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Geometrical Corrections in Angular Correlation Measurements*

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It is shown that given a point-point angular correlation of the form $W(\theta)=\Sigma_l a_lP_l(\cos\theta)$, the experimentally measured angular correlation using finite-sized detectors of arbitrary shape and efficiency distributions is given by $W(\theta)=\Sigma_{ll'mm'} a_lb_{lm}c_{lm'}g_{mm'}{}^{ll'}P_{l'}(\cos\theta)$, where b_{lm} and c_{lm} are the Legendre coefficients describing the efficiency functions of the detectors, and $g_{mm'}{}^{ll'}$ are numerical coefficients. An extensive table of these latter coefficients, sufficient for most applications, is included. The manner in which detector symmetries of various types affect the form of the measured angular correlation is discussed; in particular it is shown that if the efficiency function of both counters is invariant to reflection about both horizontal and vertical axes, the measured angular correlation will contain no P_l's of higher order than the P_l of highest order appearing in the point-point correlation. The above formula for the measured angular correlation is also shown to apply if an axially-extended source instead of a point source is used, the detector coefficients simply being replaced by a new set of suitably averaged coefficients. Tables of correction factors to fourth order in detector and axial source size are included for the special cases of rectangular and circular detectors of constant efficiency.

I. INTRODUCTION

NUCLEAR measurement techniques have in the past few years increased in accuracy and as a result the need for precise means of correcting angular measurements for the finite size of both source and detector has grown. In addition a knowledge of these corrections allows the use of "poor geometry" experiments with a resulting decrease in experiment time but without an attendant loss in measurement accuracy.

The efficiency of detection, E, of the most general detector is both a function of the coordinates (θ,φ), which locate a point on the surface of the detector, and $(\bar{\theta},\bar{\varphi})$ which specify the angle of incidence of the radiation at this point. Gamma-ray detectors of arbitrary dimensions may be described in this manner. The efficiency of charged particle detectors, however, can usually be made to be independent of the angle of incidence of the radiation and are therefore only functions of (θ,φ). Such detectors we shall denote as incidence independent detectors. If we choose to describe a gamma ray detector with a point source of radiation at the center of the coordinate system defining the angles θ, φ we will have $\theta=\bar{\theta}$, $\varphi=\bar{\varphi}$. Thus for incidence-dependent detectors, with this restriction, and for all incidence-independent detectors we may write

$$E(\theta,\varphi)=\sum_{lm}\left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}a_{lm}Y_l{}^m(\theta,\varphi)\qquad\text{(arbitrary detector).}\quad(1)$$

For the special case where the efficiency does not depend on the azimuthal angle, φ,

$$E(\theta)=\sum_{l}\left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}a_{l0}Y_l{}^0(\theta)=\sum_l\frac{2l+1}{4\pi}a_lP_l(\cos\theta)\qquad(\varphi\text{-symmetric detector}).\quad(2)$$

The special case of an incidence independent detector with constant efficiency over the detector surface we denote as the constant efficiency detector. Most charged-particle detectors are of this type.

Walter *et al.*[1] have treated the effects of certain constant-efficiency detectors to a first approximation in the detector solid angle. The φ-symmetric detector has been treated exactly by Frankel[2] in terms of an arbitrary efficiency function. Rose[3] has calculated theoretically the efficiency function to be expected in the case of an unshielded right circular scintillator exposed to gamma rays. His efficiency function formula has been tested experimentally by Klema and McGowan.[4] In many experimental arrangements, however, it is necessary to obtain the efficiency functions of the detector by a direct experimental measurement. Experimental problems that arise in determining $E(\theta)$ for gamma rays have been discussed by Church and Kraushaar,[5] Steffen,[6] and Lawson and Frauenfelder.[7]

In the present paper we shall assume that the efficiency functions of the detectors are known and are of the general form given by Eq. (1). In Sec. II the effect of such detectors, when used with a centered point source of radiation, is calculated exactly. The manner in which detector symmetries of various types affect the form of the measured angular correlation is discussed in Sec. III. In Sec. IV the case of an extended line source of radiation used in conjunction with arbitrary detectors is solved by showing that it is equivalent to the point source situation provided the efficiency function of the detectors are replaced by suitably modified functions. The proper functions to use for the

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[1] Walter, Huber, and Zunti, Helv. Phys. Acta **23**, 697 (1950).
[2] S. Frankel, Phys. Rev. **83**, 673 (1951).
[3] M. E. Rose, Phys. Rev. **91**, 610 (1953).
[4] E. D. Klema and F. K. McGowan, Phys. Rev. **92**, 1469 (1953).
[5] E. L. Church and J. J. Kraushaar, Phys. Rev. **88**, 419 (1952).
[6] R. Steffen, Phys. Rev. **91**, 443 (1953).
[7] J. S. Lawson, Jr. and H. Frauenfelder, Phys. Rev. **91**, 649 (1953).

two important cases of circular and rectangular detectors of constant efficiency are then found explicitly. These latter results are an extension of the results previously obtained by Walter *et al.*[1] and by Frauenfelder.[8]

II. ARBITRARY DETECTORS WITH POINT SOURCE

We consider two finite arbitrary detectors 1 and 2 where detector 1 is sensitive to radiation b and detector 2 is sensitive to radiation c. (The requirement that each counter be sensitive to only one radiation is easily removed by proper weighting.[7])

We wish to find the experimental angular correlation,

$$W(\theta)=\sum_l \frac{2l+1}{4\pi} h_l P_l(\cos\theta), \tag{3}$$

when the true point-point correlation is

$$W(\theta')=\sum_l \frac{2l+1}{4\pi} a_l P_l(\cos\theta'). \tag{4}$$

We assume the efficiency functions of the detectors to their respective radiations are expressed in a series of Y_l^m's in a coordinate system whose origin is a point source of radiation. This origin and a fixed point on each detector determine the Z_1 and Z_2 axes as shown in Fig. 1. We let

$$E_{b1}(\theta_1,\varphi_1)=\sum_{l'm'}\left(\frac{2l'+1}{4\pi}\right)^{\frac{1}{2}} b_{l'm'} Y_{l'}^{m'}(\theta_1,\varphi_1),$$

$$E_{c2}(\theta_2,\varphi_2)=\sum_{l''m''}\left(\frac{2l''+1}{4\pi}\right)^{\frac{1}{2}} c_{l''m''} Y_{l''}^{m''}(\theta_2,\varphi_2). \tag{5}$$

The experimental correlation will then be

$$W(\theta)=\int\int E_{b1}(\theta_1,\varphi_1)W(\theta')E_{c2}(\theta_2,\varphi_2)d\Omega_1 d\Omega_2. \tag{6}$$

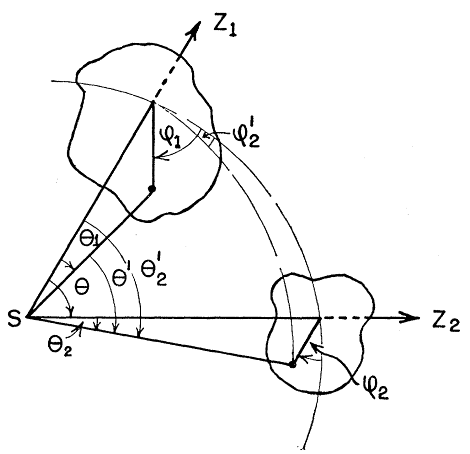

FIG. 1. Angular correlation geometry.

[8] H. Frauenfelder, Ann. Rev. Nuc. Sci. 2, 129 (1953).

We may express $P_l(\cos\theta')$ via the addition theorem[9] in terms of θ_1, φ_1 and θ_2', φ_2' (see Fig. 1):

$$P_l(\cos\theta')=\sum_n \frac{4\pi}{2l+1} Y_l^{n*}(\theta_1,\varphi_1)Y_l^n(\theta_2',\varphi_2'). \tag{7}$$

We may also express $Y_{l''}^{m''}(\theta_2,\varphi_2)$ in terms of θ_2', φ_2' by a rotation of the coordinate system through the angle θ:

$$Y_{l''}^{m''}(\theta_2,\varphi_2)=\sum_{n'} D_{n'm''}^{l''}(0,\theta,0)Y_{l''}^{n'}(\theta_2',\varphi_2')$$
$$\equiv\sum_{n'} d_{n'm''}^{l''}(\theta)Y_{l''}^{n'}(\theta_2',\varphi_2'), \tag{8}$$

where the D's are the representation coefficients[10] of the rotation group corresponding to the rotation through the Eulerian angles 0, θ, 0 which rotates the coordinate frame of detector 1 into coincidence with the coordinate frame of detector 2. From 6, 7, and 8 we obtain

$$W(\theta)=\sum_{l'm'}\sum_{l''m''}\sum_{lnn'}\frac{[(2l'+1)(2l''+1)]^{\frac{1}{2}}}{4\pi}$$
$$\times a_l b_{l'm'} c_{l''m''} d_{n'm''}^{l''}$$
$$\times\int Y_{l'}^{m'}(\theta_1,\varphi_1)Y_l^{n*}(\theta_1,\varphi_1)d\Omega_1$$
$$\times\int Y_{l''}^{n'}(\theta_2',\varphi_2')Y_l^n(\theta_2',\varphi_2')d\Omega_2'. \tag{9}$$

The Y_l^m used here satisfy the orthonormality relation:

$$\int (Y_l^m)^* Y_{l'}^{m'} d\Omega=\delta_{ll'}\delta_{mm'}. \tag{10}$$

Also

$$Y_l^m=(-1)^m(Y_l^{-m})^*. \tag{11}$$

Thus (9) becomes

$$W(\theta)=\sum_{lmm'}(-1)^m\frac{2l+1}{4\pi}a_l b_{lm} c_{lm'} d_{-m,m'}^l(\theta). \tag{12}$$

The remaining problem is to resolve the $d_{mm'}^l(\theta)$ into $P_l(\cos\theta)$. We let

$$d_{mm'}^l(\theta)=\sum_{l'}\frac{2l'+1}{2l+1}g_{mm'}^{ll'}P_{l'}(\cos\theta), \tag{13}$$

so that

$$g_{mm'}^{ll'}=\frac{2l+1}{4\pi}\int_{-1}^{+1} d_{mm'}^l(\theta)P_{l'}(\cos\theta)d(\cos\theta). \tag{14}$$

[9] We shall use throughout the normalization and phase convention as given by E. U. Condon and G. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, London, 1935).

[10] E. Wigner, *Gruppentheorie* (Friedrich Vieweg & Sohn, Braunschweig, Germany, 1931); G. Goertzel, Phys. Rev. 70, 897 (1946); G. Racah, Phys. Rev. 84, 910 (1951). The definition of D used here follows Goertzel and Racah and differs from Wigner's $\mathfrak{D}$ in that $D_{mm'}=\mathfrak{D}_{-m,-m'}$. For the definition and properties of D see the Appendix.

TABLE I. $g_{mm'}{}^{ll'}$ for $l \leq 6$; m, m' even.

$ll'mm'$	$g_{mm'}{}^{ll'}$	$ll'mm'$	$g_{mm'}{}^{ll'}$	$ll'mm'$	$g_{mm'}{}^{ll'}$
0000	1	4142	$(3\sqrt{7})/10$	5122	$121/210$
1100	1	4144	$6/5$	5140	$11/\sqrt{70}$
2020	$5/6$	4220	$3/\sqrt{10}$	5142	$121/70\sqrt{3}$
2022	$5/3$	4222	0	5144	$-11/105$
2200	1	4240	$-3\sqrt{(2/35)}$	5222	$-11/42$
2220	$-1/\sqrt{6}$	4242	$-9/10\sqrt{7}$	5242	$121/70\sqrt{3}$
2222	$1/6$	4244	$18/35$	5244	$187/210$
3022	$-7/6$	4322	$9/10$	5320	$11/\sqrt{210}$
3120	$7/\sqrt{30}$	4342	$-9/10\sqrt{7}$	5322	$-22/105$
3122	$7/15$	4344	$9/70$	5340	$-(11/3)\sqrt{(2/35)}$
3222	$14/15$	4400	1	5342	$-11/30\sqrt{3}$
3300	1	4420	$-\sqrt{(2/5)}$	5344	$242/315$
3320	$-\sqrt{(3/10)}$	4422	$2/5$	5422	$88/105$
3322	$3/10$	4440	$1/\sqrt{70}$	5442	$-88/105\sqrt{3}$
4020	$3/\sqrt{10}$	4442	$-1/5\sqrt{7}$	5444	$88/315$
4022	$9/10$	4444	$1/70$	5500	1
4040	$3\sqrt{(7/10)}$	5022	$-11/15$	5520	$-\sqrt{(10/21)}$
4042	$(9\sqrt{7})/10$	5042	$-11\sqrt{3}/15$	5522	$10/21$
4044	$9/5$	5044	$-22/15$	5540	$(1/3)\sqrt{(5/14)}$
4122	$-3/5$	5120	$11/\sqrt{210}$	5542	$-5/21\sqrt{3}$
5544	$5/126$	6242	$13/7\sqrt{30}$	6464	$-13/6\sqrt{66}$
6020	$13/2\sqrt{105}$	6244	$-13/21$	6466	$13/154$
6022	$13/21$	6260	$(-13/10)\sqrt{(11/21)}$	6522	$65/84$
6040	$(13/5)\sqrt{(2/7)}$	6262	$(-13/28)\sqrt{(11/5)}$	6542	$-65/21\sqrt{30}$
6042	$(26/7)\sqrt{(2/15)}$	6264	0	6544	$26/63$
6044	$26/21$	6266	$65/84$	6562	$(13/84)\sqrt{(5/11)}$
6060	$(13/10)\sqrt{(33/7)}$	6322	$-13/210$	6564	$(-13/126)\sqrt{(6/11)}$
6062	$(13/7)\sqrt{(11/5)}$	6342	$(221/84)\sqrt{(2/15)}$	6566	$13/924$
6064	$(13/7)\sqrt{(11/6)}$	6344	$2/9$	6600	1
6066	$13/7$	6362	$(-13/42)\sqrt{(11/5)}$	6620	$(-1/2)\sqrt{(15/7)}$
6122	$-221/420$	6364	$(-13/42)\sqrt{(11/6)}$	6622	$15/28$
6142	$-13/14\sqrt{30}$	6366	$13/42$	6640	$1/\sqrt{14}$
6144	$-13/42$	6420	$(13/6)\sqrt{(3/35)}$	6642	$-(1/28)\sqrt{30}$
6162	$(13/28)\sqrt{(11/5)}$	6422	$-13/42$	6644	$1/14$
6164	$(13/14)\sqrt{(11/6)}$	6440	$(-26/15)\sqrt{(2/7)}$	6660	$-1/2\sqrt{231}$
6166	$39/28$	6442	$-13/42\sqrt{30}$	6662	$(1/28)\sqrt{(5/11)}$
6220	$13/2\sqrt{105}$	6444	$52/63$	6664	$1/14\sqrt{66}$
6222	$143/420$	6460	$(13/10)\sqrt{(3/77)}$	6666	$1/924$
6240	$(13/10)\sqrt{(2/7)}$	6462	$13/42\sqrt{55}$		

Thus the final coefficient in the experimentally obtained angular correlation will be

$$h_l = \sum_{l'mm'} (-1)^m a_{l'} b_{l'm} c_{l'm'} g_{-m,m'}{}^{l'l}$$
$$= \sum_{l'mm'} (-1)^{m+m'+l+l'} a_{l'} b_{l'm} c_{l'm'} g_{mm'}{}^{l'l}. \quad (15)$$

The latter form is obtained by the use of (16). The problem of correction therefore reduces to evaluating the $g_{mm'}{}^{ll'}$. Below are listed some of the more important properties of the $g_{mm'}{}^{ll'}$, and a useful recursion relationship. The derivation of these formulas appears in the Appendix.

$$g_{mm'}{}^{ll'} = g_{-m',-m}{}^{ll'} = (-1)^{m+m'} g_{m'm}{}^{ll'} = (-1)^{l+l'+m} g_{m,-m'}{}^{ll'},$$

$$g_{00}{}^{ll'} = \delta_{ll'}; \quad g_{m0}{}^{ll'} = g_{0m}{}^{ll'} = 0 \text{ for } (l+l'+m) \text{ odd},$$

$$g_{mm'}{}^{ll'} = 0 \text{ for } l' > l \text{ and } (m+m') \text{ even}, \quad (16)$$

$$g_{mm'}{}^{ll'} = \sum_L \frac{2l+1}{2L+1} (ll'm0|Lm)(ll'm'0|Lm') g_{mm'}{}^{L0}. \quad (17)$$

Table I gives exact values of $g_{mm'}{}^{ll'}$ up to $l=6$ for m and m' even. (The case m and m' even is the only case of practical importance.)

The final expression (15) for arbitrary detectors shows that φ-dependent detectors mix the a_l which appear in the point-point angular correlation. If one uses φ-symmetric detectors, m and m' are zero, and from (16) we obtain the well-known result[2]

$$h_l = a_l b_l c_l. \quad (18)$$

If there is φ-symmetric scattering present in the source and the scattering distributions have the coefficients k_l and j_l the point-point correlation will be[2] $a_l' = a_l k_l j_l$. The experimental angular correlation will therefore be given by

$$h_l = \sum_{l'mm'} (-1)^{l+l'+m+m'} a_{l'} k_{l'} j_{l'} b_{l'm} c_{l'm'} g_{mm'}{}^{l'l}. \quad (19)$$

Church and Kraushaar[5] have shown that if one uses a source of annihilation radiation and φ-symmetric detectors the angular correlation that results is just $h_l = b_l c_l$. This follows from (18) if one makes use of the fact that the coefficients a_l in the expansion of the annihilation distribution, $\delta_{\theta\pi}$, are unity. This method is useful provided the energies of the gamma rays in the angular correlation measurement are close to that of annihilation quanta. It is clear from the mixing of coefficients in (15) that the annihilation method of determining correction factors experimentally cannot be used with arbitrary detectors.

III. EFFECTS OF SPECIAL DETECTOR SYMMETRIES

We first consider how special detector symmetries are evidenced in the b_{lm}. We may then determine the effect of special detector symmetries on the angular correlations. We define the following types of detector characteristics:

(A) Horizontal symmetry.—The detector is invariant to a reflection about the $\varphi=0°$, $180°$ line, i.e., $E(\theta,\varphi) = E(\theta, -\varphi)$.

(B) Vertical symmetry.—The detector is invariant to a reflection about the $\varphi=90°$, $270°$ line, i.e., $E(\theta,\varphi) = E(\theta, \pi-\varphi)$.

(C) Inversion symmetry.—The detector is invariant to inversion through the origin, i.e., $E(\theta,\varphi) = E(\theta, \pi+\varphi)$.

(D) Double symmetry.—The detector satisfies both (A) and (B); this is a special case of (C).

(E) φ-symmetry.—The detector is invariant to an arbitrary rotation about the Z axis.

In Table II we list the general properties of these detectors. In all cases $E(\theta,\varphi)$ real requires

$$b_{lm} = (-1)^m b_{l,-m}{}^*.$$

TABLE II. Characteristics of detectors of various symmetries.

Detector	Conditions on b_{lm}
Horizontal symmetry	$b_{lm} = b_{lm}{}^*$; $b_{lm} = (-1)^m b_{l,-m}$
Vertical symmetry	$b_{lm} = b_{l,-m}$
Inversion symmetry	$b_{lm} = 0$ for m odd
Double symmetry	$b_{lm} = 0$ for m odd; $b_{lm} = b_{lm}{}^*$
φ-symmetry	$b_{lm} = 0$ for $m \neq 0$

TABLE III. Characteristics of experimental angular correlations.[a]

	Horizontal symmetry	Vertical symmetry	Inversion symmetry	Double symmetry	φ-symmetry
Horizontal symmetry	δ				
Vertical symmetry	δ	β			
Inversion symmetry	δ	γ	β		
Double symmetry	δ	β, γ	β, γ	β, γ	
φ-symmetry	δ	β, γ	β, γ	β, γ	α

[a] Legend: α denotes no l's in experimental correlation *other* than those appearing in true point-point correlation. β denotes no l's in experimental correlation *higher* than those appearing in true point-point correlation. γ denotes no odd l in experimental correlation if none appear in true correlation. δ denotes no special properties such as α, β, γ.

In Table III the effect of various detector symmetries on experimental correlations is tabulated. These results follow immediately from the special form of the b_{lm} for special detector symmetries as given in Table II together with the symmetry properties of the $g_{mm'}{}^{ll'}$ as given in Eq. (16). Note that the bottom row of Table III, relating to one detector being φ-symmetric, can also be used for the case of an angular *distribution* measurement where one scatters a narrow collimated beam off a target into a detector. In this case the initial source of the beam acts like a point detector (φ-symmetric with $b_l=1$) of an angular correlation experiment.

IV. ARBITRARY DETECTOR WITH AXIAL SOURCE

In many angular correlation measurements source dimensions are not many orders of magnitude smaller than detector dimensions and it becomes necessary to introduce additional corrections to the data. The

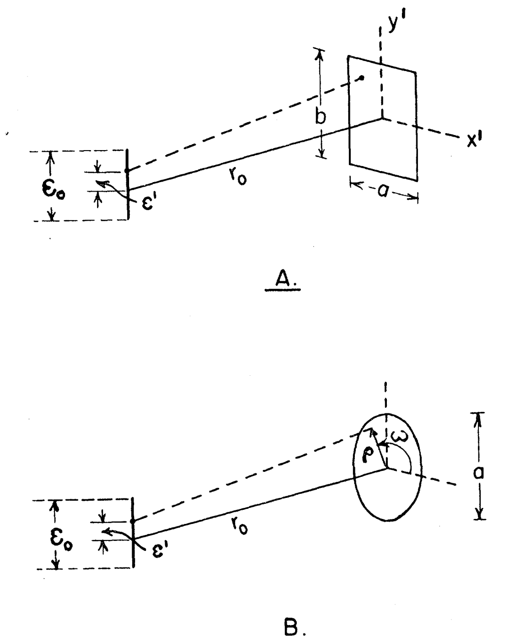

FIG. 2. (a) Geometry for transformation θ, $\varphi \rightarrow x', y', r_0$. (b) Geometry for transformation θ, $\varphi \rightarrow \rho, \omega, r_0$.

parameters specifying the corrections are $\epsilon_0/2r_0$ and $a/2r_0$, where ϵ_0 is the source size and a the detector size, r_0 being the source-detector separation. Source and detector corrections are independent to second order in these parameters but mixing occurs in the fourth order terms.

By the use of the results of Sec. II, however, it is possible to calculate an *exact* correction for an "axial" source. An axial source is a line source located at the origin and oriented perpendicular to the plane of rotation employed in the angular correlation measurement. While this source geometry is not a practical one, the fact that the correction is exact is quite valuable. Source geometry is usually at one's disposal. By making the source a long thin cylinder the off-axis corrections can be reduced at the expense of the axial correction. Since off-axis corrections are available to second order[1] and since to this order the off-axis, axial, and detector corrections are independent, corrections to almost any desired accuracy can be obtained by the procedure of suitably choosing the diameter of the cylindrical source.

To calculate the axial source correction we need only note that a displacement of the source above the plane of rotation is equivalent to an equal displacement of the detector below the plane of rotation. If the center of the detector is now defined as the original point in the plane of rotation it is clear that the b_{lm}'s of the detector have changed. The angle θ employed in Sec. II has not changed. The new b_{lm} resulting from this displacement, ϵ', of the center of the detector we call $b_{lm}(\epsilon')$. If the source is placed symmetrically about the plane of rotation,[11] is of length ϵ_0, and has the density of activity $n(\epsilon')$, we have

$$b_{lm}(\epsilon_0)=\int_{-\epsilon_0/2}^{+\epsilon_0/2} b_{lm}(\epsilon')n(\epsilon')d\epsilon' \Big/ \int_{-\epsilon_0/2}^{+\epsilon_0/2} n(\epsilon')d\epsilon'. \quad (20)$$

The new $b_{lm}(\epsilon_0)$ contain both source and counter corrections and are to be used in Eq. (17).

The determination of $b_{lm}(\epsilon')$ for an incidence dependent detector is in general a difficult problem. It can of course be done experimentally by two dimensional scanning of the detector with a narrow collimated beam for various displacements ϵ' of the detector. Alternately one might consider applying the analytical method of Rose[3] to the displaced right circular scintillator. However, the analysis would be complicated since the displaced scintillator and consequently its edge effects would be no longer φ-symmetric.

For an incidence independent detector the analysis is much simpler, in principle, since the change in its efficiency distribution is simply due to the change in

[11] It is important to point out that where sources having reflection symmetries are employed the first order terms in the source correction are zero. Displacements of the source from its true origin, however, introduce first order corrections (see reference 1). Thus a point source displaced a distance d from the origin results in larger errors than a source of dimensions d properly located.

aspect of its surface as seen from the centered point source. Thus $b_{lm}(\epsilon')$ is determined completely by the undisplaced values of the b_{lm} and the shape of the detector surface (which will be assumed to be flat in what follows). Two cases of practical interest where $b_{lm}(\epsilon_0)$ can be readily calculated are the rectangular and circular detectors of constant efficiency. The values of $b_{lm}(\epsilon_0)$ for these two cases are calculated below under the assumption that the axial source has a uniform density of activity, $n(\epsilon')=$ constant, though other situations may be handled with equal ease [see Eqs. (28), (29), and (30)].

TABLE IV. Rectangular detector correlation coefficients.

	f_1	f_2	g_2		h_1	h_2
b_{00}	$-1/2$	$3/8$	$5/12$	b_{22}	$(1/12)\sqrt{6}$	$-(1/8)\sqrt{6}$
b_{10}	$-2/3$	$5/8$	$8/12$	b_{32}	$(1/12)\sqrt{30}$	$-(3/20)\sqrt{30}$
b_{20}	-1	$9/8$	$15/12$	b_{42}	$(3/12)\sqrt{10}$	$-(11/20)\sqrt{10}$
b_{30}	$-3/2$	$21/10$	$42/12$	b_{52}	$(1/12)\sqrt{210}$	$-(9/40)\sqrt{210}$
b_{40}	$-13/6$	$30/8$	$50/12$	b_{62}	$(1/6)\sqrt{105}$	$-(11/20)\sqrt{105}$
b_{50}	-3	$51/8$	$85/12$			
b_{60}	-4	$207/20$	$23/2$			
b_{44}	—	$(1/80)\sqrt{70}$	$-(1/24)\sqrt{70}$			
b_{54}	—	$(3/80)\sqrt{70}$	$-(1/80)\sqrt{70}$			
b_{64}	—	$(3/16)\sqrt{14}$	$-(5/8)\sqrt{14}$			

A. Axial Source Corrections for Rectangular Detectors

From (5),

$$b_{lm}=\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}\int Y_l^m(\theta,\varphi)E(\theta,\varphi)d\Omega. \tag{21}$$

We transform (21) to Cartesian coordinates at the center of the detector to obtain

$$b_{lm}(\epsilon)=\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}\frac{1}{r_0^2}\int_{-\beta}^{\beta}dy\int_{-\alpha}^{\alpha}dx \times\frac{Y_l^m(x,\,y-\epsilon)E(x,y)}{[1+x^2+(y-\epsilon)^2]^{\frac{3}{2}}}, \tag{22}$$

where $x=x'/r_0$, $y=y'/r_0$, $\epsilon=\epsilon'/r_0$, $\beta=b/2r_0$, and $\alpha=a/2r_0$. The pertinent quantities are shown in Fig. 2(a). Thus

$$b_{lm}(\alpha,\beta,\epsilon_0)=\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}\frac{1}{\pi r_0^2\epsilon_0}\int_{-(\beta+\epsilon)}^{+(\beta+\epsilon)}dy''\int_{-\alpha}^{+\alpha}dx \times\frac{Y_l^m(x,y'')E(x,\,y''+\epsilon)}{[1+x^2+y''^2]^{\frac{3}{2}}}, \tag{23}$$

with $y''=y-\epsilon$. For the constant efficiency detector, $E(x,y)=1$; and defining $\gamma=\epsilon_0/2r_0$, we obtain

$$\begin{aligned}
b_{l0}&=\frac{ab}{r_0^2}\{1+f_1[\alpha^2+\beta^2+\gamma^2]+f_2[\alpha^4+\beta^4+\gamma^4+(10/3)\beta^2\gamma^2]+g_2\alpha^2[\beta^2+\gamma^2]\},\\
b_{l2}&=\frac{ab}{r_0^2}\{h_1[\alpha^2-(\beta^2+\gamma^2)]+h_2[\alpha^4-(\beta^4+\gamma^4+(10/3)\beta^2\gamma^2)]\},\\
b_{l4}&=\frac{ab}{r_0^2}\{f_2[\alpha^4+\beta^4+\gamma^4+(10/3)\beta^2\gamma^2]+g_2\alpha^2[\beta^2+\gamma^2]\}.
\end{aligned} \tag{24}$$

The values of the coefficients of the above expansion are given in Table IV. By judicious choice of γ^2 the second order term in b_{l2} may be made zero. With this choice the coefficient of h_2 becomes $(4/3)\beta^2\gamma^2$. Note that the $b_{lm}(\alpha,\beta,\gamma)$ are obtainable from the $b_{lm}(\alpha,\beta,0)$ by replacing β^2 by $(\beta^2+\gamma^2)$ and β^4 by $[\beta^4+\gamma^4+(10/3)\beta^2\gamma^2]$.

B. Axial Source Corrections for Circular Detectors

We transform to polar coordinates on the face of the detector as shown in Fig. 2(b).

$$b_{lm}(\alpha,\gamma)=\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}\int_{-\epsilon_0/2}^{\epsilon_0/2}d\epsilon\int_0^{\alpha}rdr\int_0^{2\pi}d\omega \times\frac{Y_l^m(r,\omega,\epsilon)E(r,\omega)}{[1+r^2+\epsilon^2+2r\epsilon\sin\omega]^{\frac{3}{2}}}, \tag{25}$$

where $\alpha=a/2r_0$, $r=\rho/r_0$, $\epsilon=\epsilon'/r_0$, and $\gamma=\epsilon_0/2r_0$. For the special case $E(r,\omega)=1$ it is easy to show that $b_{l0}(\alpha,\gamma)$ can be obtained from the $b_{l0}(\alpha,0)$ by replacing α^2 by $(\alpha^2+\frac{2}{3}\gamma^2)$ and α^4 by $(\alpha^4+\frac{3}{5}\gamma^4+2\alpha^2\gamma^2)$. The exact $b_{l0}(\alpha,0)$ is just

$$b_{l0}(\alpha,0)=2\pi\int_0^{\theta_0}P_l(\cos\theta)\sin\theta d\theta =2\pi\frac{P_{l+1}(\theta_0)-P_{l-1}(\theta_0)}{2l+1}. \tag{26}$$

This expression has been expanded in powers of α^2 to obtain $b_{l0}(\alpha,\gamma)$ using the above replacements. Since $b_{lm}(\alpha,0)=0$, Eq. (25) must be used to obtain $b_{lm}(\alpha,\gamma)$. For the circular detector

$$\begin{aligned}
b_{l0}&=\pi\alpha^2\{1+F_1[\alpha^2+\tfrac{2}{3}\gamma^2]+F_2[\alpha^4+\tfrac{3}{5}\gamma^4+2\alpha^2\gamma^2]\},\\
b_{lm}&=\pi\alpha^2\{H_1\gamma^2+J_1\gamma^4+J_2\alpha^2\gamma^2\}.
\end{aligned} \tag{27}$$

The pertinent coefficients are listed in Table V.

It is not necessary to restrict these calculations to constant efficiency detectors provided they are φ-sym-

TABLE V. Circular detector correction coefficients.

	F_1	F_2		H_1	J_1	J_2
b_{00}	$-3/4$	$5/8$	b_{22}	$-(1/12)\sqrt{6}$	$(1/8)\sqrt{6}$	$(5/16)\sqrt{6}$
b_{10}	-1	1	b_{32}	$-(1/12)\sqrt{30}$	$(1/4)\sqrt{30}$	$(3/8)\sqrt{30}$
b_{20}	$-3/2$	$15/8$	b_{42}	$-(1/4)\sqrt{10}$	$(11/20)\sqrt{10}$	$(11/8)\sqrt{10}$
b_{30}	$-9/4$	$7/2$	b_{44}	0	$(1/8)\sqrt{70}$	0
b_{40}	$-13/4$	$25/4$				
b_{50}	$-9/2$	$85/8$				
b_{60}	-6	$69/4$				

metric. If $E(r)$ is the detector efficiency for a circular detector, we make the following replacements. In the $\pi\alpha^2$-coefficient in (27),

$$\pi\alpha^2=\frac{2\pi}{r_0^2}\int_0^\infty E(r)rdr. \tag{28}$$

In the power series we make the replacements:

$$\alpha^2=\frac{2}{r_0^2}\int_0^\infty r^2E(r)rdr\bigg/\int_0^\infty E(r)rdr, \tag{29}$$

$$\alpha^4=\frac{3}{r_0^2}\int_0^\infty r^4E(r)rdr\bigg/\int_0^\infty E(r)rdr. \tag{30}$$

Note that for a circular gamma-ray detector $E=E(r,\omega)$, i.e., it is incident-angle dependent and the above results cannot be used. The error should however be of fourth order in ϵ_0.

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APPENDIX

Properties of $D_{mm'}{}^l(0,\theta,0)$

According to Wigner,[10] page 180, we have

$$D_{mm'}{}^l(0,\theta,0)=d_{mm'}{}^l(\theta)=\sum_k(-1)^k \times\frac{[(l+m)!(l-m)!(l+m')!(l-m')!]^{\frac{1}{2}}}{(l+m-k)!(l-m'-k)!k!(k-m+m')!} \times[\cos(\theta/2)]^{2l-m'+m-2k}[\sin(\theta/2)]^{2k-m+m'}, \tag{A1}$$

where the integral summation index k runs from the larger of the values 0 and $(m-m')$ to the smaller of the values $(l+m)$ and $(l-m')$. We restrict ourselves to the case l, m, m' integral.

From the form of the definition of d it follows immediately that

$$d_{-m',-m}{}^l(\theta)=d_{m,m'}{}^l(\theta). \tag{A2}$$

Also by the substitution $k=k'+m-m'$, it follows that

$$d_{-m,-m'}{}^l=(-1)^{m'-m}d_{mm'}{}^l(\theta)=d_{m'm}{}^l(\theta); \tag{A3}$$

and by the substitution $k=l+m-k'$,

$$d_{m,m'}{}^l(\theta)=(-1)^{l+m}d_{mm'}{}^l(\pi-\theta). \tag{A4}$$

We also have the relations (Racah):[10]

$$d_{00}{}^l(\theta)=P_l(\cos\theta),$$

$$d_{m0}{}^l(\theta)=(-1)^m d_{0m}{}^l(\theta)=\left(\frac{4\pi}{2l+1}\right)^{\frac{1}{2}}Y_l{}^m(\theta,0). \tag{A5}$$

Properties of $g_{mm'}{}^{ll'}$

From the definition of $g_{mm'}{}^{ll'}$ [Eq. (14)] and the aforementioned symmetry properties of $d_{mm'}{}^l(\theta)$, it follows that

$$g_{mm'}{}^{ll'}=g_{-m',-m}{}^{ll'}=(-1)^{m+m'}g_{m'm}{}^{ll'}=(-1)^{m+m'}g_{-m,-m'}{}^{ll'}. \tag{A6}$$

The relation

$$g_{mm'}{}^{ll'}=(-1)^{l+l'+m}g_{m,-m'}{}^{ll'} \tag{A7}$$

follows from (A4) and the definition of $g_{mm'}{}^{ll'}$ upon noting that $P_l(\cos\theta)$ is an even or odd function of $\cos\theta$ when l is even or odd respectively. The relation

$$g_{m0}{}^{ll'}=g_{0m}{}^{ll'}=0 \text{ for } (l+l'+m) \text{ odd} \tag{A8}$$

follows from (A5) and the even or odd character of $P_l(\cos\theta)$.

To prove

$$g_{mm'}{}^{ll'}=0 \text{ if } l'>l,\ (m+m') \text{ even}, \tag{A9}$$

we note that if $m+m'$ is even, then $d_{mm'}{}^l(\theta)$ will consist of a sum of integral positive powers of $\cos\theta$, the highest power occurring being $\cos^l\theta$. Therefore the expansion of $d_{mm'}{}^l(\theta)$ into $P_{l'}(\cos\theta)$'s will contain $l'=l$ as the largest possible value of l'.

The recursion relation

$$g_{mm'}{}^{ll'}=\sum_L\frac{2l+1}{2L+1}(ll'm0|Lm)(ll'm'0|Lm')g_{mm'}{}^{L0} \tag{A10}$$

may be derived as follows. We have

$$g_{mm'}{}^{ll'}=\frac{2l+1}{2}\int_{-1}^{+1}d_{mm'}{}^l(\theta)d_{00}{}^{l'}(\theta)d(\cos\theta), \tag{A11}$$

by Eq. (A5) and the definition of g. But from Eq. (16b) of Wigner,[10] page 204, we have

$$d_{mm'}{}^l(\theta)d_{00}{}^{l'}(\theta)=\sum_{L=|l-l'|}^{|l+l'|}(ll'm0|Lm)\times(ll'm'0|Lm')d_{mm'}{}^L(\theta), \tag{A12}$$

where $(ll'm0|Lm)$ is a Clebsch-Gordan (Wigner) coefficient. Substituting (A12) into (A11), we obtain (A10).