The interesting question which remains is: Wherein lies the discrepancy among the different experiments designed to study this problem? Yagoda and de Carvalho[5] suggested that the experiments which show an anomaly are those in which the alpha particles have had to penetrate a gas-liquid interface, so that the interface might be responsible. Since this experiment requires the penetration of two interfaces, it would seem that another explanation must be found.

The authors wish to acknowledge the courtesy of Drs. Ross, Shedlovsky, and Miles of the Colgate-Palmolive-Peet Company who made many helpful suggestions concerning water films and who supplied the detergent used.

PHYSICAL REVIEW VOLUME 97, NUMBER 4 FEBRUARY 15, 1955

Amplitudes in Nucleon-Nucleon Scattering*

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(Received October 11, 1954)

When scattering anomalies involving many phase shifts are to be studied, it appears desirable to treat the amplitudes directly before combining them into differential cross sections. Amplitudes suitable for the study of elastic collisions of charged and uncharged Fermi-Dirac particles of spin $\frac{1}{2}$, taking account of possible identity are, therefore, given in forms convenient for computation. The case of coupling between states of the same total but different orbital angular momentum is not discussed. Formulas using the spin functions usually denoted by χ_m are supplemented by forms based on spin functions which transform like the components of an ordinary space vector, the latter allowing more compact expressions in some cases.

I. INTRODUCTION

THE calculation of scattering of protons by protons and neutrons by protons has been the subject of many investigations. Recent experimental work in the region of several hundred Mev has made it desirable to be able to deal with scattering anomalies caused by many phase shifts. The calculations have been systematized therefore to a greater extent than has been done previously. The present paper is restricted to a nonrelativistic treatment and the introduction of coupling between states with the same total angular momentum $J\hbar$ but different orbital angular momenta $L\hbar$ is postponed to a succeeding and closely related one.

The treatment presupposes that either all collisions are elastic or else that the cross sections for inelastic collisions are so small that their damping effect may be neglected. In the approximations of this paper, therefore, the phase shifts may be taken to be real.[1] Some of the mathematical forms worked with are very similar to those of Ashkin and Wu[2] for complex phase shifts. Both in the present as well as the succeeding paper it has been found convenient to make use of the fact that the triplet spin behaves under rotations like an ordinary space vector.[3] The corresponding spin functions are denoted by ξ_1, ξ_2, ξ_3. Many formulas are more convenient in terms of amplitudes referred to these variables.

The set of angular and spin functions used is presented in Eqs. (1) to (1.4). The amplitudes are introduced in Eqs. (2), (2.1). Formulas for amplitudes for nonidentical particles and referred to magnetic quantum numbers are available in Eqs. (2.2) through (2.9). Effects of antisymmetry are introduced in Eqs. (3) through (3.3) and the modified results are collected in Eqs. (4) through (4.2). The relation to differential cross sections for unpolarized particles is as in Eqs. (5.1), (6.1). The ξ_1, ξ_2, ξ_3 modifications start with Eq. (7) with the scattering matrix S^ξ as in Eq. (7.5), cross sections as in Eqs. (8), (8.1).

II. NOTATION

α, β = nucleon spin function for states with magnetic quantum number $\frac{1}{2}$, $-\frac{1}{2}$, respectively.

$\chi_1, \chi_0, \chi_{-1}$ = triplet spin functions for two nucleons; $\chi_1=\alpha_1\alpha_2$, $\chi_0=(\alpha_1\beta_2+\alpha_2\beta_1)/2^{\frac{1}{2}}$, $\chi_{-1}=\beta_1\beta_2$.

$\chi_0{}^0=(\alpha_1\beta_2-\alpha_2\beta_1)/2^{\frac{1}{2}}$ singlet spin function for two nucleons.

r = distance between nucleons.

v = relative velocity.

M = nucleon mass.

$k=Mv/(2\hbar)$.

$$Y_{Lm}=\frac{(-)^m}{2^L L!}\left[\frac{(2L+1)(L-m)!}{4\pi(L+m)!}\right]^{\frac{1}{2}} \times e^{im\varphi}\sin^m\theta\left(\frac{d}{d\cos\theta}\right)^{L+m}(\cos^2\theta-1)^L.$$

θ = colatitude angle in polar coordinates = scattering angle.

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* This research was supported by the Office of Ordnance Research, U. S. Army.

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φ = azimuthal angle with x-axis corresponding to $\varphi=0$, y-axis to $\varphi=\pi/2$.
$\rho=kr$.
$\eta=e^2/(\hbar v)$.
$\psi_u{}^c$ = unsymmetrized Coulomb wave arising from a plane wave by adiabatic effect of e.
$\mathbf{s}, \mathbf{c}=\sin(\theta/2), \cos(\theta/2)$, respectively.
$Q(\delta)=(e^{2i\delta}-1)/(2i)$.
$\sigma_0=\arg\Gamma(1+i\eta)$.
$\sigma_{L,0}=\sigma_L-\sigma_0=\tan^{-1}(\eta/L)+\tan^{-1}[\eta/(L-1)]+\cdots+\tan^{-1}(\eta/1)$.
$e_{L0}=\exp(2i\sigma_{L0})$.
$\Phi=\rho-\eta\ln 2\rho+2\sigma_0$.
$\hbar L$ = orbital angular momentum.
$\hbar J$ = total angular momentum.
ξ_1, ξ_2, ξ_3 = triplet spin functions for two nucleons which form the components of a vector in ordinary space; $\xi_1=2^{-\frac{1}{2}}(\chi_{-1}-\chi_1)$, $\xi_2=i2^{-\frac{1}{2}}(\chi_1+\chi_{-1})$, $\xi_3=\chi_0$.
$S_{\mu m}, S_{\mu m}{}^a$ = components of the triplet scattering matrix referred to the functions χ_m, corresponding to unsymmetrized and antisymmetrized wave functions, respectively. The incident state is labeled by the second subscript, the final by the first.
S^c, S^{ac} = triplet Coulomb amplitudes.
$S^\xi, S^{a\xi}$ = triplet scattering matrix referred to the functions ξ_j.
$s_{00}, s_{00}{}^a$ = singlet scattering matrix elements.
$\boldsymbol{\sigma}, \boldsymbol{\sigma}^a$ = elastic differential scattering cross section for nonidentical and identical unpolarized particles, respectively.

III. REAL PHASE SHIFTS FOR DEFINITE L

Compounding spin and orbital angular momenta for the two nucleons, one obtains spin-angular functions

$$\mathcal{Y}^{L,J}{}_\mu=\sum_m\begin{pmatrix} L, J \\ \mu-m, m\end{pmatrix} Y_{L,\mu-m}\chi_m, \qquad (1)$$

with values of transformation coefficients

$$(2L+1)^{\frac{1}{2}}(2L+2)^{\frac{1}{2}}\begin{pmatrix} L, L+1 \\ \mu-m, m\end{pmatrix}$$
$$=\{[(L+\mu)(L+\mu+1)]^{\frac{1}{2}},\; 2^{\frac{1}{2}}[(L+\mu+1)(L-\mu+1)]^{\frac{1}{2}},\; [(L-\mu)(L-\mu+1)]^{\frac{1}{2}}\}, \qquad (1.1)$$

$$[2L(L+1)]^{\frac{1}{2}}\begin{pmatrix} L, L \\ \mu-m, m\end{pmatrix}$$
$$=\{[(L+\mu)(L-\mu+1)]^{\frac{1}{2}},\; -2^{\frac{1}{2}}\mu,\; -[(L-\mu)(L+\mu+1)]^{\frac{1}{2}}\}, \qquad (1.2)$$

$$[2L(2L+1)]^{\frac{1}{2}}\begin{pmatrix} L, L-1 \\ \mu-m, m\end{pmatrix}$$
$$=\{[(L-\mu)(L-\mu+1)]^{\frac{1}{2}},\; -2^{\frac{1}{2}}[(L+\mu)(L-\mu)]^{\frac{1}{2}},\; [(L+\mu)(L+\mu+1)]^{\frac{1}{2}}\}, \qquad (1.3)$$

the three numbers on the right corresponding to $m=1, 0, -1$, respectively. While these transformation coefficients are well known and have been given by many authors, the freedom of the choice of phase for each of the angular-spin functions makes it necessary to list the coefficients explicitly, as above, so as to avoid misunderstandings. The unsymmetrized Coulomb wave $\psi_u{}^c$ becomes modified by the presence of real phase shifts $\delta_{L,J}$ so that the scattered wave is given by

$$\rho(\psi^c{}_u\chi_m)_{Sc}\exp\{-i[\rho-\eta\ln 2\rho+2\sigma_0]\}$$
$$=-[\eta/(2\mathbf{s}^2)]\chi_m\exp\{-i\eta\ln\mathbf{s}^2\}+\sum_{\mu,J,L}\begin{pmatrix} L, J \\ 0, m\end{pmatrix}$$
$$\times\begin{pmatrix} L, J \\ m-\mu, \mu\end{pmatrix}Q(\delta_{L,J})Y_{L,m-\mu}\chi_\mu\exp(2i\sigma_{L,0}). \qquad (1.4)$$

The symbols in the formula are as in the list of notation. The quantity

$$\sigma_{L,0}=\sigma_L-\sigma_0,$$

where σ_L is the inherent Coulomb phase shift so that

$$\sigma_{L,0}=\tan^{-1}(\eta/L)+\tan^{-1}[\eta/(L-1)]+\cdots+\tan^{-1}\eta.$$

The employment of the transformation coefficients as in (1.1), (1.2), (1.3) gives now after a straightforward calculation the scattered wave which results if the incident wave is the coulombian modification of

$$\psi^0=\textstyle\sum_\mu a_\mu\chi_\mu e^{ikz}; \qquad (2)$$

i.e.,

$$\psi^{0c}=\textstyle\sum a_\mu\chi_\mu\psi^c, \qquad (2')$$

where, asymptotically at large distances,

$$\psi^c\sim\{1+\eta^2/[ik(r-z)]+\cdots\}\times\exp\{i[kz+\eta\ln k(r-z)]\}$$
$$-\{\eta/[k(r-z)]\}\{1+(1+i\eta)^2/[ik(r-z)]+\cdots\}\times\exp\{i[kr-\eta\ln k(r-z)+2\sigma_0]\}. \qquad (2'')$$

The triplet scattered wave is then

$${}^3\psi_{Sc}=\textstyle\sum_{\mu,m}\chi_\mu S_{\mu,m}a_m/r, \qquad (2.1)$$

where the $S_{\mu m}$ are as follows:

$$k(S_{1,1}-S^c)=k(S_{-1,-1}-S^c)=\alpha_2 e^{i\Phi}$$
$$=e^{i\Phi}\textstyle\sum_L \frac{1}{2}e_{L0}[(L+2)Q_{L,L+1}+(2L+1)Q_{L,L}+(L-1)Q_{L,L-1}]P_L, \qquad (2.2)$$

$$-kS_{0,1}e^{-i\varphi}=kS_{0,-1}e^{i\varphi}=-2^{-\frac{1}{2}}\alpha_1\sin\theta e^{i\Phi}$$
$$=2^{-\frac{1}{2}}e^{i\Phi}\textstyle\sum_L e_{L0}[L(L+2)Q_{L,L+1}-(2L+1)Q_{L,L}-(L^2-1)Q_{L,L-1}]\times\sin\theta P_L'/[L(L+1)], \qquad (2.3)$$

$$kS_{-1,1}e^{-2i\varphi}=kS_{1,-1}e^{2i\varphi}=\alpha_3\sin^2\theta e^{i\Phi}$$
$$=\tfrac{1}{2}e^{i\Phi}\textstyle\sum_L e_{L0}[LQ_{L,L+1}-(2L+1)Q_{L,L}+(L+1)Q_{L,L-1}]\times\sin^2\theta P_L''/[L(L+1)], \qquad (2.4)$$

$$kS_{1,0}e^{i\varphi}=-kS_{-1,0}e^{-i\varphi}$$
$$=2^{-\frac{1}{2}}\alpha_4 \sin\theta e^{i\Phi}=2^{-\frac{1}{2}}e^{i\Phi}$$
$$\times\sum_L e_{L0}(Q_{L,L+1}-Q_{L,L-1})\sin\theta P_L', \quad (2.5)$$

$$k(S_{0,0}-S^c)=\alpha_5 e^{i\Phi}=e^{i\Phi}\sum_L e_{L0}$$
$$\times[(L+1)Q_{L,L+1}+LQ_{L,L-1}]P_L, \quad (2.6)$$

where

$$e_{L0}=\exp(2i\sigma_{L,0}) \quad (2.7)$$

and

$$\Phi=\rho-\eta\ln 2\rho+2\sigma_0, \quad (2.8)$$

with σ_0, the Coulomb phase shift for $L=0$, defined in the list of notation. The quantity S^c represents Rutherford scattering and may be expressed as

$$S^c=-[\eta/(2k\mathbf{s}^2)]\exp[i(\Phi-\eta\ln\mathbf{s}^2)], \quad (2.9)$$

the superscript c standing for Coulomb. Equation (2.1) is directly applicable to $n-p$ scattering by setting $\eta=0$. For $p-p$ scattering the wave function must be antisymmetric in the two particles. This requirement can be satisfied in the usual manner by interchanging space as well as spin coordinates in the wave function, subtracting from the original and dividing by $\sqrt{2}$. The normalization of particle density of the incident wave is unaltered by this change in the wave function provided one deals with the probability of finding *a* proton rather than the first or the second proton. A precise visualization of the conditions can be obtained by the construction of wave packets. For noninteracting particles the wave function

$$\Psi_a{}^0=2^{-\frac{1}{2}}[\psi_{\rm I}(\mathbf{r}_1,s_1)\psi_{\rm II}(\mathbf{r}_2,s_2)-\psi_{\rm I}(\mathbf{r}_2,s_2)\psi_{\rm II}(\mathbf{r}_1,s_1)] \quad (3)$$

represents two particles moving toward each other in coordinate space provided the regions of space occupied by $\psi_{\rm I}$ and $\psi_{\rm II}$ move toward each other and do not overlap. Making

$$\int\sum_s|\psi_{\rm I}(\mathbf{r},s)|^2d\mathbf{r}=\int\sum_s|\psi_{\rm II}(\mathbf{r},s)|^2d\mathbf{r}=1$$

secures this normalization, provided one adds the probabilities of finding particles 1 and 2 in the regions of configuration space I, II occupied by $\psi_{\rm I}$ and $\psi_{\rm II}$, respectively. For nonidentical particles the wave function

$$\Psi_u{}^0=\psi_{\rm I}(\mathbf{r}_1,s_1)\psi_{\rm II}(\mathbf{r}_2,s_2) \quad (3.1)$$

describes the condition of having particle 1 in I and particle 2 in II, so that the intensities of the two beams are the same for Eq. (3) and for Eq. (3.1). The functions in (3) and (3.1) are distinguished by subscripts a and u so as to indicate antisymmetric and unsymmetrized functions respectively. If the $\Psi_u{}^0$ is modified by the inclusion of effects of an interaction between the particles one can follow the scattering of the wave packets I and II by each other. The Fourier analysis of $\Psi_u{}^0$ in terms of plane waves gives terms, involving

$$\exp\{i[\mathbf{k}_1\mathbf{r}_1+\mathbf{k}_2\mathbf{r}_2]\}$$

as factors. Each of these can be rearranged by means of

$$\mathbf{k}_1\mathbf{r}_1+\mathbf{k}_2\mathbf{r}_2=(\mathbf{k}_1+\mathbf{k}_2)[(\mathbf{r}_1+\mathbf{r}_2)/2]+[(\mathbf{k}_1-\mathbf{k}_2)/2](\mathbf{r}_1-\mathbf{r}_2)$$

so as to contain either the coordinates of the center of mass or the relative coordinates, $\mathbf{r}_1-\mathbf{r}_2$. The usual consideration of the scattering problem is concerned with the relative coordinates factor because it secures the formation of the wave packets of I and II receding from each other. The outgoing wave modification of each Fourier term, when substituted into the Fourier integral, gives the unsymmetrized function

$$\Psi_u(\mathbf{r}_1,s_1;\,\mathbf{r}_2,s_2). \quad (3.2)$$

The replacement of each term in the Fourier analysis by an antisymmetric combination including the factor $2^{-\frac{1}{2}}$, as in Eq. (3), gives the corresponding

$$\Psi_a(\mathbf{r}_1,s_1;\,\mathbf{r}_2,s_2)$$
$$=2^{-\frac{1}{2}}[\Psi_u(\mathbf{r}_1,s_1;\,\mathbf{r}_2,s_2)-\Psi_u(\mathbf{r}_2,s_2;\,\mathbf{r}_1,s_1)]. \quad (3.3)$$

The scattered parts of Ψ_u and Ψ_a give rise to wave packets of correlated probability for which protons 1 and 2 move in correlated directions determined by the conservation of energy and momentum. In making use of the scattered part of (3.3) it is natural and customary to calculate for any given direction and spin orientation the chance that proton 1 is moving in that direction, while 2 is moving in the correlated direction, and to add to this probability that obtained by interchanging the roles of protons 1 and 2. This procedure thus automatically includes particles which in the case of (3.2) would be called recoils. The cross section for scattering may then be pictured as the sum of a direct scattering term, a term representing recoils, and a Mott-type interference term. The modifications in the scattering matrix corresponding to the change from Ψ_u to Ψ_a are now, for the triplet amplitudes,

$$S^{ac}=2^{-\frac{1}{2}}(\eta/2k)[-\mathbf{s}^{-2}\exp(-i\eta\ln\mathbf{s}^2)$$
$$+\mathbf{c}^{-2}\exp(-i\eta\ln\mathbf{c}^2)]e^{i\Phi}, \quad (4)$$

$$S^a{}_{\mu,\nu}=2^{\frac{1}{2}}S_{\mu,\nu}, \quad (\mu\neq\nu) \quad (4.1)$$

$$S^a{}_{\mu\mu}-S^{ac}=2^{\frac{1}{2}}(S_{\mu\mu}-S^c), \quad (4.2)$$

an extra factor 2 in Eqs. (4.1), (4.2) arising from the fact that since all the L's are odd in this case the space functions are of negative parity.

For nonidentical particles the scattering matrix for singlet states is

$$k(s_{00}-S^c)=e^{i\Phi}\sum_L(2L+1)e_{L0}P_LQ_L. \quad (5)$$

For unpolarized nonidentical particles the differential collision cross section $\boldsymbol{\sigma}$ is obtained from

$$4\boldsymbol{\sigma}=|s_{00}|^2+\sum_{\mu,\nu}|S_{\mu\nu}|^2. \quad (5.1)$$

For identical particles the scattering matrix for singlet states is obtainable from

$$s^a{}_{00}-S^{sc}=2^{\frac{1}{2}}(s_{00}-S^c),$$
$$S^{sc}=2^{-\frac{1}{2}}(\eta/2k)[-\mathbf{s}^{-2}\exp(-i\eta\ln\mathbf{s}^2)-\mathbf{c}^{-2}\exp(-i\eta\ln\mathbf{c}^2)]e^{i\Phi}, \quad (6)$$

and the differential cross section for unpolarized protons is obtainable from

$$2\boldsymbol{\sigma}^a=|s^a{}_{00}|^2+\sum_{\mu,\nu}|S^a{}_{\mu\nu}|^2. \quad (6.1)$$

This cross section includes the effect of recoils as has been mentioned in connection with Eq. (3.3). The factor on the left of (6.1) is 2 rather than the 4 in the corresponding place in Eq. (5.1) because according to the convention used one needs the probability of finding both protons 1 and 2 rather than just proton 1.

The triplet spin functions χ_1, χ_0, χ_{-1} can be transformed by

$$\xi_1=(\chi_{-1}-\chi_1)/\sqrt{2}, \quad \xi_2=i(\chi_1+\chi_{-1})/\sqrt{2}, \quad \xi_3=\chi_0. \quad (7)$$

The three quantities (ξ_1, ξ_2, ξ_3) transform under rotations like an ordinary vector. The functions ξ_1, ξ_2 represent states in which the total spin is perpendicular to the x and y axes. In terms of the ξ_j the incident wave may be rewritten as

$$\psi^{0c}=\sum_j \xi_j(\mathfrak{M}a)_j\psi^c, \quad (7.1)$$

where

$$\|\mathfrak{M}_{j\mu}\|=\begin{pmatrix} -1/\sqrt{2}, & 0, & 1/\sqrt{2} \\ -i/\sqrt{2}, & 0, & -i/\sqrt{2} \\ 0, & 1, & 0 \end{pmatrix}, \quad (7.2)$$

the rows in ($\mathfrak{M}$) being labeled in the order $j=1, 2, 3$ going down and the columns in the order $\mu=1, 0, -1$ from left to right. The scattered wave for nonidentical particles, referred to in Eq. (2.1), becomes

$$^3\psi_{Sc}=\sum_{i,j}\xi_i(\mathfrak{M}S\mathfrak{M}^{-1})_{ij}a^{\xi}{}_j/r, \quad (7.3)$$

with

$$a^{\xi}{}_j=(\mathfrak{M}a)_j. \quad (7.4)$$

The incident amplitudes a_μ are thus replaced by the amplitudes $a^{\xi}{}_j$ which are the coefficients of the ξ_j in the expression for the incident wave. The matrix $\|S_{\mu\nu}\|$ is replaced by the matrix $\|(\mathfrak{M}S\mathfrak{M}^{-1})_{ij}\|$ as in Eq. (7.3). Substitution of Eqs. (2.2) through (2.6) gives for this matrix:

$$S^{\xi}=\|(\mathfrak{M}S\mathfrak{M}^{-1})_{ij}\|=(e^{i\Phi}/k)\begin{pmatrix} S^{c\prime}+\alpha_2-\alpha_3\cos 2\varphi\sin^2\theta, & -\alpha_3\sin 2\varphi\sin^2\theta, & -\alpha_4\cos\varphi\sin\theta \\ -\alpha_3\sin 2\varphi\sin^2\theta, & S^{c\prime}+\alpha_2+\alpha_3\cos 2\varphi\sin^2\theta, & -\alpha_4\sin\varphi\sin\theta \\ -\alpha_1\cos\varphi\sin\theta, & -\alpha_1\sin\varphi\sin\theta, & S^{c\prime}+\alpha_5 \end{pmatrix}, \quad (7.5)$$

with

$$S^{c\prime}=ke^{-i\Phi}S^c=-(\eta/2\mathbf{s}^2)\exp(-i\eta\ln\mathbf{s}^2); \quad (7.5')$$

the order ξ_1, ξ_2, ξ_3 applying in the labeling of rows downward and in that of columns from left to right. The matrix (7.5) contains the angular dependence in real expressions and is convenient for numerical work. The effect of antisymmetrizing the wave function is to replace the above matrix by

$$S^{a\xi}=\mathfrak{M}S^a\mathfrak{M}^{-1}. \quad (7.6)$$

According to Eqs. (4), (4.1), (4.2) this replacement is accomplished by the replacements in (7.5) of the quantities S^c, α_1, $\cdots$, α_5 by S^{ac} and

$$\alpha^a{}_1=2^{\frac{1}{2}}\alpha_1, \quad \alpha^a{}_2=2^{\frac{1}{2}}\alpha_2, \cdots, \quad \alpha^a{}_5=2^{\frac{1}{2}}\alpha_5. \quad (7.7)$$

The formulas for the cross section in terms of the S^{ξ} type matrix elements are

$$4\boldsymbol{\sigma}=|s_{00}|^2+\sum_{i,j}|S^{\xi}{}_{ij}|^2, \quad (8)$$

$$2\boldsymbol{\sigma}^a=|s^a{}_{00}|^2+\sum_{i,j}|S^{a\xi}{}_{ij}|^2, \quad (8.1)$$

similarly to Eqs. (5.1), (6.1). In all of these formulas the expressions for the matrix elements of S contain sums over L which are taken over all L in the case of nonidentical particles; for identical particles as is well known triplet states occur only with odd, singlet states only with even L. The sums of squares of matrix elements can be expressed as a trace of the square of the unitary matrix S.