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Statistics of Electromagnetic Radiation Scattered by a Turbulent Medium*

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The theory of Villars and Weisskopf is used to calculate the univariate and bivariate amplitude distributions of electromagnetic radiation scattered from turbulent fluctuations. The univariate distribution is Rayleigh, and is in excellent agreement with measurements made on a 49.6-Mc/sec ionospheric scatter link. Using the bivariate distribution, we relate the amplitude correlation function to a velocity correlation function which appears in the von Weizsäcker-Heisenberg statistical theory of turbulence. In this way the theoretical velocity correlation can be compared with experiment.

I. INTRODUCTION

FOR some time both vhf ionospheric propagation and anomalous beyond-the-horizon propagation of uhf and microwaves have been attributed to scattering by refractive index fluctuations associated with turbulence.[1,2] A detailed model of the origin of these fluctuations has only recently been made available in the work of Villars and Weisskopf.[3] These authors have applied to the electromagnetic problem results of the statistical theory of homogeneous turbulence developed by von Weizsäcker[4] and Heisenberg.[5] According to this theory, pressure fluctuations in a compressible medium produce corresponding density fluctuations, which in turn produce fluctuations in refractive index and hence a scattered field. Under steady-state conditions, the distribution of turbulent energy in wave number can be derived from similarity considerations[4,5] and leads to a corresponding wave number dependence of the scattered field. The average scattered field is found to involve only one parameter pertaining to the turbulence, namely the amount of turbulent energy dissipated per unit volume and time [see Eq. (48) of reference 3].

In this paper we use the theory of Villars and Weisskopf to derive the amplitude distribution of the scattered radiation, which we then compare with experiment. We also derive a relation between the amplitude correlation function and a velocity correlation function appearing in the Heisenberg theory.

II. THEORETICAL AMPLITUDE DISTRIBUTION

Villars and Weisskopf have shown that the scattered power is proportional to the quantity

$$c(\mathbf{k},t)=\sum_{\mathbf{k}',\mathbf{k}''}(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}{}^t)(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}{}^t)(\mathbf{k}\cdot\mathbf{v}_{-\mathbf{k}''}{}^t)(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}''-\mathbf{k}}{}^t),\quad(1)$$

where $\mathbf{v}_{\mathbf{k}'}{}^t$ is the Fourier coefficient in the $\mathbf{k}'$ direction of the turbulent velocity field $\mathbf{v}(\mathbf{r},t)$. $\mathbf{k}$ is the difference between the incident and scattered propagation vectors. The time dependence of $\mathbf{v}_{\mathbf{k}'}{}^t$ is here indicated explicitly by a superscript. The allowed values of $\mathbf{k}'$ and $\mathbf{k}''$ are those corresponding to expansion in a cube of side L_0, the linear dimension of the largest eddy. To calculate the average received power, Villars and Weisskopf find the time average of Eq. (1) and obtain

$$\langle c(\mathbf{k},t)\rangle_{\text{Av}}=2\sum_{\mathbf{k}'}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}{}^t|^2\rangle_{\text{Av}}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}{}^t|^2\rangle_{\text{Av}}.\quad(2)$$

They use the basic assumption of Heisenberg[5] that the different Fourier coefficients are independent, in particular that

$$\langle\mathbf{v}_{\mathbf{k}'}{}^t\cdot\mathbf{v}_{-\mathbf{k}'+\mathbf{k}''}{}^t\rangle_{\text{Av}}=\delta_{0\mathbf{k}''}\langle\mathbf{v}_{\mathbf{k}'}{}^t\cdot\mathbf{v}_{-\mathbf{k}'}{}^t\rangle_{\text{Av}}=\delta_{0\mathbf{k}''}\langle|\mathbf{v}_{\mathbf{k}'}{}^t|^2\rangle_{\text{Av}}.\quad(3)$$

They then proceed to evaluate Eq. (2) [see Eq. (45) of reference 3].

The univariate distribution of the random process $c(\mathbf{k},t)$ (assumed stationary) is easily found. For example, the second moment of $c(\mathbf{k},t)$ is

$$\begin{aligned}\langle c^2(\mathbf{k},t)\rangle_{\text{Av}}=&\sum_{\mathbf{k}',\mathbf{k}'',\mathbf{k}''',\mathbf{k}''''}\langle(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}{}^t)\\&\times(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}{}^t)(\mathbf{k}\cdot\mathbf{v}_{-\mathbf{k}''}{}^t)(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}''-\mathbf{k}}{}^t)\\&\times(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'''}{}^t)(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'''}{}^t)\\&\times(\mathbf{k}\cdot\mathbf{v}_{-\mathbf{k}''''}{}^t)(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}''''-\mathbf{k}}{}^t)\rangle_{\text{Av}}.\end{aligned}\quad(4)$$

Applying Eq. (3), we see that

$$\begin{aligned}\langle c^2(\mathbf{k},t)\rangle_{\text{Av}}=&8\sum_{\mathbf{k}',\mathbf{k}''}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}{}^t|^2\rangle_{\text{Av}}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}{}^t|^2\rangle_{\text{Av}}\\&\times\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}''}{}^t|^2\rangle_{\text{Av}}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}''}{}^t|^2\rangle_{\text{Av}}\\=&8\Big[\sum_{\mathbf{k}'}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}{}^t|^2\rangle_{\text{Av}}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}{}^t|^2\rangle_{\text{Av}}\Big]^2\\=&2\langle c(\mathbf{k},t)\rangle_{\text{Av}}{}^2.\end{aligned}\quad(5)$$

The numerical coefficient in Eq. (5) is obtained by noting the number of combinations of the $\mathbf{k}''$s that are compatible with condition (3). Thus, for example, $\mathbf{k}''$ can equal $\mathbf{k}'$ or $\mathbf{k}-\mathbf{k}'$, but if $\mathbf{k}''=-\mathbf{k}''''$ we get a term which averages to zero. Similarly, we get for the general

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[1] H. G. Booker and W. E. Gordon, Proc. Inst. Radio Engrs. 38, 401 (1950).
[2] Bailey, Bateman Berkner, Booker, Montgomery, Purcell, Salisbury, and Wiesner, Phys. Rev. 86, 141 (1952).
[3] F. Villars and V. F. Weisskopf, Phys. Rev. 94, 232 (1954).
[4] C. F. v. Weizsäcker, Z. Physik 124, 614 (1948).
[5] W. Heisenberg, Z. Physik 124, 628 (1948).

moment

$$\langle c^n(\mathbf{k},t)\rangle_{\mathrm{Av}}=2^n n!\Big[\sum_{\mathbf{k}'}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}{}^t|^2\rangle_{\mathrm{Av}} \times\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}{}^t|^2\rangle_{\mathrm{Av}}\Big]^n=n!\langle c(\mathbf{k},t)\rangle_{\mathrm{Av}}{}^n. \quad (6)$$

These moments are identical with the moments of the χ^2 distribution with two degrees of freedom (i.e., the distribution of the sum of the squares of two independent Gaussian variables),[6] with the frequency function

$$f(x)=\frac{1}{2\sigma^2}\exp(-x/2\sigma^2), \quad x>0 \quad (7)$$

and moments

$$\langle x^n\rangle_{\mathrm{Av}}=\int_0^\infty x^n f(x)dx=n!(2\sigma^2)^n. \quad (8)$$

Thus, to the extent that Eq. (3) is valid, we have inferred the power distribution of the scattered radiation from the form of Eq. (1), which in turn can be traced back to the nonlinear term of the Navier-Stokes equation [see Eqs. (27) and (29) of reference 3]. The distribution of the signal amplitude (envelope) is that of $\sqrt{x}$, i.e., the Rayleigh distribution.

It is interesting to note that crude (nonhydrodynamical) models of the scattering mechanism, e.g., a cloud of "independent scatterers" with a Maxwell velocity distribution,[7,8] can also lead to a Rayleigh distribution of the received amplitude.

III. EXPERIMENTAL AMPLITUDE DISTRIBUTION

It has already been observed[2] that the fading of the received amplitude in a long-distance vhf transmission seems to follow a Rayleigh distribution. In that experiment, an averaging circuit with a time constant of 12 seconds (much greater than the average fading time[9]) was used.

We have measured the received amplitude distribution of a 49.6-Mc/sec continuous wave transmission from Cedar Rapids, Iowa, to South Dartmouth, Massachusetts (a distance of 1716 km), using a totalizer which records correctly varying signals with frequencies up to about 15 cps. The totalizer gives the amount of time the signal exceeds ten preset levels. These figures, when normalized, give directly ten points on a distribution curve.

The apparatus was run for one-minute intervals in a series of tests conducted in December, 1953. During the tests, the mean signal level was about 20 db above noise. Intervals during which a strong meteoric burst caused enhancement of the signal far out of the range of the distribution were not accepted. (Such meteoric enhancement is not considered in the above theory.) Figure 1 shows a Rayleigh distribution (solid curve) and the means of the totalizer readings from four typical runs, all taken within 15 minutes. Shown also at each point is a segment representing the unbiased estimate of the standard deviation of the sample.

It is clear from Fig. 1 that the observed sample fits a Rayleigh distribution very closely. Other samples give similar fittings. On the other hand, other simple positive distributions (e.g., the χ^2 distributions of one or three degrees of freedom) fail to fit the data satisfactorily.

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IV. THE AMPLITUDE CORRELATION FUNCTION

We obtain the joint distribution of the pair of random variables $\xi=c(\mathbf{k},t)$ and $\eta=c(\mathbf{k},t+\tau)$ by considerations entirely analogous to those used above to obtain the univariate distribution of ξ. As a first step, we calculate the moments $\langle\xi^r\eta^s\rangle_{\mathrm{Av}}$ by time-averaging products of the form of Eq. (1), some taken at time $t+\tau$. Using the independence of Fourier coefficients belonging to different wave numbers and the following generalization of Eq. (3),

$$\langle\mathbf{v}_{\mathbf{k}'}{}^t\cdot\mathbf{v}_{-\mathbf{k}'+\mathbf{k}''}{}^{t+\tau}\rangle_{\mathrm{Av}}=\delta_{0\mathbf{k}''}\langle\mathbf{v}_{\mathbf{k}'}{}^t\cdot\mathbf{v}_{-\mathbf{k}'}{}^{t+\tau}\rangle_{\mathrm{Av}}, \quad (9)$$

we find after considerable manipulation that

$$\alpha_{rs}\equiv\langle\xi^r\eta^s\rangle_{\mathrm{Av}}=2^{r+s}r!s!\sum_{l=0}^{\min(r,s)}\binom{r}{l}\binom{s}{l}\sigma_0{}^{r+s-2l}\sigma_\tau{}^{2l}, \quad (10)$$

where

$$\sigma_0=\sum_{\mathbf{k}'}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}{}^t|^2\rangle_{\mathrm{Av}}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}{}^t|^2\rangle_{\mathrm{Av}},$$

$$\sigma_\tau=\sum_{\mathbf{k}'}\langle(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}{}^t)(\mathbf{k}\cdot\mathbf{v}_{-\mathbf{k}'}{}^{t+\tau})\rangle_{\mathrm{Av}} \times\langle(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}{}^t)(\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'-\mathbf{k}}{}^{t+\tau})\rangle_{\mathrm{Av}}. \quad (11)$$

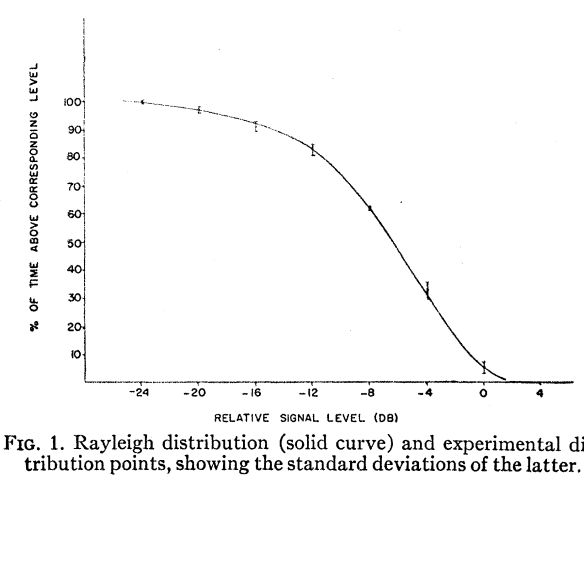

FIG. 1. Rayleigh distribution (solid curve) and experimental distribution points, showing the standard deviations of the latter.

[6] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1951), pp. 233–237.

[7] Booker, Ratcliffe, and Shinn, Trans. Roy Soc. (London) 242, 579 (1950).

[8] S. O. Rice, Proc. Inst. Radio Engrs. 41, 274 (1953).

[9] The simple consideration that an eddy is significantly changed in form after "moving its own length" (see reference 4) leads to correlation times of the order of one second.

The characteristic function[10] $\varphi(t,u)$ of the joint distribution is defined as the Fourier transform of the joint frequency function, i.e.,

$$\varphi(t,u)=\langle e^{i(\xi t+\eta u)}\rangle_{\mathrm{Av}}=\int\int_{-\infty}^{\infty} e^{i(\xi t+\eta u)}W_2(\xi,\eta)d\xi d\eta. \quad (12)$$

$W_2(\xi,\eta)d\xi d\eta$ is the probability that the received power at any time t is between ξ and $\xi+d\xi$ and that τ seconds later it is between η and $\eta+d\eta$. By comparing the expansion of $\varphi(t,u)$ in Taylor series with its expansion in moments [obtained from Eq. (12)], we find that

$$\alpha_{rs}=(-i)^{r+s}\left[\frac{\partial^r}{\partial t^r}\frac{\partial^s}{\partial u^s}\varphi(t,u)\right]_{t=u=0}. \quad (13)$$

It can be shown that

$$\varphi(t,u)=[1-2i\sigma_0 t-2i\sigma_0 u-4(\sigma_0^2-\sigma_\tau^2)tu]^{-1} \quad (14)$$

is the solution of Eqs. (10) and (13). Finally, $W_2(\xi,\eta)$ is obtained as the Fourier transform of Eq. (14), and is found to be (see Appendix)

$$W_2(\xi,\eta)=\frac{1}{4\sigma_0^2(1-\rho^2)}e^{-(\xi'+\eta')}I_0(2\rho(\xi'\eta')^{\frac{1}{2}}), \quad (15)$$

where $I_0(x)$ is the Bessel function of order zero and imaginary argument, and

$$\rho\equiv\rho_k(\tau)=\sigma_\tau/\sigma_0,$$

$$\xi'=\frac{\xi}{2\sigma_0(1-\rho^2)},\quad \eta'=\frac{\eta}{2\sigma_0(1-\rho^2)}.$$

The amplitude correlation function $A_k(\tau)$ is appropriately defined as

$$A_k(\tau)\equiv\frac{\langle(\xi\eta)^{\frac{1}{2}}\rangle_{\mathrm{Av}}-\langle\xi^{\frac{1}{2}}\rangle_{\mathrm{Av}}^2}{\langle\xi\rangle_{\mathrm{Av}}-\langle\xi^{\frac{1}{2}}\rangle_{\mathrm{Av}}^2}. \quad (16)$$

[Thus $A_k(0)=1$, since $\eta=\xi$ for $\tau=0$; and $A_k(\infty)=0$, since ξ and η become independent as $\tau\rightarrow\infty$.] Evaluating $\langle(\xi\eta)^{\frac{1}{2}}\rangle_{\mathrm{Av}}$ from

$$\langle(\xi\eta)^{\frac{1}{2}}\rangle_{\mathrm{Av}}=\int_0^\infty\int_0^\infty(\xi\eta)^{\frac{1}{2}}W_2(\xi,\eta)d\xi d\eta,$$

we find that

$$A_k(\tau)=\frac{\pi}{4-\pi}\left(\frac{1}{4}\rho_k^2(\tau)+\frac{1}{64}\rho_k^4(\tau)+\frac{1}{256}\rho_k^6(\tau)\cdots\right). \quad (17)$$

[10] H. Cramer, reference 6, pp. 265–6.

As in the case of the univariate distribution, Eq. (17) is the result predicted by the assumption of "independent scatterers."[11]

Heisenberg[5] calculates the quantity

$$\langle \mathbf{v}_{\mathbf{k}}^{t}\cdot\mathbf{v}_{-\mathbf{k}}^{t+\tau}\rangle_{\mathrm{Av}}/\langle \mathbf{v}_{\mathbf{k}}^{t}\cdot\mathbf{v}_{-\mathbf{k}}^{t}\rangle_{\mathrm{Av}}$$

and finds it to be a universal function $g(y)$ of the dimensionless variable $y=\frac{1}{6}v_0k_0^{\frac{1}{3}}k^{\frac{2}{3}}\tau$. As we have defined it, $\rho_k(\tau)$ is simply

$$\rho_k(\tau)=\frac{\sum_{\mathbf{k}'}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}^{t}|^2\rangle_{\mathrm{Av}}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}^{t}|^2\rangle_{\mathrm{Av}}\times g(\frac{1}{6}v_0k_0^{\frac{1}{3}}k'^{\frac{2}{3}}\tau)g(\frac{1}{6}v_0k_0^{\frac{1}{3}}|\mathbf{k}-\mathbf{k}'|^{\frac{2}{3}}\tau)}{\sum_{\mathbf{k}'}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}'}^{t}|^2\rangle_{\mathrm{Av}}\langle|\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}-\mathbf{k}'}^{t}|^2\rangle_{\mathrm{Av}}} \quad (18)$$

(k_0 is the wave number of the largest eddy, v_0 its mean velocity.) The denominator (aside from multiplicative constants) is just the mean power calculated by Villars and Weisskopf by an integration in bipolar coordinates. Given $g(y)$, the numerator can be calculated in exactly the same way. Heisenberg gives only a very approximate solution to the equations determining $g(y)$.[5] The authors hope to improve this approximation and then compare the $\rho_k(\tau)$ calculated from a suitable $g(y)$ with experiment.

APPENDIX

If we substitute

$$2\sigma_0 t=t'/(1-\rho^2),\quad 2\sigma_0 u=u'/(1-\rho^2), \quad (\text{A-1})$$

into the characteristic function [Eq. (14)], we get

$$\varphi(t,u)=\frac{1-\rho^2}{(1-it')(1-iu')-\rho^2}. \quad (\text{A-2})$$

The joint probability density is then

$$W_2(\xi,\eta)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\xi t+\eta u)}\varphi(t,u)dtdu$$

$$=\frac{1}{(2\pi)^2 4\sigma_0^2(1-\rho^2)}\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i(\xi' t'+\eta' u')}}{(1-it')(1-iu')-\rho^2}dt'du', \quad (\text{A-3})$$

where

$$\xi'=\frac{\xi}{2\sigma_0(1-\rho^2)},\quad \eta'=\frac{\eta}{2\sigma_0(1-\rho^2)}.$$

[11] D. E. Kerr, *Propagation of Short Radio Waves* (McGraw-Hill Book Company, Inc., New York, 1951), pp. 553–557.

Now,

$$\int_{-\infty}^{\infty}\frac{e^{-i\eta' u'}}{1-iu'-\rho^2/(1-it')}du' = 2\pi\exp\left[-\eta'+\eta'\frac{\rho^2}{1-it'}\right],\quad \eta'>0$$

$$=0,\quad \eta'<0\ (\text{since } \rho^2/(1+t^2)<1). \qquad \text{(A-4)}$$

Therefore,

$$W_2(\xi,\eta)=\frac{e^{-\eta'}}{8\pi\sigma_0^2(1-\rho^2)}\times\int_{-\infty}^{\infty}\frac{dt'}{1-it'}\exp\left[-i\xi't'+\eta'\frac{\rho^2}{1-it'}\right]. \qquad \text{(A-5)}$$

If $x=1-it'$,

$$\int_{-\infty}^{\infty}\frac{dt'}{1-it'}\exp\left[-i\xi't'+\eta'\frac{\rho^2}{1-it'}\right] = -ie^{-\xi'}\int_{1-i\infty}^{1+i\infty}\frac{dx}{x}\exp\left(\xi'x+\frac{\eta'\rho^2}{x}\right)$$

$$=2\pi e^{-\xi'}\sum_{m=0}^{\infty}\frac{1}{(m!)^2}(\xi'\eta'\rho^2)^m,$$

$$(\xi'>0,\ \text{by the Residue Theorem})$$

$$=2\pi e^{-\xi'}I_0(2\rho(\xi'\eta')^{\frac{1}{2}}), \qquad \text{(A-6)}$$

which when substituted into (A-5) gives the probability density, Eq. (15).

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λ Transition of Liquid Helium

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The partition function of liquid helium proposed by Feynman, $q=\Sigma g(L)\exp(-aTL)$, is calculated for a simple cubic lattice using an approximation corresponding to Bethe's method for the Ising model. It is shown that a second-order transition occurs at $aT_\lambda=\ln 4$, or $T_\lambda=2.9m/m'$ °K (m and m' representing the true and the effective masses of a helium atom). The nature of the approximation is discussed.

I. INTRODUCTION

THE series of papers on liquid helium which Feynman has recently published[1–3] explained many of the so far unsolved properties of this substance. However, the problem of the nature of the λ transformation still does not seem to have been settled. In his papers[1,2] Feynman proposed an expression for the partition function and solved it approximately to obtain a transition of a third order. As it is commonly accepted that the λ transformation of liquid helium is of a second order, Feynman[2] ascribed the disagreement between his results and experiment to the fact that he neglected the correlation among atoms, both in the same cyclic change and in different cyclic changes. Though Chester[4] agreed with Feynman with regard to this interpretation of the discrepancy, Rice,[5] Matsubara,[6] and ter Haar[7] expressed the view that the above neglect of the correlation is not the origin of the discrepancy.

[1] R. P. Feynman, Phys. Rev. **90**, 1116 (1953).
[2] R. P. Feynman, Phys. Rev. **91**, 1291 (1953).
[3] R. P. Feynman, Phys. Rev. **91**, 1301 (1953); Phys. Rev. **94**, 262 (1954).
[4] G. V. Chester, Phys. Rev. **93**, 1412 (1954).
[5] O. K. Rice, Phys. Rev. **93**, 1161 (1954).
[6] T. Matsubara, Busseiron Kenkyu **72**, 78 (1954).
[7] D. ter Haar, Phys. Rev. **95**, 895 (1954).

The purpose of the present paper is to show that a technique developed in the order-disorder theories can be applied to this problem to take into account the geometrical correlation, and, though the conclusion is not completely convincing (due to the approximate nature of the technique), that Feynman's partition function does give a second-order transition.

II. FREE ENERGY

The original expression for the partition function with which we begin is Eq. (7) of reference 2:

$$q=\int\sum_P\exp\left[-\frac{m'kT}{2\hbar^2}\sum_i(\mathbf{z}_i-P\mathbf{z}_i)^2\right]\times\rho(\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_N)d^N\mathbf{z}_i \qquad (1)$$

in which m' is the effective mass of a helium atom. For the derivation of this equation and the notation, readers are referred to the original paper by Feynman. When one assumes that the value of $\rho(\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_N)$ is nonvanishing only when the $\mathbf{z}$'s are located on a simple cubic lattice and that $(\mathbf{z}_i-P\mathbf{z}_i)^2$ is neglected except when its value is equal to d^2, d being the lattice constant of the hypothetical lattice, Eq. (1) reduces to Eq. (4)