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Polarization and Amplitudes in Nucleon-Nucleon Scattering*

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The effect of coupling of states of the same J but different L on the nucleon-nucleon scattering amplitudes is investigated. Formulas for the polarization of nucleons in double scattering experiments are presented in a form suitable for numerical computation. In the derivation, special attention is given to the scattering of identical particles, and it is proved that a double scattering experiment involving identical nucleons does not require the consideration of a wave function antisymmetrized in all three particles. The effect of the coupling on polarization is also discussed. In particular, it turns out that the nondiagonal elements of the coupling matrix give a contribution to the polarization in which the highest-order spherical harmonic drops out.

I. INTRODUCTION

THE present paper is written in conjunction with the immediately preceding one.[1] It makes use of the same restrictions to elastic collisions and to a nonrelativistic approximation. It amplifies the results in two respects. In the first place it includes in the calculation of the amplitudes the effect of coupling of states of the same J but different L, such as is produced for, example, by tensor coupling. These effects have been considered before in different connections but formulas for calculation are not available in the literature in a systematic form. Secondly, the present paper gives formulas for the calculation of polarization effects in double-scattering experiments. The fundamental theory of these is available in the papers of Schwinger,[2] Wolfenstein,[3] and Wolfenstein and Ashkin[4] and other publications.[5] The extension to the case of identical particles has been treated by Swanson.[6] It was felt desirable to complete the latter in some respects as a matter of clarity of presentation and to present the formulas for polarization in a general form including coupling effects between states with different L.

The paper starts out with the introduction of the coupling effects in Eqs. (1) through (3.3). The effect of the coupling on the scattering matrix in the magnetic quantum number reference system is worked out leading to Eq. (8). The equivalent form in the ξ representation of the preceding paper is Eq. (9). The effect of antisymmetry of the wavefunction on the polarization is treated in connection with Eqs. (12) through (13.3). The more symmetric forms resulting from the ξ representation are discussed in connection with Eqs. (14) and (15). Results for polarization in a form suitable for immediate numerical substitution are given in Eqs. (18) and (20). These formulas can be used for numerical work by evaluating the complex numbers $\alpha_1, \cdots, \alpha_5, \alpha_c$, or else terms can be collected so as to exhibit more explicitly the angular dependence involved. In such cases many of the terms collect themselves in the Goldfarb-Feldman[5] symbols (δ_2, δ_1) of Eq. (20.1). The additions caused by nondiagonal terms of the coupling matrix are studied in Eqs. (20.2) through (22). Terms of highest order in $\cos\theta$, where θ is the colatitude angle, are seen to disappear.

II. NOTATION

U = unitary symmetric matrix transforming uncoupled states to coupled states. First row and left-hand column refer to orbital angular momentum l, second row and right-hand column to $l+2$, where $j=l+1$ is the common total angular momentum of the states being coupled.

$\mathcal{T}$ = "coupling matrix" $=(U-1)/(2i)$.

F_L, G_L = respectively, the regular and irregular solutions of the differential equation for $r\times$radial function in a Coulomb field, normalized so as to be asymptotic at $r=\infty$ to $\sin(\rho-\frac{1}{2}L\pi-\eta\ln 2\rho+\sigma_L)$ and $\cos(\rho-\frac{1}{2}L\pi-\eta\ln 2\rho+\sigma_L)$, respectively.

$H_L=G_L+iF_L$.

$C=\mathcal{T}-Q$ where Q is a two-dimensional diagonal matrix with elements $Q(\delta_-)$ and $Q(\delta_+)$, δ_- and δ_+ being parameters of Thaler *et al.*[7]

$\mathcal{Y}^{l,j}{}_{\mu}$ = defined in Eq. (1) of I.

s_x, s_y, s_z = Pauli spin vector/2.

P = polarization. Superscripts 1, v, a denote one-particle state, state v, and antisymmetrized state, respectively.

$\mathfrak{M}$ = matrix defined in Eqs. (7.2) of I.

III. COUPLING

In the present section the effect of coupling between states of the same J but different L will be considered. The matrix transforming the two uncoupled states with orbital angular momenta l, $l+2$ to the coupled states will be referred to as U, and the matrix

$$\mathcal{T}=[(U)-1]/(2i) \tag{1}$$

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[1] G. Breit and M. H. Hull, Jr., preceding paper [Phys. Rev. 97, 1047 (1955)], referred to as I. A preliminary account of some of the results of the present paper is contained in G. Breit and J. B. Ehrman, Phys. Rev. 96, 805 (1954) and in M. H. Hull, Jr. and A. M. Saperstein, Phys. Rev. 96, 806 (1954).

[2] J. Schwinger, Phys. Rev. 69, 681 (1946); 73, 407 (1948).

[3] L. Wolfenstein, Phys. Rev. 75, 1664 (1949); 76, 541 (1949).

[4] L. Wolfenstein and J. Ashkin, Phys. Rev. 85, 947 (1952).

[5] J. V. Lepore, Phys. Rev. 79, 137 (1950); L. J. B. Goldfarb and D. Feldman, Phys. Rev. 88, 1099 (1952); R. H. Dalitz, Proc. Phys. Soc. (London) A65, 175 (1952); A. Simon and T. A. Welton, Phys. Rev. 90, 1036 (1953).

[6] D. R. Swanson, Phys. Rev. 89, 749 (1953).

[7] Thaler, Bengston, and Breit, Phys. Rev. 94, 683 (1954).

will also be used. In terms of $\mathcal{T}$, the contribution to the scattered wave arising from the presence of a term in P_l in the plane wave modified only by the Coulomb field is

$$i^l[4\pi(2l+1)]^{\frac{1}{2}}[\exp(i\sigma_l)]\rho^{-1}\begin{pmatrix} l, j \\ 0, \mu \end{pmatrix} \times[\mathcal{T}_{l,\,l}\mathcal{Y}^{l,j}{}_\mu H_l+\mathcal{T}_{l,\,l+2}\mathcal{Y}^{l+2,j}{}_\mu H_{l+2}], \quad (1.1)$$

while that arising from the presence of a term in P_{l+2} gives

$$i^{l+2}[4\pi(2l+5)]^{\frac{1}{2}}[\exp(i\sigma_{l+2})]\rho^{-1}\begin{pmatrix} l+2, j \\ 0, \mu \end{pmatrix} \times[\mathcal{T}_{l+2,\,l}\mathcal{Y}^{l,j}{}_\mu H_l+\mathcal{T}_{l+2,\,l+2}\mathcal{Y}^{l+2,j}{}_\mu H_{l+2}]. \quad (1.2)$$

Here μ is the value of the spin component of the spin function χ_μ in the incident wave which gives rise to the two expressions just written. The notation

$$H_l=G_l+iF_l \quad (1.3)$$

is used above. The matrix U is unitary and symmetric. It may be expressed[8] in terms of the reactance matrix X by means of

$$U=\frac{i-X}{i+X};\quad X=X^*=\begin{pmatrix} a, c \\ c, b \end{pmatrix}. \quad (1.4)$$

In terms of the matrix elements of X, one has

$$U=\begin{pmatrix} 1+ab-c^2-i(b-a), 2ic \\ 2ic, 1+ab-c^2+i(b-a) \end{pmatrix} \times[1-ab+c^2-i(a+b)]^{-1}, \quad (1.5)$$

$$\mathcal{T}=[1-ab+c^2-i(a+b)]^{-1}\begin{pmatrix} a-i(ab-c^2), c \\ c, b-i(ab-c^2) \end{pmatrix}.$$

Setting

$$a=\tan\varphi_a,\qquad b=\tan\varphi_b,$$
$$\tau_a=\exp(i\varphi_a),\quad \tau_b=\exp(i\varphi_b), \quad (1.6)$$

one has

$$\mathcal{T}=[\tau_a^{-1}\tau_b^{-1}+c^2\cos\varphi_a\cos\varphi_b]^{-1} \times\begin{pmatrix} \tau_b^{-1}\sin\varphi_a+ic^2\cos\varphi_a\cos\varphi_b, c\cos\varphi_a\cos\varphi_b \\ c\cos\varphi_a\cos\varphi_b, \tau_a^{-1}\sin\varphi_b+ic^2\cos\varphi_a\cos\varphi_b \end{pmatrix}. \quad (1.7)$$

If $c=0$ this matrix becomes diagonal with elements $\tau_a\sin\varphi_a$, $\tau_b\sin\varphi_b$ which are of the form of $Q(\varphi_a)$, $Q(\varphi_b)$. In the notation of Thaler, Bengston, and Breit,[7] who will be hereafter referred to as TBB,

$$\mathcal{T}=\frac{U-1}{2i}=\begin{pmatrix} Q(\delta_-), 0 \\ 0, Q(\delta_+) \end{pmatrix}+\begin{pmatrix} C_{-,-}, C_{-,+} \\ C_{+,-}, C_{+,+} \end{pmatrix}; \quad (2)$$

the matrix (C) being expressible in terms of the 4 parameters δ_-, δ_+, y, T as in their Eq. (1.4). The quantities δ_-, δ_+ are phase shifts which would exist if the coupling were removed by making the angle $T=0$. While in the model used by TBB the phase shifts δ_-, δ_+ enter in a natural manner; they are not needed for the description of coupling by means of the three parameters a, b, c or the equivalent set of parameters φ_a, φ_b, c. The parameters y, T, δ_-, δ_+ of TBB will not be used in the present discussion and the symbols Q for the coupled states will be used in a *different* sense as will be seen presently. The parameters a, b, c can be expressed in terms of y, T, δ_-, δ_+. In doing so it will be useful to rename (δ_-, δ_+) as (δ_1, δ_2), some of the relations becoming more obviously similar as a result. One has then

$$1/c=[\cos(T+\delta_1+\delta_2)+C^2\cos(T+\delta_2-\delta_1) +S^2\cos(T-\delta_2+\delta_1)]/(2CS\sin T), \quad (2.1)$$

where

$$C=\cos\epsilon=y^{\frac{1}{2}},\qquad S=\sin\epsilon=(1-y)^{\frac{1}{2}}, \quad (2.2)$$

$$a=-\tan(\Theta_2/2),\quad b=-\tan(\Theta_1/2), \quad (2.3)$$

with Θ_1, Θ_2 being obtainable from

$$\tan(\tfrac{1}{2}\Theta_i+\delta_1+\delta_2+T) =\sin(\chi_i+2\delta_i)/[(1/\mathcal{R})+\cos(\chi_i+2\delta_i)], \quad (2.4)$$

$$\mathcal{R}=[1-\sin^2(2\epsilon)\sin^2T]^{\frac{1}{2}}, \quad (2.5)$$

$$\cos\chi_1=(C^2+S^2\cos 2T)/\mathcal{R},\quad \sin\chi_1=S^2\sin 2T/\mathcal{R}, \quad (2.6)$$

$$\cos\chi_2=(S^2+C^2\cos 2T)/\mathcal{R},\quad \sin\chi_2=C^2\sin 2T/\mathcal{R}. \quad (2.7)$$

Another way of expressing a, b, c is to express c as previously mentioned and to use

$$\delta_1+\delta_2+T=\arctan\left(\frac{a+b}{1-ab+c^2}\right), \quad (3)$$

$$\delta_1-\delta_2-T+\arctan\frac{(1-y)\sin 2T}{y+(1-y)\cos 2T} =\arctan\frac{a-b}{1+ab-c^2}, \quad (3.1)$$

expressing therefore $(a+b)/[1-ab+c^2]$ and $(a-b)/[1+ab-c^2]$ in terms of δ_1, δ_2, T, y.

The employment of $\mathcal{T}$ as in Eqs. (1.1), (1.2) corresponds to the production of the following modifications by the nucleon-nucleon interaction:

$$F_l\mathcal{Y}^{l,j}{}_\mu\rightarrow F_l\mathcal{Y}^{l,j}{}_\mu+\mathcal{T}_{l,\,l}\mathcal{Y}^{l,j}{}_\mu H_l +\mathcal{T}_{l,\,l+2}\mathcal{Y}^{l+2,j}{}_\mu H_{l+2},$$
$$F_{l+2}\mathcal{Y}^{l+2,j}{}_\mu\rightarrow F_{l+2}\mathcal{Y}^{l+2,j}{}_\mu+\mathcal{T}_{l+2,\,l}\mathcal{Y}^{l,j}{}_\mu H_l +\mathcal{T}_{l+2,\,l+2}\mathcal{Y}^{l+2,j}{}_\mu H_{l+2}. \quad (3.2)$$

These modifications may be assumed to hold outside the

[8] E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **32**, 302 (1946); J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 530; J. M. Blatt and L. C. Biedenharn, Phys. Rev. **86**, 399 (1952); Revs. Modern Phys. **24**, 258 (1952).

range of nuclear forces. At a large distance, r, the asymptotic forms are accordingly

$$[-\exp(-i\varphi_l)+U_{l,\,l}\exp(i\varphi_l)]\mathcal{Y}^{l,j}{}_{\mu}+U_{l,\,l+2}\exp(i\varphi_{l+2})\mathcal{Y}^{l+2,j}{}_{\mu},$$
$$U_{l+2,\,l}\exp(i\varphi_l)\mathcal{Y}^{l,j}{}_{\mu}+[-\exp(-i\varphi_{l+2})+U_{l+2,\,l+2}\exp(i\varphi_{l+2})]\mathcal{Y}^{l+2,j}{}_{\mu}, \quad (3.3)$$

with

$$\varphi_l=\rho-\eta\ln 2\rho-(l\pi/2)+\sigma_l. \quad (3.4)$$

The physical assumptions regarding the interactions are sufficiently well summarized in the specification of these asymptotic forms for purposes of the present problem, since the asymptotic forms are the only properties of the interactions which affect scattering. Employing the transformation coefficients of Eqs. (1), (1.1), (1.2), and (1.3) of the preceding paper[1] and taking the incident wave to be

$$\psi^c\chi_\mu,$$

the factors multiplying

$$i^l[4\pi(2l+1)]^{\frac{1}{2}}\rho^{-1}[\exp(i\sigma_l)]\mathcal{T}_{l,\,l+2}H_{l+2}Y_{l+2,\,l-M}\chi_M$$

in the expression for the scattered wave are the

$$\begin{pmatrix} l+2,\ j=l+1 \\ \mu-M,\ M \end{pmatrix}\begin{pmatrix} l,\ j=l+1 \\ 0,\ \mu \end{pmatrix},$$

while the factors multiplying

$$i^{l+2}[4\pi(2l+5)]^{\frac{1}{2}}\rho^{-1}[\exp(i\sigma_{l+2})]\mathcal{T}_{l+2,\,l}H_lY_{l,\,\mu-M}\chi_M$$

in the scattered wave are the

$$\begin{pmatrix} l,\ j=l+1 \\ \mu-M,\ M \end{pmatrix}\begin{pmatrix} l+2,\ j=l+1 \\ 0,\ \mu \end{pmatrix}.$$

The combined contribution of these terms to the elements of $S_{\mu\nu}$ is expressible as

$$k\Delta S_{1,\,1}=-(l+1)(l+2)[Z^{l+2}{}_0+Z^l{}_0]e^{i\Phi}B', \quad (4)$$

$$k\Delta S_{-1,\,1}=-[(l+1)(l+2)]^{\frac{1}{2}}\{[(l+3)(l+4)]^{\frac{1}{2}}Z^{l+2}{}_2+[(l-1)l]^{\frac{1}{2}}Z^l{}_2\}e^{i\Phi}B', \quad (4.1)$$

$$k\Delta S_{0,\,1}=-[2(l+1)(l+2)]^{\frac{1}{2}}\times\{-[(l+1)(l+3)]^{\frac{1}{2}}Z^{l+2}{}_1+[l(l+2)]^{\frac{1}{2}}Z^l{}_1\}e^{i\Phi}B', \quad (4.2)$$

$$k\Delta S_{1,\,0}=-\{(l+1)[2(l+2)(l+3)]^{\frac{1}{2}}Z^{l+2}{}_{-1}-(l+2)[2l(l+1)]^{\frac{1}{2}}Z^l{}_{-1}\}e^{i\Phi}B', \quad (5)$$

$$k\Delta S_{-1,\,0}=-\{(l+1)[2(l+2)(l+3)]^{\frac{1}{2}}Z^{l+2}{}_1-(l+2)[2l(l+1)]^{\frac{1}{2}}Z^l{}_1\}e^{i\Phi}B', \quad (5.1)$$

$$k\Delta S_{0,\,0}=2(l+1)(l+2)[Z^{l+2}{}_0+Z^l{}_0]e^{i\Phi}B', \quad (5.2)$$

$$k\Delta S_{1,\,-1}=-[(l+1)(l+2)]^{\frac{1}{2}}\times\{[(l+3)(l+4)]^{\frac{1}{2}}Z^{l+2}{}_{-2}+[(l-1)l]^{\frac{1}{2}}Z^l{}_{-2}\}e^{i\Phi}B', \quad (6)$$

$$\Delta S_{-1,\,-1}=\Delta S_{1,\,1}, \quad (6.1)$$

$$k\Delta S_{0,\,-1}=-[2(l+1)(l+2)]^{\frac{1}{2}}\times\{-[(l+1)(l+3)]^{\frac{1}{2}}Z^{l+2}{}_{-1}+[l(l+2)]^{\frac{1}{2}}Z^l{}_{-1}\}e^{i\Phi}B', \quad (6.2)$$

where use is made of the abbreviations:

$$B=(\mathcal{T}_{l,\,l+2}/2)[(l+1)(l+2)]^{-\frac{1}{2}}\exp[i\sigma_{l,\,0}+i\sigma_{l+2,\,0}], \quad (7)$$

$$B'=B/[(2l+1)(2l+5)]^{\frac{1}{2}}, \quad (7.1)$$

and of a notation which can be applied to a given pair of coupled states but not in general, *viz.*,

$$Z^{l+2}{}_\mu=[4\pi(2l+1)]^{\frac{1}{2}}Y_{l+2,\,\mu}, \quad (7.2)$$

$$Z^l{}_\mu=[4\pi(2l+5)]^{\frac{1}{2}}Y_{l,\,\mu}. \quad (7.3)$$

The diagonal elements of $\mathcal{T}$ require no special consideration because according to Eq. (2) they enter in the same places as the Q's of the equations for uncoupled states.

Substituting the values of the $Y_{l,\,\mu}$, $Y_{l+2,\,\mu}$ in terms of Legendre functions through Eqs. (7.2) and (7.3) into Eqs. (4) through (6.2), one obtains

$$k\Delta\|S_{\mu\nu}\|=Be^{i\Phi}\times\begin{pmatrix} -U,\ -2^{\frac{1}{2}}W\sin\theta e^{-i\varphi},\ -V\sin^2\theta e^{-2i\varphi} \\ -2^{\frac{1}{2}}W\sin\theta e^{i\varphi},\ 2U,\ 2^{\frac{1}{2}}W\sin\theta e^{-i\varphi} \\ -V\sin^2\theta e^{2i\varphi},\ 2^{\frac{1}{2}}W\sin\theta e^{i\varphi},\ -U \end{pmatrix}, \quad (8)$$

the rows and columns of the matrix corresponding to $\mu=1,\ 0,\ -1$ in the above order, starting the labeling with 1 for rows as well as columns in the upper left-hand corner, with

$$U=(l+1)(l+2)[P_l+P_{l+2}], \quad (8.1)$$

$$V=P_l''+P_{l+2}'', \quad (8.2)$$

$$W=(l+1)P_{l+2}'-(l+2)P_l'. \quad (8.3)$$

The argument of the Legendre functions is $\cos\theta$ throughout, while the prime and double prime denote single and double differentiation with respect to the argument $\cos\theta$.

By means of $\mathfrak{M}$ and Eq. (7.4) of I, one calculates the change in kS^ξ caused by the nondiagonal element of $\mathcal{T}$ to be

$$k\Delta S^\xi=Be^{i\Phi}\times\begin{pmatrix} V\sin^2\theta\cos2\varphi-U,\ V\sin^2\theta\sin2\varphi,\ 2W\sin\theta\cos\varphi \\ V\sin^2\theta\sin2\varphi,-V\sin^2\theta\cos2\varphi-U,\ 2W\sin\theta\sin\varphi \\ 2W\sin\theta\cos\varphi,\ 2W\sin\theta\sin\varphi,\ 2U \end{pmatrix}. \quad (9)$$

Comparison with Eq. (7.5) of I shows considerable similarity of structure between ΔS^ξ and S^ξ. The effect of

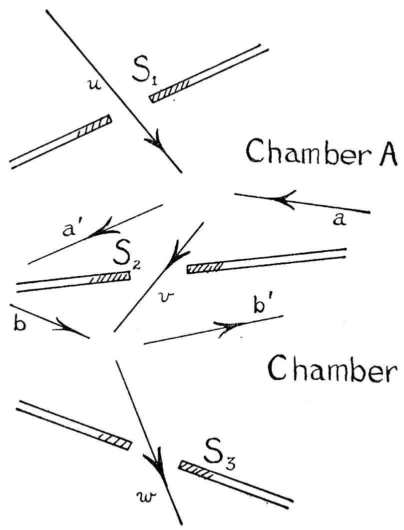

FIG. 1. Schematic diagram of a double-scattering experiment.

antisymmetrizing the wave function for identical nucleons is described by Eqs. (4), (4.1), (4.2), and (6) of I. All of these correspond to interchanging all of the coordinates, subtracting and dividing by $\sqrt{2}$. The addition to S caused by $\mathcal{T}_{l,\,l+2}$ is obtainable, therefore, by formulas similar to Eqs. (4.1), (4.2) of I, *viz.*,

$$\Delta S^{a}{}_{\mu,\,\nu}=2^{\frac{1}{2}}\Delta S_{\mu,\,\nu}, \tag{10}$$

the formula applying for $\mu=\nu$ as well as $\mu\neq\nu$. This addition can be made to the matrix S^a obtainable by means of formulas listed in I. Similarly to Eq. (10) there is the relation

$$\Delta S^{a\xi}{}_{ij}=2^{\frac{1}{2}}\Delta S^{\xi}{}_{ij}. \tag{10'}$$

The calculation of differential cross sections for unpolarized particles reduces to substitution of matrices completed through the additions designated here by prefix Δ to matrices S, S^a, S^ξ, $S^{a\xi}$ as needed in Eqs. (5.1), (6.1), (8), and (8.1) of I.

While for identical particles ($T=1$) the coupling discussed here applies only to states with odd L, states with even L may be coupled for nonidentical particles. The best known coupling of this type occurs between $^3S_1$ and $^3D_1$, as known through the work of Schwinger and Rarita.

IV. POLARIZATION, FORMAL QUESTIONS

The polarization of nucleons in double-scattering experiments has been treated by many authors.[2–6] No attempt will be made here to discuss the symmetries involved in the general problem, the viewpoint here being primarily that of supplying a general formula. However, a study of the published papers failed to reveal a complete account of the reasons for not needing to calculate with a wave function antisymmetric in all of the three particles involved in a double-scattering experiment. The tracing of the factors 2 and $\sqrt{2}$ involved in a consideration of effects of antisymmetrizing the function in a pair of colliding particles has also not been carried out[4] in a convincing manner in the literature. Since both of these tasks can be completed in a short space, they will be discussed briefly in the present section so as to make the remaining operations intelligible.

A complete theoretical description of the scattering experiments would involve a consideration of three particles and two slits, the interactions of the particles with each other and with the slits. Such detail is however unnecessary and it is inconvenient to work with a three-particle function. The fact that it suffices to work with a two-particle function for each collision is readily believed but apparently has not been demonstrated in print. A brief explanation will be given now as a matter of completeness. The two scattering chambers are schematically shown in Fig. 1 as separated by the slit S_2 with incident beam entering chamber A through slit S_1 and the finally scattered beam emerging from chamber B through slit S_3. In chamber A the incident beam in state u collides with particles in state a producing the scattered beam in state v and recoil particles in state a'. The particles in state v enter chamber B, are scattered by collision with particles in state b producing recoils in state b' and scattered particles in state w, the latter emerging through slit S_3. The initial condition is

$$\psi_1=(u,\,a,\,b), \tag{11}$$

where the symbol on the right side is meant to be

$$(x,\,y,\,z)=(3!)^{-\frac{1}{2}}\begin{vmatrix} x(1) & y(1) & z(1)\\ x(2) & y(2) & z(2)\\ x(3) & y(3) & z(3)\end{vmatrix}, \tag{11.1}$$

and it is assumed that the overlap between the functions is negligible in the calculation of the normalization integral or else that the factor $(3!)^{-\frac{1}{2}}$ is corrected for overlap. After the first scattering is taken into account in the calculations, an intermediate state

$$\psi_2=(a',\,v,\,b) \tag{11.2}$$

is obtained, and it gives rise to the final state

$$\psi_3=(a',\,b',\,w). \tag{11.3}$$

In the calculation of ψ_2, one is concerned with the production of states a', v from the states u, a. Expanding ψ_1 and ψ_2 in products of $b(1)$, $b(2)$, $b(3)$ by their cofactors in the determinants on the right side of Eqs. (11) and (11.2), one deals with the normalized determinants $(u,\,a)$ and $(a',\,v)$ which occur with protons (2, 3), (3, 1), and (1, 2) in succession. The calculation of the collision in the first chamber by means of an antisymmetric function in two particles thus supplies the co-factor multiplying one of the three quantities $b(1)$, $b(2)$, $b(3)$ in the expansion of (11.2). Similarly the passage from the intermediate state, ψ_2, to the final state, ψ_3, is obtained by finding out how the normalized determinant

(v, b) changes to (b', w). It is seen that the only complication caused by dealing with three rather than two particles consists in working with the cyclic sum

$$3^{-\frac{1}{2}} \sum u(i)(a, b)_{jk},$$

rather than with one term of the sum and factor $3^{-\frac{1}{2}}$ removed. The factor $3^{-\frac{1}{2}}$ takes care of the three ways of scattering a particle in making the complete three-particle description equivalent to considering the third proton as though it were not identical with the two protons that participate in the scattering process. It is essential in the argument just presented that the state b should not participate in the interaction giving the pair of states (a', v) out of the states (u, a), and similarly that the state a' should not be changed in the transition from ψ_2 to ψ_3.

In the first scattering chamber there is produced the cyclic sum

$$3^{-\frac{1}{2}} \sum b(i)(a', v)_{jk} = \psi_2,$$

each term of which arises from a corresponding term in

$$3^{-\frac{1}{2}} \sum b(i)(u, a)_{jk} = \psi_1.$$

It is seen that the expectation value of the y projection of the sum of spins for two particles, s_y, has a definite value for the two-particle function

$$(a', v)_{jk},$$

this value being independent of the particular pair jk. Similarly, in the second chamber the change from the two-particle state (v, b) to (b', w) produces a change in $\langle s_y \rangle$. It is seen therefore that the calculation of $\langle s_y \rangle$ for pairs of particle states (u, a) to (a', v) to (b', w) can be made by solving separate two-particle problems. From here on, one can fall back on existing theoretical treatments which give explicit formulas for the calculation of $\langle s_y \rangle$. It is desirable, however, to have a clear distinction between the cases of identical and nonidentical particles. For the latter case there is a definite meaning to the identification of a particle, for instance, a proton in distinction to a neutron. Referring to the particle scattered through the system as number 1, one has,[3] after scattering of unpolarized by unpolarized particles,

$$P^1 = 2\langle s^1{}_y \rangle = \langle \sigma^1{}_y \rangle$$
$$= 2^{\frac{1}{2}} \{ \mathrm{Im} [\textstyle\sum_m (S_{1,m} - S_{-1,m})^* S_{0,m}] \} / \{ |s_{00}|^2 + \textstyle\sum_{\mu\nu} |S_{\mu\nu}|^2 \}. \quad (12)$$

It is to the discussion of this quantity in successive scattering that the usual developments apply. Applying Eq. (5.1) of I, we obtain

$$4\boldsymbol{\sigma} P^1 = 2^{\frac{1}{2}} \, \mathrm{Im} \textstyle\sum_m (S_{1,m} - S_{-1,m})^* S_{0,m}. \quad (12.1)$$

For identical particles one cannot observe the spin of one of them. The physically meaningful quantity is $s_y = s^1{}_y + s^2{}_y$. One obtains, in this case,

$$\langle s_y \rangle = 2^{\frac{1}{2}} \{ \mathrm{Im} \textstyle\sum_m (S^a{}_{1,m} - S^a{}_{-1,m})^* S^a{}_{0,m} \} / \{ |s^a{}_{00}|^2 + \textstyle\sum_{\mu\nu} |S^a{}_{\mu\nu}|^2 \}. \quad (12.2)$$

Presupposing, for the sake of definiteness, a clean separation of the states a', v which result after the first scattering, one has

$$\langle s_y \rangle = (a'_1, s^1{}_y \, a'_1) + (v_1, s^1{}_y \, v_1)$$
$$= 2(v_1, s^1{}_y \, v_1) \equiv 2\langle s^v{}_y \rangle, \quad (12.3)$$

as follows from a calculation with an antisymmetric wave function. The equality of the two parts in the intermediate step in Eq. (12.3) is a consequence of the occurrence of the same normalization of factors multiplying the χ_μ in a' and v. The quantity $\langle s^v{}_y \rangle$ is the expectation value of the spin of a particle in state v. It does not matter, of course, which of the two particles one is talking about since v is a one-particle state. One may define now

$$P^v = 2\langle s^v{}_y \rangle = \langle s_y \rangle, \quad (12.4)$$

with $\langle s_y \rangle$ given by Eq. (12.2). According to Eq. (6.1) of I,

$$2\boldsymbol{\sigma}^a P^v = 2^{\frac{1}{2}} \, \mathrm{Im} [\textstyle\sum_m (S^a{}_{1,m} - S^a{}_{-1,m})^* S^a{}_{0,m}]. \quad (12.5)$$

The mean spin of the scattered state v which enters P^v replaces the mean spin of a particle in the considerations regarding the effect of successive scatterings, as may be seen from the discussion of Eqs. (11) to (11.3). The spin of the state v determines the angular dependence of the (v, b) scattering. It is obvious, first of all, that if the (v, b) scattering is changed to be scattering between nonidentical particles, then P^v plays the role of P^1 since it does not matter how the proton beam has been produced for incidence on neutrons b. If next, in the (v, b) scattering, both particles are protons, it proves useful to supplement the considerations of Wolfenstein and Ashkin[4] by the following.

The statistical mixture and the scattering matrix of the nonidentical-particle problem may be referred to the functions $\chi_0{}^0$, χ_1, χ_0, χ_{-1} instead of the set $\alpha_1\alpha_2$, $\alpha_1\beta_2$, $\alpha_2\beta_1$, $\beta_1\beta_2$. The scattering matrix then has 6 vanishing nondiagonal elements and breaks up into the single element s_{00} and the three-row square matrix for the triplet state. The mathematical statement of the absence of polarization of target particle 2 is simpler for the strong-field spin functions (decoupled spins) but is not very involved for the weak-field functions, the form of the density matrix being derivable for one case in terms of the other. One can write down therefore the formula for the scattered intensity in an assigned direction. With the usual reference system of Pauli's σ's, the expression for the intensity is of the form

$$\langle|s_{00}|^2\rangle_{\mathrm{Av}}{}^1\rho_{00}+\mathrm{Tr}(S^3\rho S^\dagger)$$
$$=|s_{00}|^2\,{}^1\rho_{00}+\tfrac{1}{2}\sum_\mu\{\tfrac{1}{2}(e_{\alpha\alpha}+e_{\beta\beta})\times[|S_{\mu,1}|^2+|S_{\mu,0}|^2+|S_{\mu,-1}|^2]+\tfrac{1}{2}(e_{\alpha\alpha}-e_{\beta\beta})[|S_{\mu,1}|^2-|S_{\mu,-1}|^2]+\text{terms in } e_{\alpha\beta},\, e_{\beta\alpha}\}. \quad (13)$$

Here $e_{\alpha\alpha}$, $e_{\alpha\beta}$, etc., are matrix elements of the spin density matrix of the incident polarized particle. One has

$$\tfrac{1}{2}\langle\sigma_{1z}\rangle=\tfrac{1}{2}(e_{\alpha\alpha}-e_{\beta\beta}), \quad (13.1)$$

where subscript 1 refers to the polarized incident particle. Also, for an unpolarized beam scattered from a target,

$$\langle\tfrac{1}{2}(\sigma_{1z}+\sigma_{2z})\rangle=\sum_\mu[|S_{\mu,1}|^2-|S_{\mu,-1}|^2]. \quad (13.2)$$

It is seen therefore that Eq. (7) of Wolfenstein and Ashkin (Wo-Ash),

$$I=\tfrac{1}{4}\,\mathrm{Tr}(M^\dagger M)+\tfrac{1}{4}\langle\boldsymbol{\sigma}_1\rangle_i\,\mathrm{Tr}(M^\dagger M\boldsymbol{\sigma}_1), \quad [\text{Wo-Ash } (7)]$$

can be stated also in terms of $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2$ in both places where $\boldsymbol{\sigma}_1$ enters their formula. The values of the mean spin of the first particle are seen to enter the formula for the intensity in an assigned direction only through the triplet scattering and may be replaced therefore by an expression involving the symmetric operator $\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2$. The breakup of the scattering matrix into a singlet and triplet part secures the absence of effects of elements of $\boldsymbol{\sigma}_1$ between singlets and triplets. Equation (7) of Wolfenstein and Ashkin can accordingly be stated in terms of the symmetric combinations which are the only meaningful combinations in the case of $p-p$ scattering. This statement is seen to be a consequence of the simple algebraic rearrangement mentioned in connection with Eqs. (13), (13.1), and (13.2) and need not even be based on their more elegant procedure. The parts of the matrices representing $\boldsymbol{\sigma}_1$ and $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2$ with rows and columns belonging to the triplet system are readily seen to be the same. In fact $(\chi_\mu, (\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\chi_\nu)$ is zero as a consequence of the symmetry of the χ_μ to particle interchanges. The relation between the $\langle\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\rangle$ after scattering of unpolarized particles represented by the last factor in Eq. (7) of Wolfenstein and Ashkin has been gone into in the discussion which led to Eq. (12.3), while the term in $\langle\boldsymbol{\sigma}_1\rangle_i$ after being replaced by $\langle(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)/2\rangle_i$ can be evaluated as the expectation value of $\boldsymbol{\sigma}$ for state v as follows from (12.3). The simple manipulations which express the intensity in terms of the mean spin of the incident beam and of the mean spin which would be produced in a scattering experiment are practically independent of whether one is dealing with identical or nonidentical particles. The polarization of the state v replaces the polarization of an identifiable particle and states with $T=0$ drop out, but the form of the expressions remains the same since the manipulations take place for submatrices referring to the triplet system. It is clear therefore that one may use the formula for nonidentical particles, substituting the word "state" for the word "particle" and expressing the scattering in terms of mean spin vectors, and employing the antisymmetrized scattering matrix S^a applying to identical particles. The argument just presented is very similar to that given by Swanson[6] in connection with his Eq. (2), who points out that the singlet scattering does not contribute to the "polarization term." It was nevertheless felt necessary to return to the question once more because the way in which identity of particles affects the treatment for the observed angular distribution does not appear to be contained in his paper. The "exchange nature of the interaction" might have been meant to cover the case of identical particles but the simple actual situation does not appear to have been covered.

One may now return to the calculation of P. For nonidentical particles there are Eqs. (12) and (12.1); for identical particles, Eqs. (12.2) and (12.5). Employing the same normalization of the wave function as in I and comparing formulas for $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^a$ in Eqs. (5.1), (6.1) of I, one notes that in expressing P in terms of $\boldsymbol{\sigma}$ or $\boldsymbol{\sigma}^a$ there is an extra factor of 2 entering the expression for P_{p-p}. It is connected with counting recoils in $\boldsymbol{\sigma}^a$. There is, besides, another factor 2 occurring because with the normalizations used the triplet part of S^a is $2/\sqrt{2}$ times the triplet part of S and because the S^a and S are contained quadratically in the numerators of formulas for P and P^a, where P^a is the polarization for the antisymmetrized state. When one expresses $P^a=P_{p-p}$ in terms of $\boldsymbol{\sigma}^a$ and S in P_{p-n} in terms of $\boldsymbol{\sigma}$ and S, there is a net extra factor of 4 present in the formula for P_{p-p}. This factor gives therefore the ratio

$$(P\boldsymbol{\sigma})_{p-p}/(P\boldsymbol{\sigma})_{p-n}\cong 4, \quad (13.3)$$

if one neglects the Coulomb effects in $p-p$ scattering and arbitrarily disregards the presence of $T=0$ states in the $p-n$ case.

Another way of seeing the relationship between $(P\boldsymbol{\sigma})_{p-p}$ and $(P\boldsymbol{\sigma})_{p-n}$ for vanishing Coulomb interaction is to note that in principle one can tag the two protons so that one of them is red and the other black. This identification can be made even though one uses an antisymmetric function to describe the protons. The Wolfenstein-Ashkin result can then be applied to the red protons. Since

$$\sum S^a{}_{0,\mu}(S^a{}_{1,\mu}-S^a{}_{-1,\mu})^*=2\sum S_{0,\mu}(S_{1,\mu}-S_{-1,\mu})^*,$$

there is a factor 2 in favor of $(\boldsymbol{\sigma}P)_{p-p}$. On the other hand, the expression for $\boldsymbol{\sigma}$ on the basis of the red proton is $\tfrac{1}{2}$ of the usual $\boldsymbol{\sigma}^a$, the latter presupposing that both black and red protons are counted. The Wolfenstein-Ashkin formula applies however to the red protons alone, *i.e.*, to $(P\boldsymbol{\sigma}/2)_{p-p}$. There is therefore a second factor 2 for $(P\boldsymbol{\sigma})_{p-p}$, resulting in a net factor 4. The way just presented tacitly assumes the identity of form in the formula for $P\boldsymbol{\sigma}$ whether one deals with unsymmetrized or antisymmetric functions. The justification for doing so has been given in connection with Eqs. (13), (13.1), and (13.2).

Since the case of identical particles is reducible to that of nonidentical ones as just brought out, it suffices to deal with nonidentical particles. The formula for $P\boldsymbol{\sigma}$ can then be put in a more symmetric form by means of the ξ_1, ξ_2, ξ_3 spin representation of Eqs. (7) to (8.1) of I. The transformation of the sum occurring in the numerator of Eq. (12) involves only the substitution of the $S_{\mu\nu}$ in terms of the $S^{\xi}{}_{ij}$ and yields

$$\sum S_{0,\mu}(S_{1,\mu}-S_{-1,\mu})^*=-\sqrt{2}\sum_j(S^{\xi}{}_{1,j})^*S^{\xi}{}_{3,j}, \quad (14)$$

resulting in

$$\langle 2s^1{}_y\rangle=\langle\tfrac{1}{2}(\sigma^1{}_y+\sigma^2{}_y)\rangle=-2\,\mathrm{Im}\sum_j(S^{\xi}{}_{1,j})^*S_{3,j}/[|s_{00}|^2+\mathrm{Tr}(S^{\xi\dagger}S^{\xi})], \quad (14.1)$$

the sufficiency of considerations with the triplet system being used. The simplicity of the form on the right side of Eq. (14) suggests that a direct derivation should lead to the answer more readily. In fact one may first note that the matrix for $2s^1{}_y$ in the ξ_1, ξ_2, ξ_3 representation is

$$(s_y)^{\xi}=2(s^1{}_y)^{\xi}=\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix}, \quad (15)$$

the usual reference system for Pauli's σ's being presupposed. The rows and columns of this matrix are labeled in the order 1, 2, 3 starting in the upper left-hand corner. The absence of the ξ_2, ξ_2 element in this matrix is the result of $s_y\xi_2=0$, which expresses the fact that ξ_2 is a state for which the spin is definitely perpendicular to y. An infinitesimal rotation around y can produce only an admixture of a spin function for which the spin is perpendicular to z if the initial spin condition is definitely perpendicular to x. This means that out of ξ_1 the operator s_y can produce only ξ_3. Similarly an admixture of ξ_1 is all that can result from ξ_3. The presence of the five zeros among the elements of $(s_y)^{\xi}$ is thus clear geometrically. The number i has to be worked out as is readily done. Equation (14.1) follows directly from Eq. (15).

Another way of exhibiting the relationship is to employ

$$S^{\xi}{}_{1,j}=-2^{-\frac{1}{2}}\sum_\mu(S_{1,\mu}-S_{-1,\mu})(\mathfrak{M}^{-1})_{\mu,j},\qquad S^{\xi}{}_{3,j}=\sum_\mu S_{0,\mu}(\mathfrak{M}^{-1})_{\mu,j}, \quad (15.1)$$

which lead directly to (14) on noting the unitary character of $\mathfrak{M}$.

V. FORMS FOR POLARIZATION

Substitution of the amplitudes obtained in Eqs. (2.2) to (2.9) and (4) to (5) of I, together with additions caused by coupling of states with different L but same J as in Eqs. (4) to (6.2) or in alternative form in Eqs. (8) and (9), results in forms of the $\alpha_1, \cdots\alpha_5$ corrected for coupling as follows:

$$\alpha_1=-\sum_L e_{L0}[L(L+2)Q_{L,L+1}-(2L+1)Q_{L,L}-(L^2-1)Q_{L,L-1}]_m P_L'/[L(L+1)]-2\sum_l B_{l+1}[(l+1)P_{l+2}'-(l+2)P_l'], \quad (16)$$

$$\alpha_2=\sum_L \tfrac{1}{2}e_{L0}[(L+2)Q_{L,L+1}+(2L+1)Q_{L,L}+(L-1)Q_{L,L-1}]_m P_L-\sum_l(l+1)(l+2)B_{l+1}(P_{l+2}+P_l), \quad (16.1)$$

$$\alpha_3=\sum_L \tfrac{1}{2}e_{L0}[LQ_{L,L+1}-(2L+1)Q_{L,L}+(L+1)Q_{L,L-1}]_m P_L''/[L(L+1)]-\sum_l B_{l+1}(P_{l+2}''+P_l''), \quad (16.2)$$

$$\alpha_4=\sum_L e_{L0}(Q_{L,L+1}-Q_{L,L-1})_m P_L'-2\sum_l B_{l+1}[(l+1)P_{l+2}'-(l+2)P_l'], \quad (16.3)$$

$$\alpha_5=\sum_L e_{L0}[(L+1)Q_{L,L+1}+LQ_{L,L-1}]_m P_L+2\sum_l B_{l+1}(l+1)(l+2)(P_{l+2}+P_l), \quad (16.4)$$

with B_{l+1} being as in Eq. (7), the subscript $l+1$ indicating that the value of B is taken for coupling between states of angular momentum $j\hbar=(l+1)\hbar$. The specification of j suffices for the identification of the pair of levels being coupled. In the foregoing equations, the subscript m stands for "modified" and indicates that, for any pair

$$(Q_{l,l+1},Q_{l+2,l+1})=(Q(\delta_-),Q(\delta_+)) \quad (17)$$

for which there is a nonvanishing B_{l+1}, there is understood to be a modification of these Q's by replacing them as follows:

$$Q(\delta_-)\rightarrow Q(\delta_-)+C_{--},\quad Q(\delta_+)\rightarrow Q(\delta_+)+C_{++}, \quad (17.1)$$

the Q's being in fact meaningless in the case of coupling. The modification is conveniently represented by means of Eq. (1.7) according to which

$$Q(\delta_-)\rightarrow[Q(\varphi_a)+i\tau_a\tau_b c^2\cos\varphi_a\cos\varphi_b]/[1+\tau_a\tau_b c^2\cos\varphi_a\cos\varphi_b]. \quad (17.2)$$

In this form φ_a is a kind of unmodified δ_-. The above mentioned replacements mean that the $Q(\delta_-)$, $Q(\delta_+)$ are replaced by corresponding diagonal elements of $\mathcal{T}$.

There is a chance of misunderstanding regarding the meaning of the $\alpha_1, \cdots, \alpha_5$ as introduced in Eqs. (2.2) through (2.6) of I. They are meant to be defined in terms of the $S_{\mu\nu}$ and the S^c rather than in terms of the Q's and P_L. For this reason the modifications for coupling are applied in Eqs. (16) through (16.4) to the expressions involving the Q's and P_L.

In evaluating $(P\boldsymbol{\sigma})_{p-n}$, one may set $S^c=0$ and one finds then:

$$k^2(P\boldsymbol{\sigma})_{p-n}=\tfrac{1}{2}\sin\theta\cos\varphi\,\mathrm{Im}\{\alpha_1\alpha_2{}^* -\alpha_1\alpha_3{}^*\sin^2\theta+\alpha_5\alpha_4{}^*\}. \quad (18)$$

For $p-p$ scattering it is necessary to return briefly to the comparison of Eq. (12) with Eq. (12.2). In the quantities determining $(P\boldsymbol{\sigma})_{p-p}$, there enters S^a rather than S. It has already been brought out that with neglect of Coulomb scattering one should insert a factor 4 on the right of Eq. (18). This occurred through a combination of two factors 2. The same result can be obtained working with the antisymmetrized wave function without the factor $1/\sqrt{2}$. The thus modified S^a, $S^{a'}$ is $2^{\frac{1}{2}}$ times the original one. The relation between $\boldsymbol{\sigma}$ and S is the same as between $\boldsymbol{\sigma}^a$ and the modified S^a. For odd L it reproduces the terms of S except that they are multiplied by 2. The quantity to be used with $4\sum(S_{1,\mu}-S_{-1,\mu})^*S_{0,\mu}$ in place of $S^c(12)$ of the nonidentical particle calculation is $S^c(12)-S^c(21)$. Since S^c occurs with only one of the $S_{\mu\nu}$, the modification for antisymmetry needed on S^c consists therefore in replacing in Eqs. (2.2), (2.6) of I the combinations $S_{\mu\mu}-S^c$ by $\frac{1}{2}[S^{a'}{}_{\mu\mu}-S^c(12)+S^c(21)]$, which means that the combination occurring in the formulas for $S^{a'}{}_{\mu\mu}$ are $\alpha_2+\frac{1}{2}e^{-i\Phi}[S^c(12)-S^c(21)]k$ and similarly for α_5. The above combination is

$$\alpha_2+\alpha_c,$$

with

$$\alpha_c=\tfrac{1}{4}\eta[-\mathbf{s}^{-2}\exp(-i\eta\ln\mathbf{s}^2)+\mathbf{c}^{-2}\exp(-i\eta\ln\mathbf{c}^2)]. \quad (19)$$

The same result can be obtained by calculating S^a directly and substituting in Eq. (12.2) or Eq. (12.5). It is thus found that

$$k^2(P\boldsymbol{\sigma})_{p-p}=2\sin\theta\cos\varphi\,\mathrm{Im}\{\alpha_1(\alpha_2+\alpha_c)^* -\alpha_1\alpha_3{}^*\sin^2\theta+(\alpha_5+\alpha_c)\alpha_4{}^*\}. \quad (20)$$

In these equations, the quantity $P\boldsymbol{\sigma}$ is defined in terms of s_y in a scattering experiment from unpolarized particles, the direction of incidence being the positive z axis and the direction of scattering corresponding to colatitude angle θ, azimuthal angle φ with x axis corresponding to $\varphi=0$ and y axis to $\varphi=\pi/2$.

Employment of Eqs. (18) and (20) is appreciably simpler than that of the forms in Eqs. (12) and (12.5), the only algebraic manipulations consisting in grouping terms with different L, J and taking the imaginary part. A helpful formula in these manipulations is

$$\mathrm{Im}\{Q(\delta_2)Q^*(\delta_1)\}=\sin\delta_2\sin\delta_1\sin(\delta_2-\delta_1)\equiv(\delta_2,\delta_1), \quad (20.1)$$

the symbol on the right being the abbreviation of Goldfarb and Feldman except for the substitution of ( ) for the [ ]. The latter are easily confused with brackets indicating commutators while Poisson brackets of classical dynamics are not likely to occur in the same context. If the values of $P\boldsymbol{\sigma}$ at a few angles are the only object the amplitudes can be evaluated numerically without the expansion in terms of the (δ_2, δ_1).

The evaluation of the right side of Eqs. (18) and (20) is thus quite mechanical for the terms which do not involve coupling between states of the same J but different L. If such is the case the contributions caused by the diagonal elements of $\mathcal{T}$ do not introduce much complication, involving no more than the replacement of the Q by the diagonal elements of $\mathcal{T}$ as in Eqs. (1.7), (2) above. The contribution to $k^2(P\boldsymbol{\sigma})_{p-n}$ caused by the nondiagonal element of $\mathcal{T}$ for one coupled state is

$$\begin{aligned}\Delta_B[k^2(P\boldsymbol{\sigma})_{p-n}]
&=\tfrac{1}{2}\sin\theta\cos\varphi\,\mathrm{Im}\Big\{B_{l+1}{}^*\Big\{-U\textstyle\sum_L\Big[\frac{L}{L+1}Q_{L,L+1}+\frac{2L+1}{L(L+1)}Q_{L,L}-\frac{L+1}{L}Q_{L,L-1}\Big]P_L'\\
&\quad-2W\textstyle\sum_L\Big[\frac{1}{L+1}Q_{L,L+1}-\frac{2L+1}{L(L+1)}Q_{L,L}+\frac{1}{L}Q_{L,L-1}\Big]\cos\theta P_L'\\
&\quad-(\sin^2\theta)V\textstyle\sum_L\Big[\frac{L+2}{L+1}Q_{L,L+1}-\frac{2L+1}{L(L+1)}Q_{L,L}-\frac{L-1}{L}Q_{L,L-1}\Big]P_L'\Big\}\Big\}\\
&=\sin\theta\cos\varphi P_{l+1}'\,\mathrm{Im}\{B_{l+1}{}^*\Xi\}, \quad (20.2)\end{aligned}$$

where

$$\Xi=\sum_L\Big[-\frac{2L+3}{L+1}Q_{L,L+1}+\frac{2L+1}{L(L+1)}Q_{L,L}+\frac{2L-1}{L}Q_{L,L-1}\Big]_m P_L', \quad (20.3)$$

and where the total angular momentum of the coupled state is $l+1$, while U, V, W are as in Eqs. (8.1), (8.2), (8.3). The second of the two forms in Eq. (20.2) is obtained by employing recurrence relations so as to express the derivatives of P_l, P_{l+2} in terms of P_{l+2}, P_l, P_{l-2}, $\cdots$. The highest-order terms then cancel out and the remaining terms form the P_{l+1}' in Eq. (20.2). The cancellation of the highest-order terms is a helpful simplification. Specializing to ${}^3S_1$, ${}^3D_1$ coupling reduces the expression to

$$\Delta_B[k^2(P\boldsymbol{\sigma})_{p-n}]=\sin\theta\cos\varphi\,\mathrm{Im}\{B_1{}^*\Xi\}, \quad ({}^3S_1, {}^3D_1). \quad (21)$$

Specializing further to the case of ${}^3S_1$, ${}^3D_1$ being the only important phase shifts,

$$\Delta_B[k^2(P\boldsymbol{\sigma})_{p-n}] =(9/2)\sin\theta\cos\theta\cos\varphi\,\mathrm{Im}\{B_1{}^*Q_{2,1}\}. \quad (21.1)$$

The dependence of these terms on θ and φ is seen to be the same as though there were only p term effects present. The effect of the diagonal terms in case ${}^3S_1$, ${}^3D_1$ are the only phase shifts has the same angular dependence. Combining its effect with that of Eq. (21.1) one has

$$[k^2(P\boldsymbol{\sigma})_{p-n}]({}^3S_1, {}^3D_1)$$
$$= (9/4) \sin\theta \cos\theta \cos\varphi \, \mathrm{Im}\{Q_{2,1}Q_{0,1}^* + 2B_1^*Q_{2,1}\}. \quad (21.2)$$

The presence of coupling between ${}^3S_1$ and ${}^3D_1$ does not affect the type of angular distribution which exists in the presence of ${}^3S_1$, ${}^3D_1$ phase shifts. In Eq. (21.2), it is understood that the terms $Q_{2,1}$, $Q_{0,1}$ may be modified by coupling in the sense of Eqs. (1.5), (1.7), and (2). It may be noted that the first form of Eq. (20.2) contains terms in $\sin\theta P_2(\cos\theta)P_L'(\cos\theta)$ which all cancel, so that only terms in $\sin\theta P_L'(\cos\theta)$ survive in Eq. (21), or the equivalent second form.

Taking ${}^3P_2$, ${}^3F_2$ as the only states with coupling and again starting with Eq. (20.2), one obtains

$$\Delta_B[k^2(P\boldsymbol{\sigma})_{p-n}]$$
$$= 3 \sin\theta \cos\theta \cos\varphi \, \mathrm{Im}\{B_2^*\Xi\}, \quad ({}^3P_2, {}^3F_2). \quad (22)$$

Here the $\cos\theta$ in front of the Im sign arose as $P_1(\cos\theta)$ which occurred alongside with terms in $P_3(\cos\theta)$. The latter canceled out similarly to the disappearance of terms in P_2 in Eq. (21). In both cases the coupling to terms of a higher L, $l+2$, does not introduce in the cross terms with the $Q_{l,j}$ an angular dependence not contained in the terms involving the $Q_{l,j}$ alone. The first-order effects of the nondiagonal elements of the coupling matrix are thus not introducing higher orders of Legendre functions, except through combinations with Q's which arise in addition to the $Q_{l,j}$, as is the case for example for $Q_{l+2,l+1}$, $Q_{l+2,l+2}$.

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Theory of Polarization of Nucleons Scattered Elastically by Nuclei

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A study of the nucleon polarization to be expected when nucleons are elastically scattered from nuclei is presented. The polarization effect is a consequence of the fact that the nucleon-nucleus interaction may be represented as a complex spin-dependent potential. The existence of such a potential is suggested by the nuclear shell model and the spin dependence of the nucleon-nucleon interaction. Qualitative arguments are advanced to determine this potential in terms of the nucleon-nucleon interaction. Although the polarization effect is by no means confined to elastic scattering, it is in this case particularly useful, since the large diffraction cross sections observed experimentally insure relatively high yields of polarized particles. A number of theoretical studies have been carried out, for both neutron and proton scattering, which show that almost full polarization can occur. The calculations have been carried out by using the W.K.B. approximation as usually applied to the nuclear optical model. The method has been checked by carrying out an exact phase shift analysis for a particular case. The results show that studies of nucleon polarization can illuminate some aspects of nuclear structure, since the polarization depends on the particular nucleus used as a target as well as upon the form of the interaction.

I. INTRODUCTION

THE existence of a nucleon-nucleus spin-dependent interaction is suggested by the fact that the nucleon-nucleon potential is itself spin-dependent[1]; moreover, such an interaction is an essential feature of the nuclear shell model.

Such an interaction should manifest itself in a polarization of nucleons scattered by nuclei.[2] Although the polarization effect is by no means confined to the case of elastic scattering, this process is particularly interesting and useful since the large diffraction cross sections found experimentally insure a relatively high yield of polarized particles.

The elastic scattering of nucleons by nuclei can be described by treating the nucleon-nucleus interaction as a complex potential.[3] The imaginary part of the complex potential represents the effect of all processes not leading to elastic scattering. If, in addition to a complex central potential, there exists a spin-dependent potential, the elastically scattered nucleons will be polarized.

For low-energy nucleon scattering one may expect that the polarization will reflect the characteristics of the spin-orbit potential of the shell model, but at high energies it is certainly more sensible to expect that the nucleon-nucleon potential is directly effective[4] since the incident particle can then "see" individual nucleons in the nucleus.

[1] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).
[2] E. Fermi, Nuovo cimento **11**, 407 (1954); W. Heckrotte and J. V. Lepore, Phys. Rev. **94**, 500 (1954); B. J. Malenka, Phys. Rev. **95**, 620 (1954); Snow, Sternheimer, and Yang, Phys. Rev. **94**, 1073 (1954); R. H. Sternheimer, Phys. Rev. **95**, 587 (1954).
[3] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).
[4] R. Serber, Phys. Rev. **72**, 1114 (1947).