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Anisotropy of High-Energy Cosmic Rays

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Review of the evidence indicates that a magnetic field of the order of 10^{-5} gauss probably lies along a spiral arm of the galaxy. If so, any anisotropy observed in high-energy cosmic radiation must be associated with this field. Anisotropy might be due to: (a) acceleration by Fermi's mechanism, either by his longitudinal collisions or by betatron effects; (b) diffusion along field lines toward a region where the cosmic rays escape from the galaxy; (c) inhomogeneities in cosmic-ray density normal to the field lines. From symmetry considerations theoretical expressions are developed for the cosmic-ray flux as a function of direction and for the resulting sidereal time dependence of extensive showers as a function of latitude and the orientation of the detecting apparatus. If atmospheric effects can be corrected for, the main harmonics predicted are the first and second, the second being mainly due to anisotropy produced by acceleration. In the absence of detailed calculations based on a specific theory of the origin of cosmic rays and on the way the extensive showers are detected, the amplitude of the harmonics must be determined from experiment. Preliminary reports of measurements by Cranshaw and Galbraith and by Farley and Storey seem to indicate tentatively that the magnetic field is as described above and that cosmic rays are accelerated by Fermi's mechanism; the measurements of Daudin and Daudin require some other explanation.

I. INTRODUCTION

IT is the purpose of this paper to consider theoretically the anisotropy of the cosmic-ray flux and the sidereal time dependence to be expected for extensive air showers if a galactic magnetic field of about 10^{-5} gauss lies along a spiral arm of the galaxy. Some variation with sidereal time would be expected in the output of any cosmic-ray measuring apparatus that rotates with the earth if the cosmic-ray flux here depends on its direction with respect to the galaxy. It is simplest to try to study this anisotropy by investigating extensive air showers since they are produced by very high-energy primary particles that are not affected by the magnetic field of the earth or, it will be assumed, by any field confined to the solar system. Since the assumed field cannot confine these primary particles to a region very small compared to the thickness of the galaxy, they might be expected to show more anisotropy than would lower-energy particles, which, however, could be treated in the same way if the effects of local fields are allowed for.

A number of possible causes of anisotropy are well known. If cosmic rays travel in straight lines, a distribution of sources concentrated in the plane of the Milky Way or at the center of the galaxy would lead to an anisotropic flux. Motion of the solar system with respect to the sources[1] would also produce such an effect if the rays travel in straight lines. If cosmic rays diffuse through the galaxy by scattering from small knots of magnetic field embedded in gas clouds and separated by field-free regions, anisotropy would be produced by boundary effects.[2,3] However recent information on the galactic magnetic field suggests that it extends throughout a spiral arm with quite a regular structure that must determine the character of the anisotropy.[4] If this is the case, observations on the sidereal time dependence of extensive showers should provide evidence bearing on the local direction of the galactic magnetic field and on the way in which cosmic rays receive their energy but should provide very little evidence concerning the location of the original sources of cosmic rays.

Since this entire treatment assumes the presence of a galactic magnetic field, the evidence for it should be reviewed. No method has yet been suggested by which the material in the galaxy can supply as much power to cosmic rays as to starlight, which gets its energy from nuclear sources. Since the observed flux of cosmic rays is about equal to that of starlight, cosmic rays cannot travel in straight lines. They must be held in the galaxy for times of the general order of a million years. Magnetic fields seem to be required in any mechanism that can do this. Once magnetic fields are accepted, the argument that Dungey and Hoyle[5] used to try to prove that there is no galactic magnetic field can be used to show that the field must be at least $B=6\times10^{-6}$ gauss. For unless it is this large, its energy density, $B^2/8\pi$, will be less than that of the cosmic rays, which would then push the magnetic field aside and expand until their energy density was smaller than that of the field.[5a] The observed polarization

[1] A. H. Compton and I. A. Getting, Phys. Rev. 47, 817 (1935).
[2] G. Cocconi, Phys. Rev. 83, 1193 (1951).
[3] Morrison, Olbert, and Rossi, Phys. Rev. 94, 440 (1954).
[4] L. Davis, Bull. Am. Phys. Soc. 29, No. 6, 18(A) (1954).
[5] J. W. Dungey and F. Hoyle, Nature 162, 888 (1948).
[5a] *Note added in proof.*—Dr. Arnulf Schlüter has pointed out in a private communication that this overlooks the influence of forces, such as those of gravity, that could prevent the expansion of a weaker field through their action on the interstellar gas in which the field is embedded. However, it seems significant that the forces due to gravitation, cosmic rays, random motions of the gas clouds, and magnetic fields all appear to be of the same general order of magnitude. Even though no theory yet explains this in detail, the point tends to make more plausible the assumption of a magnetic field strength of the order of 10^{-5} gauss.

of starlight suggests[6,7] that the galactic magnetic field is roughly uniform and in the plane of the galaxy, perhaps running along a spiral arm. There appears to be a root-mean-square deviation of the order of 5° to 10° in the planes of polarization of neighboring stars.[8] Hence there should be corresponding deviations in the direction of the field that must be due to magneto-hydrodynamic waves, and this implies[6,9] that the field is of the order of 10^{-5} gauss. Chandrasekhar and Fermi[9] show that such a field is required to explain the stability of the spiral arms of the galaxy; the lateral pressure due to the magnetic field can support the gravitational forces attracting the interstellar gas and dust towards the axis of the spiral arm whereas kinetic pressures are much too low. Fermi's most successful modification[10] of his theory of the origin of cosmic rays uses a magnetic field directed along the spiral arms; magneto-hydrodynamic waves in this field supplying the energy to the cosmic rays. If such fields were formed[11] by turbulence and a contraction of the galaxy from spherical to disk form in its early history, the conductivity is great enough so that they would be expected to persist for many billion years. Biermann and Schlüter[12] and Biermann[13] conclude from a study of the turbulent motion of ionized gases in a galaxy of the present shape that in a few billion years fields of this strength would be formed even if one started with no magnetic field.

These arguments make plausible the assumption of a magnetic field of about 10^{-5} gauss running along the spiral arms and suggests the investigation of its consequences, including its effect on the isotropy of cosmic rays. The main consequence for cosmic rays is that each particle is constrained to spiral around a particular line of force, following it wherever it may go with, perhaps, occasional reversals of the direction of motion along the line of force and with some very slow transverse diffusion due to inhomogeneities in the field. Even a proton whose energy is 10^{16} ev will move in a helix whose maximum radius (when the helix angle is 90°) in a field of 10^{-5} gauss is 3.5 light years, a small distance on a galactic scale.

For convenience in analysis it will be assumed throughout this paper that the magnetic field within a few light years of the solar system is uniform but is part of a larger field having a somewhat complicated structure. On a galactic scale, the magnetic lines of force will be assumed in general to lie in a parallel bundle that starts from one end of a spiral arm at the rim of the galaxy, follows the arm around several turns, connects to another arm near the galactic center, spirals out to the rim along this arm, and then spreads out to give a very weak dipole field in intergalactic space. Observationally, the distribution of stars and gas along a spiral arm is far from uniform. Thus, by the argument of Chandrasekhar and Fermi,[9] at any point where the mass density and hence the gravitational forces drop off a bit, the lateral pressure of the magnetic field, and, one might also add, of the cosmic radiation, will expand the field laterally. The remaining dust and gas will tend to slide down the lines of force away from this bulge. This further reduces the gravitational forces there and allows the field to bulge out still more. Hence it must always be borne in mind that at a number of places along the spiral arm there may be leaks where some of the lines of force escape into intergalactic space. Assume that the sun is not in such a region. Even in the most nearly uniform regions the magnetic field strength should decrease as one moves out from the axis of the spiral arm. Only then is there a gradient in the pressure of the magnetic field to support the gravitational forces. Superposed on this static field are magneto-hydrodynamic oscillations covering an extensive spectrum of wavelengths. For waves in which regions having dimensions of a few light years move in phase, the root-mean-square deviation of the lines of force appears to be[8] of the order of 0.1 radian. Although ordinary turbulence is suppressed by the strong magnetic field and the conductivity of the gas, waves should be generated by the hydrodynamic forces that ordinarily would generate turbulence from the nonrigid rotation of the galaxy. These waves may be thought of as transient inhomogeneities of all sizes superposed on the nearly uniform static field. The magnetic fields of the earth and sun will be disregarded because of the very high energies of the primary cosmic-ray particles considered.

II. CAUSES OF ANISOTROPY

The isotropy in the cosmic-ray flux depends both on the structure of the galactic magnetic field and on the origin assumed for the cosmic rays. A widely accepted theory of their origin holds that by some electromagnetic process involving varying magnetic fields large amounts of energy are given to ions, probably in the neighborhood of stars, but possibly in particularly turbulent and strongly magnetized gas clouds. There are then two possibilities. The first is that this initial process supplies substantially all the energy that the particles ever get and that their subsequent history is a slow diffusion out of the galaxy. The large number of heavy primaries shows that collisions with gas atoms are unimportant. The second possibility is that the initial process gives only an energy of the general order of a Bev per nucleon, and that further energy is supplied over periods of millions

[6] L. Davis, Phys. Rev. **81**, 890 (1951).
[7] L. Davis, *Vistas of Astronomy*, edited by A. Beer (Pergamon Press, London, to be published).
[8] G. Stranahan, Astrophys. J. **119**, 465 (1954).
[9] S. Chandrasekhar and E. Fermi, Astrophys. J. **118**, 113 (1953).
[10] E. Fermi, Astrophys. J. **119**, 1 (1954).
[11] This possibility was suggested in conversation by F. Hoyle.
[12] L. Biermann and A. Schlüter, Phys. Rev. **82**, 863 (1951).
[13] L. Biermann, in *Kosmische Strahlung*, edited by W. Heisenberg (Springer-Verlag, Berlin, 1953), second edition, pp. 47–65.

of years by interactions with the galactic magnetic field as suggested by Fermi.[10] He considered interactions in which the particle is reflected from a moving inhomogeneity in the magnetic field in a way that adds momentum along the magnetic field only. This should tend to make an anisotropic distribution of cosmic ray flux with greater intensity along the field lines than at right angles to them. Alternatively acceleration could be due to "betatron collisions";[14] i.e., to passage through regions where the magnetic field varies with time. The principal advantages of the betatron mechanism are that it seems easier to add energy at a high rate and that there is not the difficulty found in Fermi's mechanism where, when a particle gains some energy, it is difficult to gain more because it escapes from his "trap." Although the betatron mechanism initially adds momentum normal to the field, this is converted into momentum added parallel to the field when the particles spiral along the lines of force from the region where the field is increasing with time to a region where the field has its average value.[14a]

Further anisotropy will be introduced by the acceleration process if the probability of escape from the system depends on the steepness of the spirals in a different way than does the probability of acceleration. Now the steeper the helices, the more rapidly particles diffuse away along the lines of force, while the flatter the helices, the more easily a particle is accelerated both by Fermi's collision process and by the betatron process. Hence if these processes are important in supplying energy to cosmic rays, there is a tendency for the flux of particles in a given energy range to be greater normal to the field than parallel to it. Whether acceleration produces an anisotropy of this character or one of the opposite character as described in the previous paragraph depends on which tendency dominates. With either accelerating mechanism, inhomogeneities in the magnetic field whose scale is less than the radius of the helix tend to make the flux more nearly isotropic by scattering particles from one helix into another of different helix angle or pitch.

There are additional sources of anisotropy that apply both to such accelerating mechanisms and to the case in which the initial process supplies all the energy. If the particles diffuse along the lines of force out of the galaxy either at the end of a spiral arm or at a region of low mass density where there is a bulge in the field, there will be a net flux along the field lines towards the nearest point of escape. Scattering by inhomogeneities and the collisions described by Fermi will contribute to the flux in the opposite direction. Apparatus should detect a minimum flux when directed along the lines of force in a direction leading to the nearest escape point; it should detect an intermediate flux for directions at right angles to the lines of force; and should detect the maximum flux when directed along the lines of force away from the nearest escape point.

A further source of anisotropy is the fact that the cosmic-ray energy density should decrease, together with the magnetic field strength as described above, as one moves out from the axis of a spiral arm of the galaxy. This could be due to a decrease in the rate of acceleration of cosmic rays by Fermi's mechanism as the field strength decreases. It could be contributed to by diffusion normal to the lines of force caused by inhomogeneities in the field. Since particles would escape into nearly field free intergalactic space at the surface of the spiral arm, the density would decrease from the center out. The very high-energy particles with large radii of curvature should be more inhomogeneous than lower-energy particles. In the absence of all other reasons for a decrease in cosmic-ray density, the decrease in magnetic field strength will require it, since as soon as the source of cosmic rays builds up their energy density in a region to the point where the force they exert on the field is greater than that of gravity on the interstellar gas, the magnetic field will be pushed outward. This decreases the density of particles and decreases the energy of each one by the betatron effect. If for any of these reasons there is a variation in cosmic-ray density as one moves in a plane normal to the lines of force, there will be a corresponding anisotropy observed in the cosmic-ray flux. The flux observed along the lines of force and in the directions in which the density changes most rapidly will have an intermediate value. The maximum and minimum fluxes, as shown in Fig. 1, will be measured along the line orthogonal to both these directions.

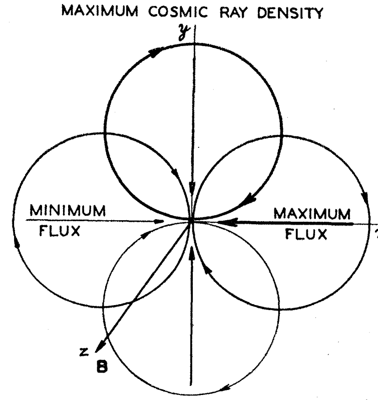

FIG. 1. Anisotropy in flux due to inhomogeneity in density. The circles are projections on the xy plane of helices; their widths indicate the number of particles involved. The straight arrows indicate the flux in the different directions.

[14] L. Davis, Phys. Rev. **93**, 947 (1954).

[14a] The author is indebted to Dr. Schlüter for pointing out that, as a consequence of the theorem that the magnetic moment of a charged particle in a magnetic field is an adiabatic invariant for changes in the field both with time and with position, the betatron mechanism has the same effect on the isotropy as Fermi's collision mechanism.

III. DESCRIPTION OF THE ANISOTROPY

The anisotropy produced in these various ways will now be described analytically. Introduce Cartesian coordinates with origin at the point of interest, for us the solar system. Take Oz along $\mathbf{B}$, the direction of the locally uniform magnetic field, and Oy in the direction in which the cosmic ray density increases most rapidly. Each particle describes a helix about a line parallel to Oz. Six quantities are required to completely specify the motion of a particle. Let them be R, β, ψ, x_A, y_A, t_A; where, as shown in Fig. 2, x_A, y_A are the coordinates of the point A at which the axis of the helix intersects the xy plane, β is the constant angle, called the helix angle, between $\mathbf{B}$ and the tangent to the trajectory, $R\sin\beta$ is the radius of the cylinder on which the helix is wound, $\psi-\frac{1}{2}\pi$ is the azimuthal angle of the line from A to the point P at which the trajectory intersects the xy plane, and t_A is the time at which the particle in question crosses the plane. Thus R, which is the radius of the circle in which the particle moves when $\beta=90°$, fixes the kinetic energy E of a particle of given charge and rest mass. If one measures the flux of cosmic radiation as a function of direction, β and ψ are the polar and azimuthal angles, respectively, of the axis of the narrow cone within which the apparatus receives the particles. They locate $-\mathbf{v}$, where $\mathbf{v}$ is the velocity of the particle.

A complete description of the cosmic-ray flux is given by defining

$$N(R,\beta,\psi,x_A,y_A,t_A)\sin\beta\cos\beta dRd\beta d\psi dx_A dy_A dt_A \quad (1)$$

to be the number of particles with parameters in the indicated range, dt_A being taken large enough so that N can be regarded as a continuous function. The trigonometric factors are chosen to make N a constant in the homogeneous isotropic case. In the steady state

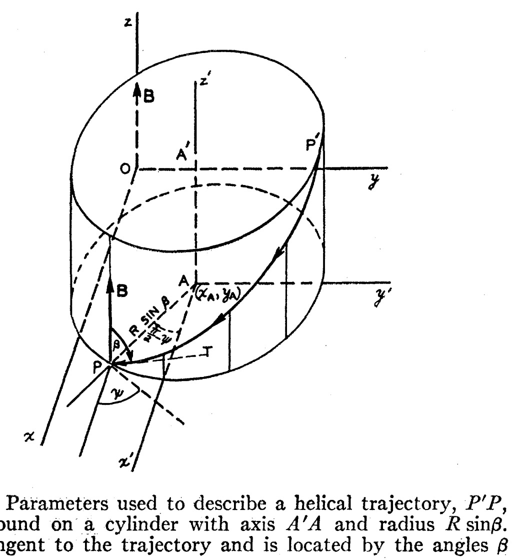

FIG. 2. Parameters used to describe a helical trajectory, $P'P$, shown wound on a cylinder with axis $A'A$ and radius $R\sin\beta$. PT is tangent to the trajectory and is located by the angles β and ψ.

N is not a function of t_A. The axes were oriented so that N is not a function of x_A. Strictly this might be impossible for more than one energy; assume it is possible for all energies of interest. Assume that N is independent of ψ; this is equivalent to the assumption that there would be no periodic variation if O were displaced along $\mathbf{B}$. Finally, require that P fall on O so that

$$y_A=R\sin\beta\cos\psi. \quad (2)$$

Thus $N(R,\beta,y_A)$ can be expanded in the series

$$\begin{aligned} N(R,\beta,y_A)&=\Sigma b_{kn}(R)y_A{}^kP_n(\cos\beta)\\ &=\Sigma b_{kn}(R)R^k(\sin\beta\cos\psi)^kP_n(\cos\beta), \end{aligned} \quad (3)$$

where P_n is the usual Legendre polynomial.

The analysis is started with a definition of N since it is what will be determined by any theory of the accelerating process and of the interactions with the inhomogeneities in the galactic magnetic field. However, the quantity of experimental interest is the value at the position O and the time t of $F(E,\beta,\psi,t)dE$, the number of particles in the energy range dE received per unit time, solid angle, and area normal to the beam in a narrow cone whose axis is directed along β,ψ. The connection between F and a completely general N may be found to be

$$\begin{aligned} &F(E,\beta,\psi,t)\\ &=(dR/dE)N(R,\beta,\psi,-R\sin\beta\sin\psi,R\sin\beta\cos\psi,t), \end{aligned} \quad (4)$$

either by evaluation of the Jacobian of the coordinate transformation or by consideration of the geometrical factors involved. The factor dR/dE in (4) allows for the change from R to E. The factor $\sin\beta d\beta d\psi$ in (1) measures the solid angle, and the factor $\cos\beta$ in (1) allows for the fact that for F area is taken normal to the beam while for N it is taken in the xy plane. For an N which, as discussed above, depends only on its first, second, and fifth arguments and which can be expanded in the series (3), F is easily expressed in a similar series. If only low-order terms need be considered at present, the series can be written

$$\begin{aligned} F(E,\beta,\psi)=\bar{F}(E)[1+\Delta_D\cos\beta\\ -(4/3)\Delta_AP_2(\cos\beta)+\Delta_I\sin\beta\cos\psi], \end{aligned} \quad (5)$$

where

$$\begin{aligned} \bar{F}(E)=(dR/dE)b_{00},\quad \Delta_D=b_{01}/b_{00},\\ \Delta_A=-3b_{02}/4b_{00},\quad \Delta_I=Rb_{10}/b_{00}. \end{aligned} \quad (6)$$

Throughout the rest of this paper all terms arising from those given in (5) will be retained but all higher-order terms omitted in (5) will be ignored. No attempt will be made here to deduce theoretical values of the Δ's; they will be regarded as parameters to be determined by experiment.

The new coefficients, which are evidently functions of E, are defined as in (6) in order to have simple physical meanings. The identification of the terms in the series with the different sources of anisotropy is easily made from the symmetry properties of each. $\bar{F}$ is essentially the mean cosmic ray flux averaged over all directions. [To make this hold precisely when more terms in the series are used, one must replace $(y_A/R)^k$ in (3) by $T_k(\sin\beta\cos\psi)$, where T_k is the Tschebyscheff polynomial of the first kind.[15]] The term in Δ_A gives the main contribution to the anisotropy resulting from an acceleration uniformly distributed throughout the spiral arms of the galaxy. The sign is so chosen that Δ_A is positive if the likelihood of greater acceleration from the flatter helices is more important than the increase in steepness produced by acceleration. In the reverse case, Δ_A is negative. The term in Δ_D gives the main contribution to the anisotropy resulting from any net diffusion along a line of force. The coefficient Δ_D will be positive when the Oz axis from which β is measured is directed toward the region where most of the cosmic rays originate and away from the nearest region where they escape from the galaxy. The term Δ_I gives the main contribution of any inhomogeneity in the cosmic ray density; its coefficient is positive for particles having a positive charge if Oz is directed along $\mathbf{B}$ and Oy along the direction in which the density increases most rapidly. In each case the numerical factor is so chosen that 2Δ is the maximum possible variation with direction in $F/\bar{F}$ due to the term in question.

If observation yields values of these coefficients, it will be important to know to what extent each coefficient arises solely from the indicated mechanism. Terms having the symmetry of those in Δ_D and Δ_I can arise from the acceleration mechanism only if there is some inhomogeneity, such as a limited region of vigorous acceleration, and in this case interpretation of Δ_D and Δ_I as due to diffusion and inhomogeneity is still justified. Diffusion will tend to produce mostly odd harmonics; hence it would not be expected to contribute much to the coefficient of $P_2(\cos\beta)$. If the density of cosmic rays varies reasonably smoothly over distances of the order of the radius of a spiral arm, one might expect that in (3) quadratic terms in y_A would be less important than linear terms until R approached the radius of the spiral arm. Thus neither diffusion nor inhomogeneity should produce terms having the symmetry of P_2, and if such terms are found, they must be due to acceleration. Diffusion contributes terms in $P_n(\cos\beta)$, i.e., in powers of the cosine of the angle between the axis of the detecting apparatus and $\mathbf{B}$. Inhomogeneity contributes terms in $(\sin\beta\cos\psi)^k$; i.e., in powers of the cosine of the angle between the axis of the detecting apparatus and $\mathbf{e}_T$, a transverse unit vector along Ox that may be defined independently of coordinate systems as being in the direction of $(\text{grad } N)\times\mathbf{B}$ for positive particles. Thus it seems impossible to distinguish observationally between anisotropy due to diffusion along the lines of force and that due to inhomogeneity at right angles to the lines unless one has some other means of determining the direction of $\mathbf{B}$ or of $\mathbf{e}_T$. Of course, from one point of view there is very little difference between the two sources of anisotropy since a difference in cosmic-ray energy density will be associated with each of them.

[15] W. Magnus and F. Oberhettinger, *Special Functions of Mathematical Physics* (Chelsea, New York, 1949), p. 78.

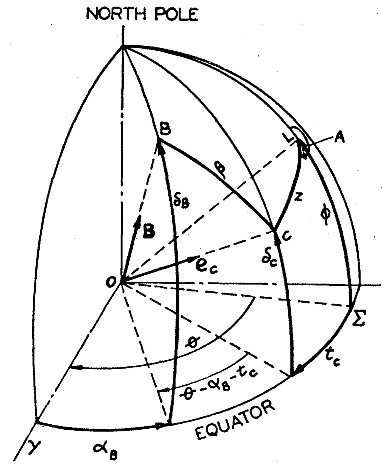

FIG. 3. The celestial sphere seen from the outside. O is the center of the earth, OL is in the direction of the radius vector to the laboratory, OB is the direction of $\mathbf{B}$, OC is the direction of the axis of the apparatus, and γ is the vernal equinox. As the earth rotates, γ and B remain fixed, C, L, and Σ move together to the right.

IV. SIDEREAL TIME DEPENDENCE

This anisotropy in cosmic ray flux means that any apparatus that rotates with the earth will have an output that varies with sidereal time provided adequate corrections for atmospheric effects can be made and provided sufficient data are available to make the statistical fluctuations small enough. To connect the above description of the anisotropy with expressions for the output of the apparatus it is necessary to use the usual astronomical coordinate systems as shown in Fig. 3. Let α_B and δ_B be the right ascension and declination, respectively, of $\mathbf{B}$ and let α_T and δ_T be those of $\mathbf{e}_T$. Let $\mathbf{e}_C$, the axis of the narrow cone in which the detecting apparatus measures the flux of incoming cosmic rays, be located by t_C and δ_C, the local hour angle and declination, respectively. The averaging procedures to be used when the cone is not narrow will be considered below. In terms of the more convenient zenith angle z, (angle from the vertical to $\mathbf{e}_C$) and azimuth, A, (measured eastward from the northern horizon), t_C and δ_C are given by

$$\begin{aligned} \sin\delta_C &= \sin\varphi\cos z + \cos\varphi\sin z\cos A, \\ \cos\delta_C\cos t_C &= \cos\varphi\cos z - \sin\varphi\sin z\cos A, \\ \cos\delta_C\sin t_C &= -\sin z\sin A, \end{aligned} \qquad (7)$$

where φ is the latitude of the observer. For vertical observation, $z=0$, $\delta_C=\varphi$, and t_C may be taken to be zero. It may be seen from Fig. 3 that, in general,

$$\cos\beta=\sin\delta_B\sin\delta_C+\cos\delta_B\cos\delta_C\cos(\theta-\alpha_B-t_C), \quad (8)$$

where θ is the local sidereal time. In a corresponding way

$$\sin\beta\cos\psi=\mathbf{e}_C\cdot\mathbf{e}_T$$
$$=\sin\delta_T\sin\delta_C+\cos\delta_T\cos\delta_C\cos(\theta-\alpha_T-t_C). \quad (9)$$

When these are put into (5), the result can be simplified to give the observed flux as a function of local sidereal time.

$$F(E,\theta)=\bar{F}(E)\{1-3\Delta_A(\sin^2\delta_B-\tfrac{1}{3})(\sin^2\delta_C-\tfrac{1}{3})$$
$$+(\Delta_I\sin\delta_T+\Delta_D\sin\delta_B)\sin\delta_C$$
$$-\Delta_A[\sin2\delta_B\sin2\delta_C\cos(\theta-\alpha_B-t_C)$$
$$+\cos^2\delta_B\cos^2\delta_C\cos2(\theta-\alpha_B-t_C)]$$
$$+\Delta_D\cos\delta_B\cos\delta_C\cos(\theta-\alpha_B-t_C)$$
$$+\Delta_I\cos\delta_T\cos\delta_C\cos(\theta-\alpha_T-t_C)\}. \quad (10)$$

In general, what is observed is the cosmic-ray flux in a wide cone and a wide energy range. Accordingly the flux at the top of the atmosphere given by (10) must be multiplied by $S(E,z,A)$, which can be regarded as the effective area of response to primaries of energy E coming from a direction defined by z, A, and the result integrated over A, z, and E. Since this effective area depends in a complicated way on the geometry of the experimental apparatus, its depth in the atmosphere, and the structure of the shower produced by the primary particle, only a partial treatment will be attempted. Assume that the apparatus detects coincidences in an array of some kind of counters in a horizontal plane. Suppose it to have a horizontal axis of symmetry whose azimuth measured eastward from north is A_S. Then since rotation of the apparatus through 180° about a vertical axis should not change the counting rate, the effective area is of the form

$$S(E,z,A)=\sum S_n(E,z)\cos2n(A-A_S). \quad (11)$$

The predicted mean rate of response when statistical fluctuations are averaged out is therefore

$$\Phi(\theta)=\int\int\int S(E,z,A)F(E,\theta)\sin z\,dA\,dz\,dE. \quad (12)$$

Substitute (11), (10), and (7) in (12), introduce the abbreviation

$$M_n(E,z)=2\pi S_n(E,z)\bar{F}(E)\sin z, \quad (13)$$

and evaluate the integrals over A to get

$$\Phi(\theta)=\int\int M_0[1+(\Delta_D\sin\delta_B+\Delta_I\sin\delta_T)\sin\varphi\cos z-3\Delta_A(\sin^2\delta_B-\tfrac{1}{3})(1-\tfrac{3}{2}\sin^2 z)(\sin^2\varphi-\tfrac{1}{3})]dz\,dE$$
$$-\left[\int\int M_0\Delta_A(1-\tfrac{3}{2}\sin^2 z)dz\,dE\right][\sin2\delta_B\sin2\varphi\cos(\theta-\alpha_B)+\cos^2\delta_B\cos^2\varphi\cos2(\theta-\alpha_B)]$$
$$+\left[\int\int M_0\Delta_D\cos z\,dz\,dE\right]\cos\delta_B\cos\varphi\cos(\theta-\alpha_B)+\left[\int\int M_0\Delta_I\cos z\,dz\,dE\right]\cos\delta_T\cos\varphi\cos(\theta-\alpha_T)$$
$$+\frac{1}{4}\left[\int\int M_1\Delta_A\sin^2 z\,dz\,dE\right][-(3\sin^2\delta_B-1)\cos^2\varphi\cos2A_S+\sin2\delta_B\sin2\varphi\cos2A_S\cos(\theta-\alpha_B)$$
$$-\cos^2\delta_B(1+\sin^2\varphi)\cos2A_S\cos2(\theta-\alpha_B)+2\sin2\delta_B\cos\varphi\sin2A_S\sin(\theta-\alpha_B)$$
$$-2\cos^2\delta_B\sin\varphi\sin2A_S\sin2(\theta-\alpha_B)]. \quad (14)$$

It is of interest to note that only the first two terms of (11) affect the result, and that the only second harmonics are those due to Δ_A. Even when the counters lie on a single straight line, no second harmonics are introduced by Δ_D or Δ_I.

The above treatment omits one factor that could be important. $S(E,z,A)$ depends on solar time because of variations in the state of the atmosphere. It is well known that if the corrections for this are not adequate, S will also have first and second harmonics on sidereal time, even when averaged over a year. This is because of seasonal effects. Hence observation of first and second harmonics on sidereal time of the cosmic-ray flux implies anisotropy only if the corrections for atmospheric effects are known to be adequate or if a differential method of observing is used that cancels out atmospheric effects.

V. INTERPRETATION OF OBSERVATIONS

There are several types of observations that can be made to test this theory and perhaps to obtain information on the galactic magnetic field and the origin of cosmic rays. One procedure is to determine the first and second harmonics on sidereal time of the flux of nearly vertical extensive showers, correcting for atmospheric effects, which have periods of a solar day and a year. If this is done with identical apparatus at different latitudes so that the dependence on φ can be used to partially disentangle the terms in (14), it is possible in principle to determine the values of α_B, δ_B, an integral involving Δ_A, and two integrals involving the remaining quantities. Variation of the range of z, the zenith angle, over which the apparatus responded might be helpful. From shower theory one could get $S(E,z,A)$ and by examination of the integrals, one might get some estimate of the Δ's.

A second procedure, that would largely eliminate the need for corrections for atmospheric effects, would be to make simultaneous continuous measurements of the flux in two or three narrow cones for which z was the same but for which δ_C, as given by (7), had quite different values. A particularly simple case would be that in which $\varphi=z=45°$, $\delta_C=0$, 45°, and 90°. In principle, measurement of the constant term and the first and second harmonics on sidereal time of the flux differences for three values of δ_C would determine Δ_A, α_B, δ_B, and three combinations of the other terms from which it is not quite possible to determine Δ_D, Δ_I, α_T, and δ_T. Observations in more directions or at more latitudes do not provide new algebraic equations; they just improve the statistical accuracy of the information obtained from three directions. When it is stated here that Δ_A is determined, it is meant of course that an integral over E of Δ_A times $S(E,z,A)$ as given by shower theory is determined. The above-mentioned constant terms alone, that is, the differences in the average values of the fluxes for three directions, would give the values of $\Delta_A(\sin^2\delta_B-\frac{1}{3})$ and $(\Delta_D \sin\delta_B + \Delta_D \sin\delta_I)$, information that would be very significant if it were not that $\sin^2\delta_B$ is expected to be about $\frac{1}{3}$.

A third procedure that would give data at a more rapid rate than the second and that might be less difficult to carry out would be to detect extensive showers for all azimuths and over as wide a range of z as possible, preferably at mountain top elevations where oblique showers are more common, and to record the direction of each shower either with cloud chambers, by accurate timing measurements, or perhaps by other means.

Sufficient experimental data do not seem to be available for any of these methods to be used at present. The procedure that will now be followed is to take the few astronomical hints available and estimate the directions of **B** and $\mathbf{e}_T$. By using this as a working hypothesis, to be modified or discarded if the results are unsatisfactory, Eqs. (10) and (14) will be tested for consistency with the available observational results.

As discussed in Sec. I, the galactic magnetic field is likely to run along the spiral arms of the galaxy with local deviations of perhaps 5° or 10°. Astronomical evidence[16] indicates that the sun is within a spiral arm, but not on its axis, and that in our neighborhood this spiral arm runs in a direction whose galactic coordinates are $l=40°$, $b=0°$ and hence[17] whose right ascension and declination are

$$\alpha_B=20 \text{ hr}, \quad \delta_B=35°. \tag{15}$$

Following the spiral arm in this direction leads towards the center of the galaxy. This estimate of the direction of **B** is corroborated by the observation[18] that near this direction not many stars show interstellar polarization, and those that are polarized have essentially random planes of vibration as would be expected[7] if one looked along the lines of force and observed the component of **B** normal to the line of sight due to small waves in **B**. Although the spiral arm has considerable structure if one looks at stars and dust, it may not have as much structure if one looks at **B** and cosmic rays. Assuming this, one can make a very tentative statement as to the direction of $\mathbf{e}_T$. Assume that the spiral arm is roughly elliptical in cross section, with major axis extending 3000 light years in the galactic plane and minor axis 1000 light years normal to this plane. The sun is, say, 50 light years north (on the side of the north galactic pole) of the plane and 1000 light years from the axis of the spiral arm. One might therefore expect that $\mathbf{e}_T$ would be tangent at the sun to an ellipse whose plane is normal to the axis of the spiral arm and which is similar to the cross section of the arm. Thus it should be within about 10° of the direction $b=90°$, which is the same as

$$\alpha_T=12.3 \text{ hr}, \quad \delta_T=28°. \tag{16}$$

Since nothing is known as to the sense of **B**, $\mathbf{e}_T$ is just as likely to have the opposite direction, which is

$$\alpha_T=0.3 \text{ hr}, \quad \delta_T=-28°. \tag{17}$$

Without a knowledge of the $\Delta(E)$'s and of $S(R,z,A)$ it is impossible to evaluate the integrals in (14). However, a reasonable estimate for comparison with experiment can be obtained by assuming that S is zero except for z near zero and that $S\Delta$ is zero except in a narrow band of energy. Then the relative amplitudes and the phases of the various harmonics can be predicted with the results shown in the bottom half of Table I, where it must be remembered that the Δ's may depend on E. These results can be compared with the experimental results—shown in the top half of the table—obtained by Cranshaw and Galbraith[19]

[16] The structure described here is an idealized model mainly based on information supplied by Professors Guido Münch and J. L. Greenstein. Any errors made in the interpretation of the information they supplied seems likely to be less important than the other uncertainties involved.

[17] Landolt-Bornstein, *Zahlenwerte und Funktionen* (Springer, Berlin, 1952), sixth edition, Vol. 3.

[18] W. Hiltner, Astrophys. J. **114**, 241 (1951), Fig. 7b.

[19] T. E. Cranshaw and W. Galbraith (to be published).

TABLE I. Comparison of experimental and theoretical variations in the rate of extensive air showers, expressed as Fourier series with fundamental period one sidereal day. The theoretical values are based on (15) and (16).

Location	Harwell[d]		Pic du Midi[e]		Aukland[f]	
Latitude, φ	52°		43°		−37°	
Harmonic	1st	2nd	1st	2nd	1st	2nd
Experimental results						
Energy of primary (ev)	5×10^{16}		6×10^{14}		10^{14}	
Amplitude (%)	4.9 ± 1.5[a]	3.5 ± 1.5	1.5 ± 0.75	Present	1.43 ± 0.38[b]	0.41 ± 0.25[b]
Time of maximum (hr)[c]	10.5 ± 1.3	4.5 ± 1.0	21.0		17.6 ± 1.0	2.0 ± 2.3
Theoretical results						
Effects of acceleration						
Amplitude (%)	$0.91\Delta_A$	$0.25\Delta_A$	$0.94\Delta_A$	$0.36\Delta_A$	$0.90\Delta_A$	$0.43\Delta_A$
Time of maximum (hr)	8	2	8	2	20	2
Effects of diffusion						
Amplitude (%)	$0.51\Delta_D$	Absent	$0.60\Delta_D$	Absent	$0.65\Delta_D$	Absent
Time of maximum (hr)	20		20		20	
Effects of inhomogeneity						
Amplitude (%)	$0.54\Delta_I$	Absent	$0.65\Delta_I$	Absent	$0.70\Delta_I$	Absent
Time of maximum (hr)	12		12		12	

[a] All errors given are standard deviations.
[b] The second harmonic does not appear to have been corrected for a seasonal effect that changed the first harmonic from an amplitude of (1.10 ± 0.26) percent with maximum at (19.8 ± 0.9) hr to the values given. The method of correction used seems to be an analytically convenient development of the procedure of J. L. Thompson, Phys. Rev. 55, 11 (1939). Its basic assumption is that there is no annual variation in the phase of the diurnal component on solar time of the atmospheric effect. This assumption is known to be false for barometric pressure and hence presumably is false for other atmospheric effects. However, if no other correction for such effects is made, this correction appears to be desirable since the seasonal variation in their phase is probably less important than the variation in amplitude. The residual error seems likely to be of the same order of magnitude as the correction.
[c] All times are local sidereal time.
[d] See reference 19.
[e] See reference 20.
[f] See reference 21.

at Harwell, by Daudin and Daudin[20] at Pic du Midi, and by Farley and Storey[21] at Auckland. The energies listed are the approximate energies of the primaries that produce most of the showers recorded. Similar results, but with lower statistical significance, are obtained at neighboring energies by Cranshaw and Galbraith and by Daudin and Daudin. It is at once apparent that the standard deviations in the experimental results are small enough so that they must be taken seriously, but are not quite small enough, particularly if one considers the uncertainties introduced by the atmospheric diurnal effects, to make a conclusive test of the theory. The results of Cranshaw and Galbraith and those of Farley and Storey could be explained as due exclusively to acceleration with $\Delta_A>0$ which implies that the preferential acceleration from the flatter helices dominates. They could also be explained as a mixture of roughly equal parts of acceleration and diffusion since this would augment the first harmonic relative to the second in the Southern Hemisphere and decrease it in the Northern. Unless the Δ's vary strongly with energy, there seems to be no way to explain simultaneously the three sets of experimental results, mainly because the phases obtained at Harwell seem inconsistent with those from Pic du Midi. The results of Daudin and Daudin taken together with those of Farley and Storey would imply that the dominant effect is that of diffusion if it were not that then the second harmonics are hard to explain. Of course they could be due to atmospheric effects or to large even spherical harmonics produced by diffusion. It will be noted that if the direction of $\mathbf{e}_T$ is given by (16), the resulting phases in the last row of the table agree rather poorly with the observations. If the direction is given by (17), the amplitudes are unchanged but the times of maximum are advanced 12 hr, which does not improve the agreement. If the increased pitch on acceleration dominates the production of anisotropy, i.e., if $\Delta_A<0$, and if α_B were 8 hr rather than the value given in (15), the only change in the table would be that the second harmonics would have their maxima at 8 hr and 20 hr local sideral time. Since the observations seem to indicate that it is the minima rather than the maxima of the two harmonics that coincide, and since this is an astronomically implausible direction for **B**, it appears unlikely that $\Delta_A<0$.

It appears, therefore, that until the statistical effects in the observational data are further reduced, until corrections for atmospheric effects are known with considerable assurance, and until the allowances to be made for the variation in the energy selected at the different laboratories are settled, it will be difficult to decide with confidence whether or not the theory described above fits the observations and to determine which factors dominate in the production of anisotropy. If it were not for the results of Daudin and Daudin, one would say that at present the observations tend to support the above theory and to favor the assumption that the dominant term is a positive Δ_A, or perhaps

[20] A. Daudin and J. Daudin, Proceedings of the Bagneres-de Bigorre Conference, July, 1953 (unpublished).
[21] F. J. M. Farley and J. R. Storey, Nature **173**, 445 (1954); also (private communication).

an equal mixture of Δ_A and Δ_D. If this is the case, it can be regarded as evidence favoring a relatively uniform galactic magnetic field directed along a spiral arm of the galaxy. The field strength could not be an order of magnitude less than 10^{-5} gauss or there would be more anisotropy in the highest energy particles observed by Cranshaw and Galbraith. It is also implied then that cosmic rays are accelerated by the Fermi mechanism rather than exclusively by processes taking place at the original ion sources.

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Age-Dependent Branching Stochastic Processes in Cascade Theory*

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A brief introduction to the recent Bellman-Harris theory of branching stochastic processes is given in the nomenclature of cascade theory; and a simple model in cascade theory formulated as an age-dependent branching process is given.

INTRODUCTION

The theory of branching stochastic processes has been used on many occasions in the development of mathematical models of cascade phenomena (e.g., cosmic-ray showers, neutron multiplication, etc.).[1–6] Recently Bellman and Harris[7] have developed a theory of age-dependent branching processes which appears to have important applications in the physical and biological sciences. The purpose of this communication is twofold: first, to give a brief introduction to the Bellman-Harris theory in the nomenclature of cascade theory; and second, to present a simple model for the electron population of a cosmic-ray shower. The model considered is a modification of the Furry process.

In the Bellman-Harris theory the distance or thickness, say τ, travelled by a particle (electron, neutron, etc.) from its formation until it is transformed is a random variable with general distribution $G(\tau)$, $0<\tau<\infty$; i.e., $G(\tau)$ is the integral distribution for all paths of length less than or equal to τ. At the end of its path of travel the particle is transformed into n particles with probabilities q_n, $n=0, 1, \cdots$, each particle having the same distribution $G(\tau)$ for the distance it will travel before being transformed. For example, q_0 is the probability of absorption, q_1 is the probability that one new particle will be formed, the original one being absorbed, q_2 is the probability that two new particles will be formed (the original one being absorbed), etc. The random variable τ measures the distance to the next point of regeneration. We remark that the age-dependence is only for the total cross section, the branching ratio being age-independent.

The Bellman-Harris process is formulated as follows: Let $X(t)$ be an integer-valued random variable representing the number of particles at thickness t; and define $p(x,t)=Pr(X(t)=x)$, $x\geqslant 0$. Let

$$\pi(s,t)=\sum_{x=0}^{\infty} p(x,t)s^x, \quad |s|<1 \tag{1}$$

be the generating function for the probabilities $p(x,t)$ starting with one particle at thickness zero. $[\pi(s,t)]^n$ is the generating function if the process starts with $n>1$ particles at thickness zero. In treating both cases the assumption is made that the particles do not interact with one another. The generating function (1) has been shown to satisfy the nonlinear Stieltjes functional equation

$$\pi(s,t)=\int_0^t h[\pi(s,\,t-\tau)]dG(\tau)+s[1-G(t)], \tag{2}$$

where

$$h(s)=\sum_{n=0}^{\infty} q_n s^n, \tag{3}$$

that is, $h(s)$ is the generating function for the transformation probabilities q_n. The equation for the gener-

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[7] R. Bellman and T. E. Harris, Ann. Math. **55**, 280 (1953).