Variable Mass Equations

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Isotopic spin-space is generalized. The equations of field theory then describe systems of interacting particles as in ordinary isotopic spin-theory, but the particles and their interactions are more complex. The different systems which arise from different generalizations are systematically discussed.

1. INTRODUCTION

THE aim of this paper is to systematize field equations which describe particles having variable masses, and to make possible the description of families of such particles by one general theory.

It is shown that a desire for conservation equations of the usual charge-current type leads to the study of a generalized isotopic spin space. This space is Euclidean and has an arbitrary number of dimensions. It is because it has the same kind of structure as ordinary coordinate space, that the concept of spin occurs in it.

The general theory allows, but does not necessarily contain, multiply-charged particles. If the concepts of mesonic charge and mesonic charge conservation are introduced, a use for theories with multiply-charged particles is found. Apparently different particles can be thought of as differing only in mesonic charge.

It is shown that for interaction, a modified Dyson graph picture is valid. Renormalization is possible, but this is not shown.

2. GENERAL THEORY

We shall consider the customary boson and fermion field equations, but we shall replace the c-number masses by matrices, whose nature is unspecified save that they are Hermitian.

The ψ's and ϕ's are given an extra suffix, and using the summation convention, the free-field equations are

$$\frac{\partial}{\partial x_\mu}\gamma_{\alpha\beta}{}^{\mu}\psi_{\beta a}+M_{ab}\psi_{\alpha b}=0, \tag{1}$$

$$\Box^2\phi_a+(m^2)_{ab}\phi_b=0. \tag{2}$$

The matrix multiplication will often just be implied.

When there is interaction, it is assumed that the Hamiltonian contains a term

$$ig\bar{\psi}_{\alpha a}\gamma_{\alpha\beta}\Lambda_{ab}{}^{c}\psi_{\beta b}(\phi_c+\phi_c{}^{\dagger}), \tag{3}$$

where the γ is an ordinary γ matrix, e.g., γ^5, and the Λ^c are a set of matrices in the same space as M, whose nature is unknown. Equation (3) may be written as

$$ig\bar{\psi}\gamma\Lambda^c\psi(\phi_c+\phi_c{}^{\dagger}).$$

The fermion charge-current vector is taken as

$$\bar{\psi}\gamma^\mu J\psi,$$

where J is a matrix in the same space as M, and the boson charge-current vector is taken as

$$i\left[\frac{\partial\phi^\dagger}{\partial x_\mu}j\phi-\phi^\dagger j\frac{\partial\phi}{\partial x_\mu}\right],$$

where j is a matrix in the same space as m. Strictly, charge conjugates should be added in, but these are temporarily ignored. They make no essential difference.

The equation for total charge-current conservation, which it is necessary to have, is

$$\frac{\partial}{\partial x_\mu}\left\{i\bar{\psi}J\psi+\phi^\dagger j\frac{\partial\phi}{\partial x_\mu}-\frac{\partial\phi^\dagger}{\partial x_\mu}j\phi\right\}=0. \tag{4}$$

By satisfying this equation, we can partially determine the unknown matrices.

From (1),

$$\bar{\psi}J\gamma^\mu\frac{\partial\psi}{\partial x_\mu}+\bar{\psi}JM\psi=0,$$

$$\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma^\mu J\psi-\bar{\psi}MJ\psi=0;$$

so for conservation for a free-fermion field,

$$[M,J]=0. \tag{5a}$$

Similarly, for a free-boson field,

$$[m^2,j]=0. \tag{5b}$$

We now satisfy the conservation equation for the interacting fields. The coupled equations are

$$\left.\begin{aligned}\frac{\partial}{\partial x_\mu}\gamma^\mu\psi+M\psi&=g\gamma\Lambda^\alpha\psi(\phi_\alpha+\phi_\alpha{}^\dagger),\\ -\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma^\mu+\bar{\psi}M&=g\bar{\psi}\gamma\Lambda^\alpha(\phi_\alpha+\phi_\alpha{}^\dagger),\end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned}\Box^2\phi_\alpha+(m^2)_{\alpha\beta}\phi_\beta&=ig\bar{\psi}\gamma\Lambda^\alpha\psi,\\ \Box^2\phi_\alpha{}^\dagger+\phi_\beta{}^\dagger(m^2)_{\beta\alpha}&=ig\bar{\psi}\gamma\Lambda^\alpha\psi,\end{aligned}\right\}. \tag{7}$$

(In the last equation, an infinite constant appears, but as usual disappears when the charge conjugate part is added in.) Thus,

$$\frac{\partial}{\partial x_\mu}(i\bar{\psi}J\gamma^\mu\psi)=ig\bar{\psi}\gamma[J,\Lambda^\alpha]\psi(\phi_\alpha+\phi_\alpha{}^\dagger), \quad (8)$$

and

$$\frac{\partial}{\partial x_\mu}\left(\phi j^\dagger\frac{\partial\phi}{\partial x_\mu}-\frac{\partial\phi^\dagger}{\partial x_\mu}j\phi\right) =ig\phi_\alpha{}^\dagger j_{\alpha\beta}\bar{\psi}\gamma\Lambda^\beta\psi-ig\bar{\psi}\gamma\Lambda^\alpha\psi j_{\alpha\beta}\phi_\beta. \quad (9)$$

Adding, the required conservation equation (4) is obtained if

$$[J,\Lambda^\alpha]=-j_{\alpha\beta}\Lambda^\beta=\Lambda^\beta j_{\beta\alpha}. \quad (10)$$

In this equation of course, J, Λ^α, Λ^β are matrices (in the same space as M) and $j_{\alpha\beta}$ is a number, being an element of matrix j. A trivial solution $J=1$, $j=0$, always exists and this is the equation

$$\frac{\partial}{\partial x_\mu}(\bar{\psi}\gamma^\mu\psi)=0,$$

which expresses the conservation of the number of fermions interacting.

In general, Eqs. (10) define j, given J and the Λ^α, but they also require $j_{\alpha\beta}=-j_{\beta\alpha}$. J and Λ^α Hermitian implies j Hermitian, as is necessary.

It is necessary now to postulate some relationship between J and the Λ^α. When the Λ^α are given, the interaction theory can be set up, and it is expected that a conserved J will depend upon the Λ^α. We suppose J is a linear combination of the Λ^α. If Eq. (10) is satisfied by all linear combinations of the Λ^α, it is necessary that

$$[\Lambda^\gamma,\Lambda^\alpha]=\Lambda^\beta t_{\beta\alpha}{}^\gamma, \quad (11)$$

where

$$t_{\beta\alpha}{}^\gamma=-t_{\alpha\beta}{}^\gamma.$$

This equation may be satisfied by taking for the Λ^α, the members of the group of generators of infinitesimal rotations in a Euclidean space of arbitrary dimensions. These are usually written

$$I^{ij},\quad i\neq j;\quad I^{ij}=-I^{ji},$$

and satisfy

$$[I^{ab},I^{cd}]=i\delta_{ac}I^{bd}+i\delta_{bd}I^{ac}-i\delta_{bc}I^{ad}-i\delta_{ad}I^{bc}. \quad (12)$$

These evidently satisfy (11), for (11) is certainly satisfied if I^{ab}, I^{cd}, commute, and if they do not commute they have a common suffix, and it is seen that

$$[I^{ab},I^{ac}]=iI^{bc},$$
$$[I^{ab},I^{bc}]=-iI^{ac},$$

which is what is required.

As a particular example, the spin-$\frac{1}{2}$ infinitesimal generators in 3 dimensions are just half the Pauli matrices, so it is possible to write the interaction as

$$\tfrac{1}{2}ig\bar{\psi}\gamma\tau_\alpha\psi(\phi_\alpha+\phi_\alpha{}^\dagger).$$

This form for the interaction is well known.

3. ALGEBRAIC PROPERTIES OF MATRICES

Suppose A is a linear combination of the Λ^α. Then the fermion matrix A corresponds to the boson matrix a, where

$$[A,\Lambda^\alpha]=-a_{\alpha\beta}\Lambda^\beta. \qquad \text{from (10)}$$

If similarly $B\leftrightarrow b$, then evidently

$$A+B\leftrightarrow a+b, \quad (13)$$

and a short calculation shows

$$[A,B]\leftrightarrow[a,b]. \quad (14)$$

Equation (14) shows that the boson matrices t^α satisfy the same commutation relations as the Λ^α, and so are just another representation of the group to which Λ^α belongs.

It is shown now that $(t^\alpha)^3=t^\alpha$. Thus since the eigenvalues of t, from the group property, are $\pm k$, $\pm(k-1)$, $\cdots$ ($2k$ an integer), the eigenvalues are $0, \pm1$.

$$[I^{1\alpha},[I^{1\alpha},I^{2\alpha}]]=I^{2\alpha}.$$

Thus, for given θ, $[\Lambda^\theta,[\Lambda^\theta,\Lambda^\beta]]=\Lambda^\beta$ if θ and β have a superscript in common, and zero otherwise.

Now

$$[\Lambda^\theta,\Lambda^\alpha]=\Lambda^\beta t_{\beta\alpha}{}^\theta,$$

and so $t_{\beta\alpha}{}^\theta=0$ if θ and β do not have a superscript in common. Thus, summing on β,

$$[\Lambda^\theta,[\Lambda^\theta,\Lambda^\beta]]t_{\beta\gamma}=\Lambda^\beta t_{\beta\gamma};$$

so

$$\Lambda^\beta(t^3)_{\beta\gamma}=\Lambda^\beta t_{\beta\gamma},$$
$$t^3=t.$$

Hence, the eigenvalues of t are 0, ±1. This is a welcome result because if one of the t's is interpreted as boson charge, as is expected, the boson charge values are just 0, ±1.

It is interesting to observe that if the Dirac γ^μ are considered as 4 of the 10 infinitesimal rotation generators in 5 space,[1] in the appropriate representation, and if

$$[I^\gamma,I^\alpha]=S_{\alpha\beta}{}^\gamma I^\beta$$

the 10, 10×10 matrices S^γ will just be the 10×10 β matrices of Kemmer. In fact the γ's and β's are related in a way similar to the way in which Λ^α's and t^α's are related.

4. GAUGE TRANSFORMATIONS

The parts of ψ and ϕ on which the Λ's and the t's operate respectively, are not completely determined. It is possible to have a series of infinitesimal unitary

[1] See, for example, H. J. Bhabha, Revs. Modern Phys. 17, 200 (1945).

transformations of the type

$$\psi \to \psi' = e^{iS\delta\theta}\psi,$$
$$\phi \to \phi' = e^{is\delta\theta}\phi;$$

where S is a linear combination of the Λ's, and

$$[S,\Lambda^\alpha] = \Lambda^\beta s_{\beta\alpha} = -s_{\alpha\beta}\Lambda^\beta.$$

This may be regarded as a rotation in isotopic spin space. It is the precise analog of the similar gauge transformation which operates on the part of ψ on which the γ matrices act. In this case, of course, it is familiarly recognized as a Lorentz rotation.

A trivial calculation shows that the interaction

$$ig\bar{\psi}\gamma\Lambda^\alpha\psi(\phi_\alpha + \phi_\alpha^\dagger)$$

is invariant under such rotations.

The relationship (10) between J and j is also invariant, as is necessary, for (13) and (14) show that

$$J \leftrightarrow j$$

implies

$$J + i\delta\theta[J,S] \leftrightarrow j + i\delta\theta[j,s];$$

that is,

$$e^{-i\delta\theta S}Je^{+i\delta\theta S} \leftrightarrow e^{-i\delta\theta s}je^{+i\delta\theta s}.$$

It has been possible, by starting with conservation equations, to produce an isotopic spin space and to obtain invariance of the interaction under rotations in it. Of course, it would have been possible to have started with a generalized isotopic spin space and to have assumed invariance, and then, vice-versa, the conservation equations would have followed.

5. MASS MATRICES

It is unnecessarily restrictive to demand that the field equations should be invariant under rotations in spin space. (Since the proton mass does not equal the neutron mass, this is evidently true.) We take

$$M = F(\lambda_\alpha\Lambda^\alpha), \quad m = f(\lambda_\alpha t^\alpha),$$

where F and f are arbitrary functions. $\lambda_\alpha\Lambda^\alpha$ and J are chosen so that $[\lambda_\alpha\Lambda^\alpha, J] = 0$. Thus, by (14), $[\lambda_\alpha t^\alpha, j] = 0$, and hence

$$[M,J] = 0, \quad [m^2, j] = 0.$$

Thus, the conservation condition for free fields is satisfied. F, an arbitrary function, merely means that the eigenvectors of M are the eigenvectors of $\lambda_\alpha\Lambda^\alpha$, but that the eigenvalues are arbitrary.

Suitable forms for Λ^α, M, m, J, j have now been found. The theory can now be applied.

Example: The simplest case is obtained by taking the spin-$\frac{1}{2}$ representation in 3 dimensions, for the matrices Λ^α. The Λ^α are then the three matrices $\sigma_\alpha = \frac{1}{2}\tau_\alpha$ where τ_α are Pauli matrices.

It is of interest because it gives two internal fermion states and three internal boson states, and may be used to describe the neutron, proton, π-meson system.

The σ's have eigenvalues $\pm\frac{1}{2}$ and the t's corresponding to them have eigenvalues 0, ± 1. The matrices t can be obtained explicitly at once. For example,

$$t_3 = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The most general form for $\lambda_\alpha\Lambda^\alpha$, allowing a rotation, is just $\lambda\sigma_3$. Thus J is taken as $e(\frac{1}{2}+\sigma_3)$. (There is an extra term $\frac{1}{2}e$ here, which was not in the theory and which has been put in solely to give the customary zero of the fermion charge; since $\frac{1}{2}e\bar{\psi}\gamma^\mu\psi$ is separately conserved as has been shown, this addition is unimportant.) Also j is taken as et_3. Hence, there are two fermion states having charge 0, e and three boson states having charge 0, $\pm e$. The masses corresponding to these states are arbitrary. Explicitly for example, M may be written as $a + b\sigma_3$ where a and b are arbitrary.

The current and charge has been introduced as something which is conserved. So far there has been no electrical interaction. To introduce it, a term $X_\mu A_\mu$ is taken in the total Hamiltonian, where A_μ is the customary electric vector potential, and X_μ is the current vector previously defined. Since $\partial X_\mu/\partial x_\mu = 0$, there is gauge invariance under the transformation $A_\mu \to A_\mu + \partial\phi/\partial x_\mu$.

6. INTERACTION AND THE S MATRIX

It will be shown that the usual Dyson graph picture is valid, but that for each vertex of the graph there is an additional factor occurring in the appropriate S matrix term, and that this factor depends upon the internal states of the particles interacting at that vertex. This factor multiplied by g will be termed the effective coupling constant.

The free fermion equation is

$$\gamma^\mu\partial\psi/\partial x_\mu + M\psi = 0.$$

A solution is $\psi = e^{ip_\mu(m)x_\mu}u(p(m)m)v(m)$, where u and v are column matrices operated upon by the γ's and by M respectively, and $Mv(m) = mv(m)$ (v normalized).

We proceed as usual by taking a general linear combination of such solutions as the general solution, the coefficients being annihilation and creation operators. For a creation, it is necessary to specify the internal state of the particle created and similarly for a destruction. Thus,

$$\psi = \sum_{mp} a[p(m)m]e^{ip_\mu(m)x_\mu}u(p(m)m)v(m) + \text{conjugate part},$$

where

$$[a(p(m)m), \quad a^\dagger(q(n)n)] = \delta_{mn}\delta_{pq}.$$

ϕ is dealt with in the same way. Suppose the column eigenvectors which arise here and correspond to the v's are called $w(p)$, the p labelling the eigenvectors, as m labelled the eigenvectors of M. The interaction term in the Hamiltonian is

$$ig\bar{\psi}\gamma\Lambda^{\alpha}\psi(\phi_{\alpha}+\phi_{\alpha}^{\dagger}).$$

There is, thus, an extra factor at each vertex, as compared with usual theory:

$$v^{\dagger}(m_1)\Lambda^{\alpha}v(m_2)w_{\alpha}(p).$$

This particular factor comes from a vertex where there are three lines with the additional labels m_1, m_2, and p which describe the internal states of the particles. The effective coupling constant is

$$f(m_1m_2p)=gv^{\dagger}(m_1)\Lambda^{\alpha}v(m_2)w_{\alpha}(p).$$

(It is determined only to within a phase factor. However since w's always occur in conjugate pairs, this is unimportant.)

As an example, consider the 3-dimensional spin-$\frac{1}{2}$ case:

$$\sigma_3=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix},$$

$$t_2=\begin{pmatrix}0 & -i & 0\\ +i & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

so (labelling eigenvectors by eigenvalues of σ_3 now),

$$v(\tfrac{1}{2})=\begin{pmatrix}1\\0\end{pmatrix},\quad v(-\tfrac{1}{2})=\begin{pmatrix}0\\1\end{pmatrix},$$

$$w(1)=1/\sqrt{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix},\quad w(-1)=1/\sqrt{2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix},\quad (0)=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

The nonvanishing f's are

$$f(\tfrac{1}{2}\tfrac{1}{2}0)=\tfrac{1}{2}g,$$
$$f(-\tfrac{1}{2}-\tfrac{1}{2}0)=-\tfrac{1}{2}g,$$
$$f(\tfrac{1}{2}-\tfrac{1}{2}+1)=g/\sqrt{2},$$
$$f(-\tfrac{1}{2}+\tfrac{1}{2}-1)=g/\sqrt{2}.$$

The zero f's are those at which charge is not conserved.

By using effective coupling constants, it is perhaps possible to describe by one theory many different mesonic decays. The apparent differences in coupling constants necessary to account for these decays, may just be differences in the relevant effective coupling constants.

If effective coupling constants are renormalized, renormalization theory can be applied. There is a difficulty however, since in the customary theory coupling constants are given their observed values after renormalization. It is not clear to what extent renormalized *effective* coupling constants are arbitrary.

Charge conservation may be shown directly from the theory in a very simple way. J is related to j by

$$[J,\Lambda^{\alpha}]=\Lambda^{\beta}j_{\beta\alpha}.$$

Let J have eigenvalues J_r, J_s and vectors $v(J_r)$, $v(J_s)$. Let j have eigenvalue j_1 and vector $w_{\alpha}(j_1)$. Then,

$$w_{\alpha}v^{\dagger}(J_r)[J\Lambda^{\alpha}]v(J_s)=v^{\dagger}(J_r)\Lambda^{\beta}v(J_s)j_{\beta\alpha}w_{\alpha},$$

giving

$$(J_r-J_s-j_1)v^{\dagger}(J_r)\Lambda^{\alpha}v(J_s)w_{\alpha}=0.$$

Thus, charge is conserved at those vertices where the effective coupling constant does not vanish, and other vertices just do not occur.

7. TWO-FAMILY FERMION SYSTEM

Certain interesting features arise when we consider higher dimensions for the spin space. As an example consider 4 dimensions, and take the representation in which

$$I^{01}=\rho_3\sigma_3,\quad I^{23}=\sigma_3,$$
$$I^{02}=\rho_3\sigma_1,\quad I^{31}=\sigma_1,$$
$$I^{03}=\rho_3\sigma_2,\quad I^{12}=\sigma_2,$$

where the σ's are the previous σ's and ρ_3 is a Pauli matrix commuting with the σ's. Take

$$M=F(\alpha I^{01}+\beta I^{23}),$$
$$m=f(\alpha t^{01}+\beta t^{23}),$$
$$J=eI^{01}+\tfrac{1}{2}e,$$
$$j=et^{01}.$$

The eigenvalues of J are thus 0 and e, and the eigenvalues of j, 0, and $\pm e$. The eigenvalues of $\alpha I^{01}+\beta I^{23}$ are $\pm\frac{1}{2}\alpha\pm\frac{1}{2}\beta$. The eigenvalues of $\alpha t^{01}+\beta t^{23}$ are 0 (twice) and $\pm\alpha\pm\beta$. The latter are calculated by writing down explicitly the relevant matrices. Since $\alpha I^{01}+\beta I^{23}$ represents a fermion "current" and $\alpha t^{01}+\beta t^{23}$ represents a boson "current" which is conserved, there can be no transitions from the fermion state $\frac{1}{2}(\alpha+\beta)$ to the state $\frac{1}{2}(\alpha-\beta)$, for this needs a boson with "charge" β which does not exist. Thus the fermions split into two separately conserved families, those whose mass is $F[\pm\frac{1}{2}(\alpha+\beta)]$, and those whose mass is $F[\pm\frac{1}{2}(\alpha-\beta)]$. Each of the families possesses electrically charged and uncharged particles.

There are six kinds of boson; two of charge $\pm e$ interacting with the first family only; two of charge $\pm e$ interacting with the second family only, and two with zero charge interacting with both families.

8. MULTIPLY-CHARGED FERMIONS

In three dimensions, spin-k representation, there will be three Λ's, m_1, m_2, m_3, whose eigenvalues will be $\pm k$, $\pm(k-1)\cdots$, $2k$ an integer. As in spin-$\frac{1}{2}$ theory, J is taken as m_3. There are thus $2k+1$ fermion charge states. Here and in general, the effect of altering the representation of the Λ's is to change the number of fermion charge states.

9. MULTIPLY-CHARGED BOSONS

In order to obtain a theory with multiply-charged bosons, the interaction term is taken as

$$ig\bar{\psi}\gamma\Lambda^{\alpha}\Lambda^{\beta}\psi(\phi_{\alpha\beta}+\phi_{\alpha\beta}^{\dagger}),$$

where the Λ^{α} are matrices of the type previously discussed.

This is just the analogue of ordinary tensor coupling. Equation (10) is now

$$\begin{aligned}[J,\Lambda^{\alpha}\Lambda^{\beta}]&=-j_{(\alpha\beta)(\gamma\delta)}\Lambda^{\gamma}\Lambda^{\delta}\\ &=\Lambda^{\gamma}\Lambda^{\delta}j_{(\gamma\delta)(\alpha\beta)}\end{aligned} \tag{15}$$

and this is satisfied so long as J is just a linear combination of the Λ^{α}. The whole theory works as before.

The interesting difference is this. If j is defined as in the ordinary theory by

$$[J,\Lambda^{\alpha}]=-j_{\alpha\beta}\Lambda^{\beta}$$

and the eigenvalues of $j_{\alpha\beta}$ are $0\pm e$, then the eigenvalues of $j_{(\alpha\beta)(\gamma\delta)}$ are $\pm 2e$, $\pm e$, 0.

For, from (15), $j_{(\alpha\beta)(\gamma\delta)}=j_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\gamma}j_{\beta\delta}$, and so the eigenvectors of $j_{(\alpha\beta)(\gamma\delta)}$ are just $\phi_{\gamma}(j_1)\phi_{\delta}(j_2)$, where $\phi_{\gamma}(j_1)$ is an eigenvector of $j_{\alpha\gamma}$ belonging to eigenvalue j_1. Thus eigenvalues of $j_{(\alpha\beta)(\gamma\delta)}$ are j_1+j_2 which gives the result.

By taking higher order "tensor" coupling, more highly charged bosons can be obtained. Highly charged bosons may be unable to interact, because they may be unable to lose their charge. Such bosons will be stable.

10. MESONIC CHARGE

There is at the moment no use for theories with multiply-charged particles, so long as this charge is considered as electric charge. However if it is considered as mesonic charge (whose only property is that it is conserved) there may be a use for such theories. To describe electric and mesonic charge together a product theory is needed. The generalization is simple.

The interaction term is taken as

$$ig\bar{\psi}\Lambda^{(1)\alpha}\Lambda^{(2)\beta}\psi(\phi_{\alpha\beta}+\phi_{\alpha\beta}^{\dagger}),$$

where the Λ^{α}'s are matrices of the type previously discussed, but where the $\Lambda^{(1)}$ and the $\Lambda^{(2)}$ are in different spaces.

Everything works as before. J is of the form

$$\begin{aligned}J&=\xi_{\alpha}\Lambda^{(1)\alpha}+\eta_{\beta}\Lambda^{(2)\beta},\\ j&=\xi_{\alpha}t^{(1)\alpha}+\eta_{\beta}t^{(2)\beta};\end{aligned}$$

M is of the form

$$\begin{aligned}M&=F(\lambda_{\alpha}\Lambda^{(1)\alpha},\mu_{\beta}\Lambda^{(2)\beta}),\\ m&=f(\lambda_{\alpha}t^{(1)\alpha},\mu_{\beta}t^{(2)\beta}),\end{aligned}$$

where

$$\begin{aligned}[J,\lambda_{\alpha}\Lambda^{(1)\alpha}]&=0,\\ [J,\mu_{\beta}\Lambda^{(2)\beta}]&=0.\end{aligned}$$

As an example consider the spin-$\frac{1}{2}$, 3-dimensional theory for the fermions, in both spaces.

This scheme means that there are two electric charge states and two mesonic charge states for the fermions, and three electric and three mesonic charge states for the bosons. This might be useful in describing the proton, neutron V-particle system, if the V particle is considered as a mesonically charged nucleon.

State vectors in this theory are just products of the state vectors for the two simple theories considered separately. No new features arise. For example the electric charge current vector is taken as

$$\bar{\psi}\gamma^{\mu}(\tfrac{1}{2}e+e\sigma_3^{(1)})\psi,$$

and the mesonic vector as

$$\bar{\psi}\gamma^{\mu}(\tfrac{1}{2}f+f\sigma^{(2)})\psi.$$

It is possible however to vary this slightly. Electric fermion charge can be taken as

$$e\bar{\psi}\gamma^{\mu}(\sigma_3^{(1)}+\sigma_3^{(2)})\psi.$$

Then there will be greater symmetry and the zero of electric charge will depend upon the mesonic charge.

The fermion mass is taken as

$$M=a+b\sigma_3^{(1)}+c\sigma_3^{(2)}+d\sigma_3^{(1)}\sigma_3^{(2)}. \tag{16}$$

It may be felt that the natural generalization of M to a product theory should be

$$M=F(\lambda_{\alpha}\Lambda^{(1)\alpha}+\mu_{\beta}\Lambda^{(2)\beta}+\nu_{\alpha\beta}\Lambda^{(1)\alpha}\Lambda^{(2)\beta}).$$

We consider as an example of this more general form:

$$M=a+b\sigma_3^{(1)}+c\sigma_3^{(2)}+d\sigma_3^{(1)}\sigma_1^{(2)}. \tag{17}$$

There is here an essential difference. The eigenvectors of M are now no longer just products, but are sums of products of eigenvectors of $\sigma_3^{(1)}$ and $\sigma_3^{(2)}$, and the mesonic charge conservation equation is lost. If d is small, however, in Eq. (17), mesonic charge is almost conserved. This means that when there is interaction, there exist weak transitions which violate mesonic charge conservation, i.e., the absolute selection rule which previously existed is replaced by a weak selection rule. If this second scheme is used to describe proton, neutron, V-particle system, V-particle decay may perhaps be explained in this way.

11. PAIS' THEORY[2]

It is possible to consider a product theory in which the $\Lambda^{(1)\alpha}$, $\Lambda^{(2)\beta}$ are in different spaces, but in which a rotation in one space implies a rotation in the other. We consider the 3-dimensional case and take the spin-$\frac{1}{2}$ representation for the $\Lambda^{(1)\alpha}$ and the spin-k representation for $\Lambda^{(2)\alpha}$. We take

$$M=a+b(\Lambda^{(1)1}\Lambda^{(2)1}+\Lambda^{(1)2}\Lambda^{(2)2}+\Lambda^{(1)3}\Lambda^{(2)3}).$$

Since the spaces rotate together, M is invariant. This is almost the Pais form for the fermion mass. He takes,

[2] A. Pais, Physica **19**, 869 (1953).

instead of $\Lambda^{(2\ \alpha}$, the angular momentum operator in his ω space, $m_3=i(\omega_1\partial/\partial\omega_2-\omega_2\partial/\partial\omega_1)$, etc. The only difference is that his theory allows transitions between states with different k's. Here there is only one k, which is given. In both cases, an M is found which is invariant for rotations in spin-space. This seems unnecessary, but if it is desired, this is a possible M.

In order that $[J,M]=0$, for a conservation equation, we must take

$$J=e(\Lambda^{(1)3}+\Lambda^{(2)3}).$$

There is only one conserved quantity which must be interpreted as electric charge. The eigenvalues of J are

$$\pm(k+\tfrac{1}{2}),\quad \pm(k-\tfrac{1}{2}),\quad \cdots.$$

There are thus inevitably multiply-charged particles.

12. CHARGE CONJUGATION

$j_{\beta\alpha}$ is imaginary as has been shown. Thus, with our previous notation,

$$j_{\beta\alpha}\phi_\alpha{}^\dagger=-j\phi_\alpha{}^\dagger.$$

This expresses the familiar fact that by interchanging the roles of creation and annihilation operators in ϕ, the boson current is reversed.

The Λ^α and the $-\Lambda^{T\alpha}$ have the same commutation relations. Thus, as usual,

$$-\Lambda^{T\alpha}=T\Lambda^\alpha T^{-1},$$

where

$$TT^\dagger=1$$

$$T=\pm T^T.$$

It follows that

$$Jv(J_r)=J_rv(J_r)$$

implies

$$J(T^{-1}v^\dagger)=-J_r(T^{-1}v^\dagger).$$

The eigenvalues of J occur in pairs $\pm J_r$ with eigenvectors v and $T^{-1}v^\dagger$. This is just what is needed for fermion charge conjugation.

Usually,

$$\psi'=C^{-1}\bar{\psi}.$$

Here we have

$$\psi'=T^{-1}C^{-1}\bar{\psi}.$$

13. CONCLUSION

A possible framework for dealing with families of apparently different particles has been formulated. Perhaps it may prove possible to describe in this way what is actually observed to occur.

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Proper-Time Electron Formalism*

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A classical and first quantized formalism is presented which gives a complete description of electrons in a given electromagnetic field, including real and virtual pair processes. The Feynman viewpoint of electrons propagating through space-time is adopted throughout. Interactions between electrons are considered only in the classical theory, and a nonlocal interaction is assumed to make all effects finite. Consideration of interactions in the quantized theory is reserved to field quantization, which will be presented in a following paper. The calculation of transition probabilities gives the results of hole theory as interpreted by Feynman.

I. INTRODUCTION

USE has been made of the proper time[1] in quantum electrodynamics as a means of rewriting the Dirac equation.[2–4] It has provided covariant methods of calculation and prescriptions for the evaluation of divergent terms.

There are several reasons for investigating the possibilities of a more extensive use of this parameter. Feynman graphs[5] in field-theoretical calculations suggest the interpretation of the electron motion as evolving in four-space in the course of proper time. Also, the introduction of a covariant nonlocal interaction between the electron and electromagnetic field[6–8] suggests a shape to the electron in four-space at each value of the proper time. As a consequence, a quantum electrodynamics has been formulated in which the proper time plays an essential role throughout.

The concepts to be employed are first introduced

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[1] "Proper time" is used in the general sense of an invariant parameter describing the motion of the electron.

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