-invariant expression of the classical mechanics in a form suitable for quantization.

III. FIRST QUANTIZATION

6. Two-Component Wave Equation

We quantize this classical theory by associating operators to dynamical variables such that the commutator of two operators corresponds to i times the corresponding classical Poisson Bracket, i.e.,

$$U\rightarrow u,\quad V\rightarrow v,\quad [U,V]\rightarrow i\{u,v\}, \tag{6.1}$$

where $[U,V]=UV-VU$ and $\{u,v\}$ is the Poisson Bracket defined by (5.6).

As in the classical theory, the state refers to a single τ, and not to a single time. The state at a τ is specified by the wave function, $\psi(q)$, of the Q representation. In this representation, the operators Q_μ and Π_μ, corresponding to q_μ and π_μ of the classical theory, are:

$$\begin{aligned} Q_\mu\psi(q)&=q_\mu\psi(q),\\ \Pi_\mu\psi(q)&=(-i\partial/\partial q_\mu-e\bar{A}_\mu)\psi(q). \end{aligned} \tag{6.2}$$

The anti-Hermitian character of Q_4 and Π_4 causes no difficulty.

The numerical value of the classical Hamiltonian equals $-\omega$ (or $-m^2/2$), and for an eigenstate of the Hamiltonian operator,

$$H\psi=-\omega\psi. \tag{6.3}$$

In the Schrödinger representation, ψ develops in τ in accordance with

$$H\psi=i\partial\psi/\partial\tau. \tag{6.4}$$

Were we to choose for the Hamiltonian operator the direct analog of (5.4), $H=\frac{1}{2}\pi_\mu\pi_\mu$, we would have a description of a scalar particle. Instead, we use the Pauli 2×2 spin operators,

$$\begin{aligned} \sigma_\mu&=(\sigma_x,\sigma_y,\sigma_z,i),\quad \sigma^\mu=(\sigma_x,\sigma_y,\sigma_z,-i),\\ \sigma_\mu{}^\nu&=\sigma_\mu\sigma^\nu, \end{aligned} \tag{6.5}$$

with $\sigma_\mu{}^\nu+\sigma_\nu{}^\mu=2\delta_{\mu\nu}$. Classically, π_μ and π_ν commute, so that the classical Hamiltonian can be written:

$$H=\tfrac{1}{2}\pi_\mu\pi_\nu\sigma_\mu{}^\nu. \tag{6.6}$$

The Hamiltonian operator applicable to the electron corresponds to (6.6) instead of (5 4), giving

$$H=\tfrac{1}{2}\Pi_\mu\Pi_\nu\sigma_\mu{}^\nu=\tfrac{1}{2}\Pi_\mu\Pi_\mu+\tfrac{1}{4}ie\bar{f}_{\mu\nu}\sigma_\mu{}^\nu. \tag{6.7}$$

This Hamiltonian would not be self-adjoint if the inner product between states were defined as

$$(\psi_b,\psi_a)=\int d^4q\psi_b{}^*\psi_a. \tag{6.8}$$

However, $\Pi_\mu\sigma^\mu$ as well as $\Pi_\mu\sigma^\mu H$ would be Hermitian. We can therefore make the Hamiltonian of (6.7) self-adjoint by the inner product of

$$(\psi_b,\psi_a)=\int d^4q\psi_b{}^*\Pi_\mu\sigma^\mu\psi_a. \tag{6.9}$$

This inner product has the properties

$$(\psi_b,\psi_a)^*=(\psi_a,\psi_b),\quad (\psi_b,H\psi_a)=(H\psi_b,\psi_a). \tag{6.10}$$

The metric is indefinite, as (ψ_a,ψ_a) can be of either sign. Though an indefinite metric usually raises the serious physical objections of negative probabilities, here no physical significance is associated with measurements at a particular τ.

In Sec. 8, we will give explicitly a complete orthonormal set of wave functions that diagonalize the field-free Hamiltonian. In the presence of fields, we assume that there are still complete, orthonormal sets of wave functions, with

$$\begin{aligned} (\psi_b,\psi_a)&=\delta_{ab}\epsilon(a),\\ \textstyle\sum_a \epsilon(a)\psi_{a,\rho}(q)\bar{\psi}_{a,\sigma}(q')&=\delta_{\rho\sigma}\delta^4(q-q'). \end{aligned} \tag{6.11}$$

$\bar{\psi}(q)$ is equal to $(\Pi_\mu\sigma^\mu\psi)^*$. States with $\epsilon(a)$ equal to plus or minus one will be called electron and positron states, respectively, since for the eigenstates of H with eigenvalue equal to $-m^2/2$, this reduces to the usual nomenclature.

The expectation value of an operator, A, is $(\psi,A\psi)$, for states normalized by (6.11). In particular, the expectation value for the c-number, e, is $+e$ for an electron state and $-e$ for a positron state. The expectation value for the energy, P_0, is $(\psi,P_0\psi)$ and is positive for the customary electron and positron states.

The τ-independent wave equation of (6.3) can be used for solution of eigenvalue problems. It is the second order Dirac equation, $\boldsymbol{\Pi\Pi}\psi=-m^2\psi$, for an eigenstate of γ_5 $(=-\gamma_1\gamma_2\gamma_3\gamma_4)$ with eigenvalue -1, and therefore the usual eigenvalue relations follow.

The τ-dependent equation of (6.4) can be used for obtaining transition probabilities. It is mathematically more convenient to rewrite it first as a first order, four-component wave equation. In this form, it will be shown that the results are those of Dirac hole theory expressed in the form of Feynman graphs,[5] suitably modified by the nonlocal interaction.

7. Four-Component Spinors

The τ-dependent equation of (6.4) is inconvenient to work with because it is of second order both in the coordinate derivatives and in the potentials. In place of the two second-order equations it represents (one for each spinor component) it can be written as four first-order equations. This will be only a rewriting, and there is a one-to-one correspondence between the solutions of each set.

We will here denote the two-component spinor of Sec. 6 by u. The τ-dependent wave equation of (6.4)

can be rewritten as

$$-i\,\Pi_\mu\,\sigma^\mu u=v,\quad \tfrac{1}{2}i\,\Pi_\nu\,\sigma_\nu v=i\dot{u}. \tag{7.1}$$

The first equation of (7.1) defines v, and $-iv^*$ is the $\bar{\psi}$ of Sec. 6. These equations can be combined into a single equation by defining a four-component spinor by

$$\psi=2^{-\frac{1}{2}}\begin{pmatrix} v \\ u \end{pmatrix}. \tag{7.2}$$

The Hermitian Dirac matrices, γ_μ, can be represented in terms of the σ_μ and σ^μ by

$$\gamma_\mu=\begin{pmatrix} 0 & \sigma^\mu \\ \sigma_\mu & 0 \end{pmatrix}. \tag{7.3}$$

These satisfy the usual anticommutation relations,

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}. \tag{7.4}$$

In this representation, γ_5 is diagonal and equal to

$$\gamma_5=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.5}$$

We introduce the operator, M,

$$M=\begin{pmatrix} 1 & 0 \\ 0 & -2i\partial/\partial\tau \end{pmatrix}. \tag{7.6}$$

Its adjoint, $M^\dagger$, will be shown to equal

$$M^\dagger=\begin{pmatrix} -2i\partial/\partial\tau & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.7}$$

Together, they have the properties:

$$\begin{gathered} M\gamma_\mu=\gamma_\mu M^\dagger,\quad M\gamma_\mu\gamma_\nu=\gamma_\mu\gamma_\nu M, \\ MM^\dagger=M^\dagger M=-2i\partial/\partial\tau. \end{gathered} \tag{7.8}$$

The last equation of (7.8) justifies denoting this operator by M, since classically, $m^2=-2H$. It is a "mass operator" only in a trivial sense, for it has nothing to do with the effects of coupling to the radiation field.

The equations of (7.1) can now be combined to give

$$(i\mathbf{\Pi}+M)\psi=0, \tag{7.9}$$

where $\mathbf{\Pi}=\pi_\mu\gamma_\mu$. The inner product between two states of (6.9) in terms of the ψ of (7.2) is from (7.1) and (7.2),

$$(\psi_b,\psi_a)=\int d^4q\psi_b{}^*\gamma_4\psi_a. \tag{7.10}$$

For ψ_a and ψ_b sayisfying the τ-dependent wave equation of (7.9), the operator $M^\dagger$ of (7.7) is the adjoint of M, since

$$(\psi_b,M\psi_a)=(M^\dagger\psi_b,\psi_a). \tag{7.11}$$

Equation (7.9) differs from the Dirac equation in that the operator M replaces m, the electron mass. If ψ is an eigenfunction of H with eigenvalue $-m^2/2$, then these equations are equivalent. For such a state, $-2i\dot{\psi}=m^2\psi$, and

$$M=\begin{pmatrix} 1 & 0 \\ 0 & m^2 \end{pmatrix}. \tag{7.12}$$

Under the non-unitary transformation $\psi'=T^{-1}\psi$, $M'=T^*MT$, $\mathbf{\Pi}'=T^*\mathbf{\Pi}T$, with

$$T=\begin{pmatrix} m^{\frac{1}{2}} & 0 \\ 0 & m^{-\frac{1}{2}} \end{pmatrix}, \tag{7.13}$$

$M'=m$, and $\mathbf{\Pi}'=\mathbf{\Pi}$, so that (7.9) transforms into the Dirac equation.

The four-component wave equation of (7.9) and the expression for inner products of (7.10) will be used instead of the corresponding equations, (6.4) and (6.9).

The transformation properties of the wave equation (7.9) under a Lorentz transformation, $x_\mu'=a_{\mu\nu}x_\nu$, are the usual ones.[12] M commutes with the product of any two γ's; and only such products appear in the matrix S giving the transformed spinor, $\psi'=S\psi$. S is determined by

$$S^{-1}\gamma_\mu S=a_{\mu\nu}\gamma_\nu,\quad \gamma_4S^*\gamma_4=S. \tag{7.14}$$

Under a gauge transformation, $A_\mu'=A_\mu+\partial\Lambda/\partial x_\mu$, the transformation for ψ involves the average of Λ over the charge distribution, and is

$$\psi'=\exp(ie\langle\Lambda\rangle_{Av})\psi,\quad \langle\Lambda\rangle_{Av}=\int d^4xF(x-q)\Lambda(x). \tag{7.15}$$

8. Field Free-Motion

As an illustration of this approach, we consider the field free motion of the electron, and choose the gauge $A=0$. Then, a complete set of self-adjoint, commuting constants of the motion are P_0, P_1, P_2, P_3, and σ_p, where σ_p is the spin operator in the direction of the three-vector, $\mathbf{p}$. For those states with $\mathbf{p}=0$, σ_p is in the direction of an arbitrary three vector. They commute with the Hamiltonian of (6.7), which in this case is just $\frac{1}{2}P_\mu P_\mu$.

From the τ-independent equation of (6.3), the eigenvalues of H and P_μ satisfy

$$-p_\mu{}^2=2\omega=m^2. \tag{8.1}$$

The two-component eigenfunction with eigenvalues p_μ and λ for P_μ and σ_p is

$$\psi_{p,\lambda}(q)=e^{ipq}u_{p,\lambda}, \tag{8.2}$$

where $u_{p,\lambda}$ are the normalized eigenvectors of σ_p with eigenvalues $\lambda=\pm1$. The states can be normalized by

$$(\psi_{p,\lambda},\psi_{p',\lambda'})=\delta^4(p-p')\delta_{\lambda\lambda}\epsilon(p,\lambda) \tag{8.3}$$

[12] W. Pauli in *Handbuch der Physik*, editors H. Geiger and K. Scheel (J. Springer, Berlin, 1926), Vol. 24, part I, p. 259.

to give

$$\psi_{p,\lambda}(q)=(2\pi)^{-2}|p_0+\lambda\mathbf{p}|^{-\frac{1}{2}}e^{ipq}u_{p,\lambda},$$
$$\epsilon(p,\lambda)=\text{sign}(p_0+\lambda\mathbf{p}). \tag{8.4}$$

For $p^2=0$ (states of zero mass), the two spin states cannot be orthonormal in the sense of (8.3). Since for the electron, $-p^2=m^2$, these states enter only into virtual transitions, in which case integrals over p are taken. In this sense (8.3) can be fulfilled. For states with the electron mass, $|\mathbf{p}|$ is less than $|p_0|$, and ϵ is equal to ±1 depending on the sign of p_0.

A wave packet localized in space-time at a single τ can be formed by superposing eigenstates of P_μ whose eigenvalues range over a small four dimensional region about p_μ. Such a packet combines states of slightly differing masses. If the dimensions of the packet are large compared to the wavelengths involved, the packet moves through four-space with the group four-velocity, p. In the presence of slowly varying potentials, the packet is deflected in accordance with the classical equations of motion of (3.6). This is the four-dimensional analog of Ehrenfest's theorem.

9. Green's Functions and Feynman Graphs

The development of the state in τ is given by the τ-dependent wave equation of (6.4), or its equivalent, (7.9). The wave functions at a later τ is determined by that at an earlier one and can be evaluated with the use of the appropriate Green's functions. If, initially, the state is an eigenstate of the electron Hamiltonian, it will remain so for all τ since the potentials, A_μ are independent of τ. We need only impose the initial condition on the wave function that its eigenvalue of H correspond to the observed electron mass to insure that $\omega=m^2/2$ will always be the observed one.

For the solution of the wave equation which is equal to the given one at the initial τ, the boundary condition on the Green's function is that it gives propagation forward in τ. When used for the evaluation of transition probabilities, it gives the results of Feynman,[5] and hence of hole theory.

The field-free Green's function, $S_+(q,\tau)$, satisfies the inhomogeneous wave equation corresponding to (7.9)

$$(\gamma_\mu\partial/\partial q_\mu+M)S_+(q,\tau)=-i\delta^4(q)\delta(\tau) \tag{9.1}$$

and gives propagation forward in τ,

$$S_+(q,\tau)=0, \quad \tau<0. \tag{9.2}$$

The choice of the numerical constant in (9.1) is made to simplify subsequent equations. S_+ can be explicitly evaluated in terms of the Δ function of (2.2). We first define Δ_+ by

$$\Delta_+(q,\tau)=\Delta(q,\tau), \quad \tau>0$$
$$=0, \quad \tau<0; \tag{9.3}$$

then

$$S_+(q,\tau)=\tfrac{1}{2}(-\gamma_\mu\partial/\partial q_\mu+M^\dagger)\Delta_+(q,\tau) \tag{9.4}$$

satisfies (9.1) and (9.2). Substituting (9.4) into (9.1), we obtain

$$(\gamma_\mu\partial/\partial q_\mu+M)S_+=\tfrac{1}{2}(-\partial^2/\partial q_\mu{}^2-2i\partial/\partial\tau)\Delta_+. \tag{9.5}$$

Δ satisfies the homogeneous wave equation (2.4), and is equal to the four dimensional Dirac delta function in the neighborhood of $\tau=0$. From the discontinuity in Δ_+ of (9.3), the inhomogeneous equation (9.1) follows. The properties of (2.5) and (2.6) apply to Δ_+, provided all the τ are positive.

The ω frequency component of the τ dependence of Δ_+ is the Feynman propagation function Δ_F,[13]

$$\Delta_F(q)=\int_{-\infty}^{+\infty}d\tau e^{-i\omega\tau}\Delta_+(q,\tau)=\Delta_+(q,\omega). \tag{9.6}$$

Similarly, the ω component of S_+ from (9.4) differs only by the transformation of (7.13) from $\frac{1}{2}S_F(q)$. For a state which initially has the eigenvalue of H equal to $-\omega$, only these components of S_+ will be involved in the perturbation expansion of the Green's function.

In the presence of fields, the Green's function G satisfies

$$(i\mathbf{\Pi}+M)G(q,q';\tau)=-i\delta^4(q-q')\delta(\tau),$$
$$G(q,q';\tau)=0, \quad \tau<0. \tag{9.7}$$

This can be written,[14]

$$(\gamma_\mu\partial/\partial q_\mu+M)G(q,q';\tau)$$
$$=ieA(q)G(q,q';\tau)-i\delta^4(q-q')\delta(\tau). \tag{9.8}$$

To obtain G as a perturbation expansion in powers of e, we treat the term involving e as a perturbation, and set $G=\sum_n G_n$, with G_n of nth order in e. Then,

$$G_n(q,q';\tau-\tau')=e^n\int d\Omega^{(1)}\cdots d\Omega^{(n)}S_+(q-q^{(1)},\tau-\tau^{(1)})$$
$$\times A(q^{(1)})\cdots A(q^{(n)})S_+(q^{(n)}-q',\tau^{(n)}-\tau). \tag{9.9}$$

The volume elements of integration, $d\Omega^{(i)}$, are equal to $d^4q^{(i)}d\tau^{(i)}$ and the integrals are made over all $q^{(i)}$ and τ^i. Since the S_+ vanish for negative τ, there will be contributions to the integral of (9.9) only when τ, $\tau^{(1)}$, $\cdots$, $\tau^{(n)}$, τ', are in the order of decreasing τ. Equation (9.9) can readily be interpreted as a Feynman graph with one electron line and n vertices, each following the next in τ.

We can express the solution of wave equation satisfying the boundary condition that it equal a given ψ at the initial τ by

$$\psi(q,\tau)=\int d^4q'G(q,q';\tau-\tau')(1-\gamma_5)\psi(q',\tau'). \tag{9.10}$$

$(1-\gamma_5)$ is twice the projection operator which selects

[13] See, for example, F. J. Dyson, Phys. Rev. **75**, 486 (1952).
[14] $A(q)=A_\mu(q)\gamma_\mu$.

the last two components of ψ. From (7.1) and (7.2), these two components are sufficient to determine all of ψ at the initial τ, and also for all τ. These two components are just the original two component spinor, and since the four-component equation is only a rewriting of the two-component equation, one might expect only these components to determine the initial state, and hence appear in (9.10).

The wave function at a τ can be expressed as a linear combination of an orthonormal set, ψ_a, by

$$\psi(q,\tau)=\sum_a c_a\psi_a(q,\tau). \tag{9.11}$$

Then

$$c_a=(\psi_a,\psi)\epsilon(a). \tag{9.12}$$

The inner product (ψ_a,ψ) of (7.10) can be rewritten, in view of (7.1) and (7.2), as

$$(\psi_a,\psi)=\int d^4q\psi_a{}^*\gamma_4\psi=\int d^4q\bar{\psi}_a(1-\gamma_5)\psi, \tag{9.13}$$

where

$$\bar{\psi}=(\gamma_4M\psi)^*. \tag{9.14}$$

$\bar{\psi}$ satisfies the adjoint wave equation,

$$\bar{\psi}(i\mathbf{\Pi}+M)=0. \tag{9.15}$$

The differential operators operate on $\bar{\psi}$ with their sign reversed. In terms of the original two-component u and v,

$$\bar{\psi}=2^{-\frac{1}{2}}(-2u^*,\ -iv^*). \tag{9.16}$$

The transition amplitude c_a at τ for an initial state $\psi_{a'}(\tau')$, is

$$c_a=\epsilon(a)\int d^4qd^4q'\bar{\psi}_a(q,\tau)(1-\gamma_5)\times G(q,q';\tau-\tau')(1-\gamma_5)\psi_{a'}(q',\tau'). \tag{9.17}$$

We set $\psi_{a'}(q',\tau')$ equal to the solution of the field free wave equation equal to $\psi_{a'}$ at the initial τ'; and $\bar{\psi}_a(q,\tau)$, the corresponding solution equal to $\bar{\psi}_a$ at the final τ. Then, since the terms in the perturbation expansion of (9.9) all begin and end with the field free propagation functions, we obtain for the transition amplitude from the initial state, a', to the final state, a,

$$c_a=\epsilon(a)\textstyle\sum_n e^nc_a{}^{(n)} \tag{9.18}$$

with

$$c_a{}^{(n)}=\int d\Omega^{(1)}\cdots d\Omega^{(n)}\bar{\psi}_a(q^{(1)},\tau^{(1)})A(q^{(1)})\times S_+(q^{(1)}-q^{(2)},\ \tau^{(1)}-\tau^{(2)})\cdots\times A(q^{(n)})\psi_{a'}(q^{(n)},\tau^{(n)}). \tag{9.19}$$

The limits of integration on $\tau^{(1)}$ and $\tau^{(n)}$ are τ and τ'. All other integrations range from $\pm\infty$. However, since S_+ vanishes for negative τ, all the τ integrals give contributions only from the range τ to τ'.

If the states a and a' are eigenfunctions of the field free Hamiltonian with eigenvalues $-\omega_a$ and $-\omega_{a'}$, then $\psi_{a'}$ and $\bar{\psi}_a$ depend on τ in an exponential manner, $e^{i\omega_{a'}\tau'}$ and $e^{-i\omega_a\tau}$, respectively. When the difference $\tau-\tau'$ is large compared to the periods of the ω_a frequencies, each successive τ integration selects the ω_a component of S_+ until the last is reached. The last integral introduces a factor which can be written either as $2\pi\delta(\omega_a-\omega_{a'})$ or $(\tau-\tau')\delta_{\omega_a\omega_{a'}}$. The transition probabilities per unit τ follow from (9.19).

For a plane-wave initial state, the transition probability per unit time is $|p_0|^{-1}$ times the transition probability per unit τ, since p_0 is the time-like component of the four velocity. Also

$$\delta(\omega-\omega')=E^{-1}[\delta(p_0-E)+\delta(p_0+E)], \tag{9.20}$$

where $E=(m^2+\mathbf{p}^2)^{\frac{1}{2}}$ and $m^2=-p'^2$ for the initial state.

Therefore, the transition probability per unit time is $2\pi|p_0p_0'|^{-1}[\delta(p_0-E)+\delta(p_0+E)]|\mathfrak{M}|^2$, where $\mathfrak{M}$ is given by the sum of the expressions of (9.19) with the last τ integration omitted. Since the ω components of S_+ are just the Feynman propagation functions, S_F, the equivalence of this method to the usual hole theory is established for an electron in a given electromagnetic field.

In the evaluation of specific terms, the properties of (2.6) and (2.7) are of value if the q integrations in (9.19) are made first.