an equal mixture of Δ_A and Δ_D. If this is the case, it can be regarded as evidence favoring a relatively uniform galactic magnetic field directed along a spiral arm of the galaxy. The field strength could not be an order of magnitude less than 10^{-5} gauss or there would be more anisotropy in the highest energy particles observed by Cranshaw and Galbraith. It is also implied then that cosmic rays are accelerated by the Fermi mechanism rather than exclusively by processes taking place at the original ion sources.

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Age-Dependent Branching Stochastic Processes in Cascade Theory*

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A brief introduction to the recent Bellman-Harris theory of branching stochastic processes is given in the nomenclature of cascade theory; and a simple model in cascade theory formulated as an age-dependent branching process is given.

INTRODUCTION

THE theory of branching stochastic processes has been used on many occasions in the development of mathematical models of cascade phenomena (e.g., cosmic-ray showers, neutron multiplication, etc.).[1–6] Recently Bellman and Harris[7] have developed a theory of age-dependent branching processes which appears to have important applications in the physical and biological sciences. The purpose of this communication is twofold: first, to give a brief introduction to the Bellman-Harris theory in the nomenclature of cascade theory; and second, to present a simple model for the electron population of a cosmic-ray shower. The model considered is a modification of the Furry process.

In the Bellman-Harris theory the distance or thickness, say τ, travelled by a particle (electron, neutron, etc.) from its formation until it is transformed is a random variable with general distribution $G(\tau)$, $0<\tau<\infty$; i.e., $G(\tau)$ is the integral distribution for all paths of length less than or equal to τ. At the end of its path of travel the particle is transformed into n particles with probabilities q_n, $n=0, 1, \cdots$, each particle having the same distribution $G(\tau)$ for the distance it will travel before being transformed. For example, q_0 is the probability of absorption, q_1 is the probability that one new particle will be formed, the original one being absorbed, q_2 is the probability that two new particles will be formed (the original one being absorbed), etc. The random variable τ measures the distance to the next point of regeneration. We remark that the age-dependence is only for the total cross section, the branching ratio being age-independent.

The Bellman-Harris process is formulated as follows: Let $X(t)$ be an integer-valued random variable representing the number of particles at thickness t; and define $p(x,t)=Pr(X(t)=x)$, $x\geqslant 0$. Let

$$\pi(s,t)=\sum_{x=0}^{\infty} p(x,t)s^x, \quad |s|<1 \tag{1}$$

be the generating function for the probabilities $p(x,t)$ starting with one particle at thickness zero. $[\pi(s,t)]^n$ is the generating function if the process starts with $n>1$ particles at thickness zero. In treating both cases the assumption is made that the particles do not interact with one another. The generating function (1) has been shown to satisfy the nonlinear Stieltjes functional equation

$$\pi(s,t)=\int_0^t h[\pi(s, t-\tau)]dG(\tau)+s[1-G(t)], \tag{2}$$

where

$$h(s)=\sum_{n=0}^{\infty} q_n s^n, \tag{3}$$

that is, $h(s)$ is the generating function for the transformation probabilities q_n. The equation for the gener-

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ating function can be derived as follows. By definition,

$$p(x,t)=Pr(X(t)=x)=\int_0^t Pr(X(t)=x\,|\,\tau)dG(\tau),$$

where $Pr(X(t)=x\,|\,\tau)$ is the probability of x particles at thickness t from a single particle at thickness zero which is known to have branched at $t=\tau$. Now

$$Pr(X(t)=x\,|\,\tau)$$
$$=\sum_{n=0}^{\infty} q_n\{\sum_{i_1+\cdots+i_n=x} p(i_1,t-\tau)\cdots p(i_n,t-\tau)\},$$

where the term in braces is the coefficient of s^x in the expansion of

$$[\sum_{x=0}^{\infty} p(x,t-\tau)s^x]^n=\pi^n(s,t-\tau).$$

(The reasoning used above is the same as that used in the theory of compound probability distributions.) Multiplying $p(x,t)$ by s^x and summing over x we obtain, after adding the term for $p(1,t)=1-G(t)$, Eq. (2). If $G(t)$ has a density function $g(\tau)$ of bounded total variation, we can write (2) as

$$\pi(s,t)=\int_0^t h[\pi(s,t-\tau)]g(\tau)d\tau+s[1-G(t)].$$

Differentiation of (2) with respect to s yields intergal equations of the renewal type for the moments of $X(t)$, the properties of which can be studied using well-known methods.[8] For example, the expected value of $X(t)$ is

$$EX(t)=m(t)=K\int_0^t m(t-\tau)g(\tau)d\tau+1-G(t), \quad (4)$$

where

$$K=\sum_{n=0}^{\infty} nq_n=\left(\frac{dh}{ds}\right)_{s=1}.$$

In a forthcoming publication we plan to treat models of cascade phenomena which involve more than one type of particle, and in which energy considerations are taken into account. We plan also to discuss the diffusion equations associated with age-dependent branching processes and their physical applications.

AN AGE-DEPENDENT MODEL

Consider a single particle of the soft component of cosmic radiation falling on a material slab (e.g., lead). and its effect being multiplied to form a cascade shower, The model we consider assumes (i) $G(t)=1-e^{-\lambda t}$, and (ii) $h(s)=q_0+q_1s+q_2s^2$. With $G(t)$ thus defined the state of the cascade at any thickness t depends only upon the number of particles at that thickness and is independent of their previous paths; hence, the stochastic process we consider is a Markov process with an enumerable number of states. If we disregard the photons in the electron-photon cascade the probability that an electron will be transformed between t and $t+\delta t$ is $\lambda\delta t+o(\delta t)$, where λ is defined as the birth constant or birth-rate. From the definition of $h(s)$, (ii) states that an electron after travelling a distance of random length τ has probability q_0 of being absorbed, probability q_1 of being absorbed and producing one new electron, and probability q_2 of being absorbed and producing two new electrons. Each new electron then has probability $G(\tau)$ of travelling a distance τ before being transformed again.

The generating function in this case satisfies the equation,

$$\pi(s,t)=\int_0^t\{\sum_{n=0}^{2} q_n\pi^n(s,t-\tau)\}\lambda e^{-\lambda\tau}d\tau+se^{-\lambda t}. \quad (5)$$

This integral equation can be written as the differential equation,

$$(1/\lambda)\partial\pi/\partial t=q_2\pi^2+(q_1-1)\pi+q_0,$$

or

$$(1/\lambda)\partial\pi/\partial t=q_2\pi^2-(q_0+q_2)\pi+q_0. \quad (6)$$

If we put $q_0=q_1=0$, Eq. (6) becomes the generating function of the Furry distribution.[1] The solution to (6) with initial condition $\pi(s,0)=s$ is

$$\pi(s,t)=\frac{q_0-q_2[(q_2s-q_0)/(q_2s-q_2)]e^{\lambda(q_0-q_2)t}}{q_2\{1-[(q_2s-q_0)/(q_2s-q_2)]e^{\lambda(q_0-q_2)t}\}}. \quad (7)$$

By (1), the probabilities $p(x,t)$ can now be found by taking the coefficient of s^x in the powers series expansion of $\pi(s,t)$. We first rewrite (7) as

$$\pi(s,t)=\frac{A+Bs}{C-Ds}=\frac{1}{C}(A+Bs)\sum_{x=0}^{\infty}\left[\frac{D}{C}\right]^x s^x, \quad (8)$$

where

$$A=q_0q_2e^{\lambda(q_0-q_2)t}-q_0q_2,$$
$$B=q_0q_2-q_2^2e^{\lambda(q_0-q_2)t},$$
$$C=q_0q_2e^{\lambda(q_0-q_2)t}-q_2^2,$$
$$D=q_2^2e^{\lambda(q_0-q_2)t}-q_2^2.$$

Therefore,

$$p(x,t)=\left[\frac{A}{C}+\frac{B}{D}\right]\left[\frac{D}{C}\right]^x, \quad x\geqslant 1. \quad (9)$$

Using Eq. (4) we can write the integral equation for the expected, or mean, value of $X(t)$. We have

$$EX(t)=m(t)=(1-q_0+q_2)$$
$$\times\int_0^t m(t-\tau)\lambda e^{-\lambda\tau}d\tau+e^{-\lambda t}. \quad (10)$$

[8] W. Feller, Ann. Math. Stat. 12, 243 (1941).

Equation (10) reduces to the differential equation

$$\partial m(t)/\partial t=\lambda(q_2-q_0)m(t), \tag{11}$$

whose solution, when $X(0)=1$, is

$$m(t)=\exp\{\lambda(q_2-q_0)t\}. \tag{12}$$

It is clear that the behavior of $EX(t)$ will depend on the values of q_0 and q_2. As t approaches infinity we have

$$\lim_{t\to\infty} EX(t)=\begin{cases}0, & q_0>q_2\\ \infty, & q_2>q_0\\ 1, & q_0=q_2.\end{cases} \tag{13}$$

Whenever the probability of absorption is introduced into a model of cascade phenomena it is of interest to determine the probability $p(0,t)$ that the cascade will die out, that is that all particles will for some thickness t be absorbed. From (1) and (8) we see that this probability is given by

$$p(0,t)=\frac{A}{C}=\frac{q_0e^{\lambda(q_0-q_2)t}-q_0}{q_0e^{\lambda(q_0-q_2)t}-q_2}. \tag{14}$$

The probability that all electrons will eventually be absorbed is given by

$$\lim_{t\to\infty} p(0,t)=\begin{cases}1, & q_0>q_2\\ q_0/q_2, & q_0<q_2.\end{cases} \tag{15}$$

Should the cascade start with $n>1$ electrons at thickness zero, the expected number of electrons in the cascade can be obtained by multiplying (12) by n. Since independence is assumed, this is equivalent to considering n independent processes with associated random variables $X_1(t)$, $X_2(t)$, $\cdots$, $X_n(t)$, and since

$$E(\sum_{i=1}^{n} X_i(t))=\sum_{i=1}^{n} EX_i(t)$$

the above result follows. An explicit expression for $p(x,t)$ can be obtained by expanding $\pi^n(s,t)$ as a power series in s and proceeding as before. Hence

$$\pi^n(s,t)=\left(\frac{A+Bs}{C}\right)^n\left(1-\frac{D}{C}s\right)^{-n}$$
$$=\left(\frac{A+Bs}{C}\right)^n\sum_{x=0}^{\infty}\binom{-n}{x}\left[-\frac{D}{C}\right]^x s^x. \tag{16}$$

We have, therefore,

$$p(x,t)=\left\{\sum_{i=0}^{n}\binom{n}{i}\left[\frac{A}{C}\right]^{n-i}\left[\frac{B}{D}\right]^i\right\}$$
$$\times\binom{-n}{x}\left[-\frac{D}{C}\right]^{x-i}, \quad x\geqslant n. \tag{17}$$

This distribution is of the negative binomial type. The asymptotic behavior of the mean is the same as before, except that now $EX(t)=n$ when $q_0=q_2$. In addition, we have

$$p(0,t)=(A/C)^n, \tag{18}$$

and

$$\lim_{t\to\infty} p(0,t)=\begin{cases}1, & q_0>q_2\\ (q_0/q_2)^n, & q_0<q_2.\end{cases} \tag{19}$$