# Model for Nuclear Reactions with Neutrons*

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A simple model is proposed for the description of the scattering and the compound nucleus formation by nucleons impinging upon complex nuclei. It is shown that, by making appropriate averages over resonances, an average problem can be defined which is referred to as the "gross-structure" problem. Solution of this problem permits the calculation of the average total cross section, the cross section for the formation of the compound nucleus, and the part of the elastic-scattering cross section which does not involve formation of the compound nucleus. Unambiguous definitions are given for the latter cross sections.

The model describing these properties consists in replacing the nucleus by a one-body potential which acts upon the incident nucleon. This potential $V=V_0+iV_1$ is complex; the real part represents the average potential in the nucleus; the imaginary part causes an absorption which describes the formation of the compound nucleus. As a first approximation a potential is used whose real part $V_0$ is a rectangular potential well and whose imaginary part is a constant fraction of the real part $V_1=\zeta V_0$.

This model is used to reproduce the total cross sections for neutrons, the angular dependence of the elastic scattering, and the cross section for the formation of the compound nucleus. It is shown that the average properties of neutron resonances, in particular the ratio of the neutron width to the level spacing, are connected with the gross-structure problem and can be predicted by this model.

The observed neutron total cross sections can be very well reproduced in the energy region between zero and 3 Mev with a well depth of 42 Mev, a factor $\zeta$ of 0.03, and a nuclear radius of $R=1.45\times10^{-13}A^{\frac{1}{3}}$ cm. The angular dependence of the scattering cross section at 1 Mev is fairly well reproduced by the same model. The theoretical and experimental values for the ratios of neutron width to level distance at low energies and the reaction cross sections at 1 Mev do not agree too well but they show a qualitative similarity.

## I. INTRODUCTION

THIS paper deals with the interaction of nuclear particles with complex nuclei in nuclear reactions. A model is proposed for the description of the energy exchange between the incoming particle and the target nucleus. The considerations are restricted to neutron reactions with incident energies between 0 and 20 Mev.

One usually describes the interaction of nuclear particles with complex nuclei by means of the concept of a compound nucleus which is formed after the nucleon has entered the nucleus. Before the striking success of the nuclear shell model was known, it was generally assumed that the quantum state formed by the particle entering the nucleus is one in which the motions of all particles are intimately coupled. We will refer to this assumption as the "strong-coupling model." These ideas led to certain general qualitative conclusions in regard to the cross sections for nuclear reactions. Several authors[1] have attempted in previous papers to derive approximate expressions for the cross sections of nuclear reactions with a minimum of special assumptions in addition to the main assumption of the validity of the strong-coupling model. We summarize the main results of these qualitative considerations.

*1. Particle widths.*—The particle widths of nuclear resonances with respect to particle emission are related in a general way to the average spacing $D$ of the levels of the compound nucleus. For example, the width for the emission of neutrons with zero-orbital angular momentum is given approximately by

$$\Gamma_n\approx(2/\pi)(k/K)D, \tag{1.1}$$

and the widths for the emission of other particles are equal to the above expression multiplied by the penetration factor of the potential barrier. Here $k$ is the wave number of the incoming particle, and $K$ is the wave number in the interior of the nucleus; $K$ is of the order of $10^{13}$ cm$^{-1}$.

*2. Potential scattering.*—The elastic scattering arises from a superposition of a resonance amplitude and a slowly varying potential scattering amplitude. The former is important only in the immediate vicinity of the resonance; the latter is equal to the scattering amplitude of an impenetrable sphere of a radius approximately equal to the nuclear radius.

*3. Neutron total cross section.*—The neutron total cross section averaged over resonances is equal to the total cross section of a spherical potential well whose depth is such as to give rise to an internal wave number $K\sim10^{13}$ cm$^{-1}$ and which possesses an absorption for the incoming waves such that the waves are absorbed inside within distances of the order $K^{-1}$. These conditions were expressed approximately by Feshbach and Weisskopf[2] in the form of a boundary condition on the incoming wave function $u/r$ at the nuclear boundary:

$$du/dr=-iKu.$$

Formula (1.1) and the other consequences of the strong-coupling model have been found correct as to the order of magnitude. However, as a consequence of point (3), the neutron total cross sections when averaged

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[1] H. Bethe, Phys. Rev. **57**, 1125 (1940); Feshbach, Peaslee, and Weisskopf, Phys. Rev. **71**, 145 (1947); H. Feshbach and V. F. Weisskopf, Phys. Rev. **76**, 1550 (1949); E. P. Wigner, Phys. Rev. **73**, 1002 (1948); E. P. Wigner, Am. J. Phys. **17**, 99 (1949).

[2] H. Feshbach and V. F. Weisskopf, Phys. Rev. **76**, 1550 (1949).

over resonances should all be smooth functions of the energy which decrease monotonically with increasing energy and whose form is rather similar for all atomic numbers $A$. Also the dependence on $A$ at constant energy should show a continuous, slowly increasing trend with increasing $A$. The measurements of neutron total cross sections by the Wisconsin group and by others have clearly demonstrated that this is not so. The neutron total cross sections exhibit typical deviations from the predictions of the strong-coupling model. (See Fig. 3.) The shape of the energy dependence of the neutron cross sections changes significantly over the range of $A$; however, this change is not random but gradual. Nuclei with small differences in $A$ show almost the same behavior. One concludes, therefore, that these characteristic shapes do not depend on detailed features of nuclear structure but on some general properties which vary slowly with $A$, say, the nuclear radius.

The success of the shell model has cast some doubt upon the fundamental assumptions of the strong-coupling model. Does the particle necessarily form a "compound state" after entering the nucleus? The shell structure furnishes much evidence that a nucleon can move freely within the nucleus without apparently changing the quantum state of the target nucleus. This is a consequence of observations made at the ground state and at low excitation energies, and it is questionable whether this apparent absence of interaction between one nucleon and the rest is valid also at those excitation energies ($\sim$8 Mev) which are created in nuclear reactions with neutrons of a few Mev. Furthermore, there is some reason to believe that at higher energies, 15 Mev and up, the interaction between the entering nucleon and the target is appreciable, since the reaction cross sections at those energies have been found[3] to be equal to the geometrical cross sections $[\pi(R+\lambda\!\!\!^{-})^2]$. Hence, for such energies it happens rarely that a neutron enters the nucleus and leaves it again without sharing its energy with the rest.

It seemed worth while, therefore, to investigate the consequences of a reduced interaction between the nucleons for the theory of nuclear reactions in the energy region of a few Mev. This reduced interaction will manifest itself in the following way: The incident nucleon can penetrate into the nucleus and move within the boundaries of the nucleus *without* forming a compound state. Hence, in this case, the target nucleus acts upon the incoming nucleon as a potential well. The actual formation of a compound state occurs only with a probability smaller than unity, once the particle has entered the nucleus. It has a finite chance of leaving the nucleus without having formed a state in which it has exchanged energy or momentum with the rest of the nucleus. The formation of the compound state then would have the aspect of an absorption. Hence, the effect of the nucleus upon the incident particle could be described as the effect of a potential well with absorption, where the absorption coefficient within the well would be an adjustable parameter. It is obvious that this description represents an oversimplification which naturally cannot reproduce all features of nuclear reactions. Specifically, it will not reproduce any resonance phenomena which are connected with the many possible quantum states of the compound system. We therefore expect that this model will at best describe only the features of nuclear reactions after averaging over the resonances of the compound nucleus.

The formulation of this attempt to construct a simple model for nuclear reactions requires a study of the definitions of the various cross sections; in particular, the meaning of the cross section for the formation of a compound nucleus must be clarified.

We introduce the following cross sections: $\sigma_t$, the total cross section, which can be split into

$$\sigma_t=\sigma_{el}+\sigma_r,$$

where $\sigma_{el}$ is the elastic scattering cross section, and $\sigma_r$ is the "reaction cross section." The former is defined as the cross section for scattering without change of the quantum state of the nucleus. The particle leaves by the same channel by which it has entered. The elastic scattering has an angular dependence which we express by the differential cross section $d\sigma_{el}/d\Omega$,

$$\sigma_{el}=\int\frac{d\sigma_{el}}{d\Omega}(\theta)d\Omega.$$

The reaction cross section includes all processes in which the residual nucleus is different from or in a state different from that of the target nucleus. These are all processes whose exit channels differ from the entrance channel. It will be practical later on to subdivide the elastic cross section into two parts:

$$\sigma_{el}=\sigma_{se}+\sigma_{ce}.$$

We call the second part, $\sigma_{ce}$, the "compound elastic" cross section. It is the part of the elastic scattering which comes from the formation of the compound nucleus and the subsequent emission of the incident particle into the entrance channel. The first part we call "shape elastic" cross section; this is the part of the elastic scattering which occurs without the formation of a compound. The exact definition of this split will be given in Sec. II. We note that such definitions will be possible only for the average cross sections, averaged over an energy interval containing many resonances, if such resonances are present.

On the basis of the compound nucleus assumption, we consider all actual reactions to occur after compound formation. Hence, we introduce a cross section $\sigma_c$ of compound nucleus formation:

$$\sigma_c=\sigma_{ce}+\sigma_r,$$

[3] Phillips, Davis, and Graves, Phys. Rev. 88, 600 (1952).

and obtain, naturally,

$$\sigma_t=\sigma_{se}+\sigma_c.$$

The nuclear model which we propose here is expected to predict only the cross sections $\sigma_{se}$ and $\sigma_c$. It considers only the conditions in the entrance channel, that is, in that part of the phase space in which the target nucleus is in its initial state. Hence, the compound nucleus formation is considered as an *absorption* of the incident beam, although part of it, namely $\sigma_{ce}$, leads to an elastic scattering process. The model consists in describing these conditions by means of a one-particle problem. The nucleus is replaced by a complex potential,

$$V=V_0+iV_1, \tag{1.2}$$

acting upon the incoming neutron. The scattering which the neutron suffers in (1.2) should reproduce the shape elastic scattering $\sigma_{se}$; and the absorption which is caused by the imaginary part $V_1$ should reproduce the compound nucleus formation.

It is probable that the potential functions in (1.2) vary somewhat with the incident energy. For example, one might expect an increase of the imaginary part with increasing energy. If an approximate description of the facts is possible by means of a potential (1.2), the shape of the potential will be indicative of the type of nuclear interaction which a neutron suffers in the nucleus. The real part $V_0$ would describe the average potential energy of the neutron within the nucleus, and its shape would give indications as to the form of the potential "well" inside the nucleus. It is similar to the potential encountered in the shell model of the nucleus, although we do not pretend that an incident neutron of several Mev is faced with exactly the same potential which acts upon the nucleus in the ground state. The imaginary part $V_1$ would indicate the strength and location of the processes that lead to an energy exchange between the incoming neutron and the target nucleus.

We expect the potential $V$ to depend in a simple way upon the mass number $A$. Its dependence on $r$ should be similar for all nuclei. The simplest choice would be a square-well potential:

$$V_0=-U \quad \text{for} \quad r<R,$$
$$V_0=0 \quad \text{for} \quad r>R,$$
$$V_1=\zeta V_0.$$

In general, we might express it in the form $V=V(r/R)$, $R=r_0A^{\frac{1}{3}}$. However, there might be a region near $r=R$ in which the features depend on $r$ and not on $(r/R)$; the thickness of that part of the potential which represents the surface might be independent of the radius.

With a given $V(r)$ and its dependence on $A$, it is possible to calculate the cross sections $\sigma_t$, $\sigma_{se}$, and $\sigma_c$, each as functions of energy and mass number, and also the angular dependence of the scattering. The next section contains the definitions of the cross sections involved, and the following sections describe the technique of calculating the cross sections and their comparison with experimental material.

## II. THEORY OF AVERAGE CROSS SECTIONS

All nuclear cross sections exhibit strong fluctuations with energy which are generally referred to as resonances, especially in the lower part of our energy range. As the energy increases, the width of the resonances increases too; and, for not too light nuclei, the width becomes comparable or larger than the level distance at energies above a few Mev. Hence, we find the cross sections at higher energies to be smooth functions of energy with little fluctuation. We will refer to the lower-energy region as the "resonance region" and the upper as the "continuum region."

The behavior of the cross sections in the resonance region does not lend itself to a description by a simple one-particle potential (1.2) because of the rapid fluctuations with energy. However, the averages of the cross sections taken over an interval $I$, which includes many resonances, will be shown to be the cross sections belonging to a new scattering problem with slowly varying phases, which we will call the "gross-structure" problem. In this problem it is possible to define cross sections for the formation of a compound nucleus which also includes the compound elastic scattering. *It is this gross-structure problem and not the actual rapidly varying cross sections which we intend to describe by means of a one-particle problem with the potential (1.2).*

We bombard a nucleus $X$ with particles $a$ and consider the total cross section $\sigma_t$, the elastic cross section $\sigma_{el}$, and the reaction cross section $\sigma_r$. $\sigma_t=\sigma_{el}+\sigma_r$. Each of these cross sections will be subdivided into their parts coming from different angular momenta $l$, e.g.,

$$\sigma_t=\sum_l \sigma_t^{(l)}. \tag{2.1}$$

These cross sections can be expressed in terms of the amplitudes of the wave which describes the situation in the entrance channel. We consider the subwave $u_l/r$ in the entrance channel with the orbital angular momentum $l$ ($r$ is the channel coordinate), and we write the wave in the form for

$$r\to\infty,$$
$$\varphi_l\to\text{const}[\exp(-i(kr-\tfrac{1}{2}l\pi)) - \eta_l\exp(+i(kr-\tfrac{1}{2}l\pi))]. \tag{2.2}$$

The complex reflection factor $\eta_l$ is connected with the complex phase shift $\varphi_l$ by $\eta_l=\exp(2i\varphi_l)$, and the cross sections are given by the well-known expressions for the elastic cross section:

$$\sigma_{el}^{(l)}=\pi\lambda^2(2l+1)|1-\eta_l|^2, \tag{2.3}$$

and for the reaction cross section,

$$\sigma_r^{(l)}=\pi\lambda\!\!\!^{-2}(2l+1)(1-|\eta_l|^2), \tag{2.3a}$$

where $\lambda\!\!\!^{-}$ is the wavelength of the incoming particle divided by $2\pi$.

The reflection factor $\eta_l$ is a complicated function of the energy of the incoming particle. It exhibits rapid fluctuations coming from the numerous close-spaced resonances of the compound nucleus. We will make the assumption that one can average over these fluctuations; that is, we assume that the average reflection factor,

$$\bar{\eta}_l(\epsilon)=\frac{1}{I}\int_{\epsilon-I/2}^{\epsilon+I/2}\eta_l(\epsilon')d\epsilon', \tag{2.4}$$

is a smooth function of $\epsilon$ if the interval $I$ contains many close-spaced resonances. We also define average cross sections in the same way, and we can write

$$\begin{aligned}\bar{\sigma}_{el}^{(l)}&=\pi\lambda\!\!\!^{-2}(2l+1)\overline{|1-\eta_l|^2},\\ \bar{\sigma}_r^{(l)}&=\pi\lambda\!\!\!^{-2}(2l+1)(1-\overline{|\eta_l|^2}),\end{aligned} \tag{2.5}$$

where the bar over an expression signifies its average over the interval $I$. It is also assumed that $I$ is much smaller than the energy $\epsilon$ such that slowly varying functions of $\epsilon$, like $\lambda\!\!\!^{-2}$, need not be averaged.

One can easily verify the following relations:

$$\bar{\sigma}_{el}^{(l)}=\pi\lambda\!\!\!^{-2}(2l+1)\{|1-\bar{\eta}_l|^2-|\bar{\eta}_l|^2+\overline{|\eta_l|^2}\}, \tag{2.6}$$

and especially

$$\bar{\sigma}_t^{(l)}=\pi\lambda\!\!\!^{-2}(2l+1)\{|1-\bar{\eta}_l|^2+1-|\bar{\eta}_l|^2\}. \tag{2.7}$$

Hence, the average total cross section depends only upon the average reflection factor (2.4). [This follows directly from the fact that the total cross section is a *linear* function of the real part of the phase $\eta_l$.]

We now divide the average elastic cross section into two parts, the "shape elastic" cross section $\sigma_{se}^{(l)}$ and the "compound elastic" cross section[4] $\sigma_{ce}^{(l)}$, by writing

$$\begin{aligned}\sigma_{se}^{(l)}&=\pi\lambda\!\!\!^{-2}(2l+1)|1-\bar{\eta}_l|^2,\\ \sigma_{ce}^{(l)}&=\pi\lambda\!\!\!^{-2}(2l+1)\{\overline{|\eta_l|^2}-|\bar{\eta}_l|^2\}.\end{aligned} \tag{2.8}$$

Furthermore, we combine $\sigma_{ce}^{(l)}$ and $\bar{\sigma}_r^{(l)}$ into a new cross section $\sigma_c^{(l)}$, which we call the cross section for the formation of the compound nucleus

$$\sigma_c^{(l)}=\sigma_{ce}^{(l)}+\bar{\sigma}_r^{(l)}=\pi\lambda\!\!\!^{-2}(2l+1)\{1-|\bar{\eta}_l|^2\}. \tag{2.9}$$

We can see from (2.3) and (2.3a) that $\sigma_{se}^{(l)}$ and $\sigma_c^{(l)}$ have just the form of a scattering and a reaction cross section of a new and different problem, whose phase is the slowly varying function $\bar{\eta}_l$. In other words, by replacing $\eta_l$ with $\bar{\eta}_l$, we obtain a new problem, which we have called the "gross-structure problem." The elastic scattering cross section $\sigma_{se}$ of *this* problem is only part of the actual scattering; it is the "shape elastic" scattering. The other part, the "compound elastic," appears incorporated into the absorption or reaction cross section $\sigma_c$ of the gross-structure problem together with the actual reaction cross section.

One is therefore led to consider the "compound elastic" scattering as that part which comes from the formation of the compound nucleus and its subsequent decay into the entrance channel, hence its incorporation into $\sigma_c$. After the averaging, $\sigma_{ce}$ appears as part of the absorption from the incoming beam, which corresponds to the idea that the formation of the compound nucleus can be considered as an absorption whatever happens afterwards, re-emission or not.

It is the gross structure problem which we intend to reproduce by the interaction of the incident particle with the potential (1.2). The resulting scattering cross section should represent the shape elastic scattering, and the resulting absorption cross section should represent the compound formation. The latter contains the part $\sigma_{ce}$ of the actual scattering.

When the energy is high enough above the resonance region that the continuum region is reached, the cross sections and phases are no longer rapidly varying functions of energy. Then the gross-structure problem is equal to the actual one and $\bar{\eta}_l=\eta_l$. It follows from (2.9) that $\sigma_{ce}=0$. One also can see this from an application of the compound nucleus assumption to the continuum region. The overlap of the resonances can be interpreted as a consequence of the fact that the probability $\Gamma_\alpha^s$ of the decay of the compound nucleus in the state $s$ into the entrance channel $\alpha$ is much smaller than the probability of the decay into other channels. This follows from the well-known relation that any channel width $\Gamma_\alpha^s$ cannot be larger than $D/2\pi$ ($D$ is the distance between resonances of the same $J$ value). Hence, if the total width is much larger than $D$, the contribution to $\Gamma$ from decays other than the one through $\alpha$ must be overwhelming. In the continuum region, therefore, the cross section for the formation of the compound nucleus is identical to the average reaction cross section $\bar{\sigma}_r$, and $\sigma_{ce}$ is negligible.

We now illustrate the averaging process described above by using cross sections as given by the Breit-Wigner formula. We consider a nucleus with resonances at the energies $\epsilon_s$, and we restrict our considerations to neutrons with $l=0$. We also restrict the discussion to low energies so that the following two magnitudes are small: One is $kR$ and the other is $\Gamma/D$, with $R$ the nuclear radius, and $\Gamma$ and $D$ the average values of the total width of and the distance between neutron resonances.

[4] This terminology will become obvious later on. B. T. Feld [*Experimental Nuclear Physics*, edited by E. Segrè (John Wiley and Sons, Inc., New York, 1953), Vol. 2] calls $\sigma_{ce}$ the "capture elastic" cross section.

We have derived in the Appendix exact and approximate expressions for the scattering amplitude $\eta_0$ and for the cross sections in this energy region. For the present purposes, we will use the following form (see A.14b), which is valid in a region $D_s$ including a resonance $\epsilon_s$ as indicated:

$$\eta_0=e^{-2ikR'(\epsilon)}\left(1-\frac{i\Gamma_\alpha{}^s}{\epsilon-\epsilon_s+i\Gamma^s/2}\right)+\eta_0{}^*,$$

$$\eta_0{}^*=e^{-2ikR'(\epsilon)}\left(\frac{\Gamma_\alpha{}^s}{\epsilon-\epsilon_s+i\Gamma^s/2}G_1+iG_2+G_3\right), \quad (2.10)$$

for

$$\epsilon_s+\epsilon_{s-1}<2\epsilon<\epsilon_{s+1}+\epsilon_s.$$

Here $R'$ is a length and a slowly varying function of the energy. (A function is slowly varying if it changes value appreciably only over intervals large compared to $D$.) The length $R'$ is of the order of magnitude of nuclear dimensions. It plays the role of a scattering length and takes on both positive and negative values. The quantities $\Gamma_\alpha{}^s$ and $\Gamma^s$ are the partial width and the total width, respectively. The terms $G_1$, $G_2$, and $G_3$ are real functions of $\epsilon$ of the following order of magnitude:

$$G_1\sim\Gamma/D,\ G_2\sim(\Gamma_\alpha/D)[(\Gamma/D)+kR],\ G_3\sim\Gamma_\alpha/D, \quad (2.11)$$

where the omission of the superscript signifies the average value of the magnitude in the interval $I$.

The first term in $\eta_0$ incorporates the contribution from the resonance level $\epsilon_s$, whereas $\eta_0{}^*$ contains the contribution from the other resonances; the first term in $\eta_0{}^*$ represents interference effects between the resonance $\epsilon_s$ and other resonances. It will appear later that $\eta_0{}^*$ contributes negligibly to the average of $\eta_0$.

The cross sections in the immediate neighborhood of the resonance ($|\epsilon-\epsilon_s|\ll D_s$) follow from (2.5) and (2.10) by neglecting $\eta_0{}^*$, since, in that region, they contribute terms much smaller than the others.

$$\sigma_r{}^{(0)}=\pi\lambda\!\!\!^{-2}\frac{\Gamma_\alpha{}^s(\Gamma^s-\Gamma_\alpha{}^s)}{(\epsilon-\epsilon_s)^2+(\Gamma^s/2)^2},$$

$$\sigma_{el}{}^{(0)}=\pi\lambda\!\!\!^{-2}\left|(e^{2ikR'}-1)+\frac{i\Gamma_\alpha{}^s}{\epsilon-\epsilon_s+i\Gamma^s/2}\right|^2, \quad (2.12)$$

$$|\epsilon-\epsilon_s|\ll D.$$

The reaction cross section is just the sum over $\beta$ of the one-level Breit-Wigner cross sections

$$\sigma_{\alpha\beta}{}^{(0)}=\frac{\Gamma_\alpha{}^s\Gamma_\beta{}^s}{(\epsilon-\epsilon_s)^2+(\Gamma^s/2)^2}\pi\lambda\!\!\!^{-2} \quad (2.13)$$

for the reaction leading from the entrance channel $\alpha$ to an exit channel $\beta$. ($\Gamma_\beta{}^s$ is the partial width of decay into the channel $\beta$.)

The elastic cross section contains a "potential" scattering amplitude

$$P=e^{2ikR'(\epsilon)}-1,$$

which corresponds to a scattering at a hard sphere of a radius $R'$, where $R'$ is not identical to but only of the order of magnitude[5] of the nuclear radius and is a slowly varying function of the energy.

We now determine the average value of the scattering amplitude $\eta_0$ over the resonances in the interval $I$:

$$\bar{\eta}_0=\frac{1}{I}\int_I\eta_0 d\epsilon=\left\langle\frac{1}{D_s}\int_{D_s}\eta_0(\epsilon)d\epsilon\right\rangle_I,$$

where the symbol $\langle\ \rangle_I$ signifies an average taken over all resonances within $I$. The random position of resonances allows us to write

$$\bar{\eta}_0=\frac{1}{D}\int_{\epsilon_s-D/2}^{\epsilon_s+D/2}\eta_0(\epsilon)d\epsilon, \quad (2.14)$$

where $D$ is the average level distance within the interval $I$; typical average values of $\Gamma^s$ and $\Gamma_\alpha{}^s$ should be used in the expression (2.10) for $\eta_0$.

Evaluation of (2.14) gives

$$\bar{\eta}_0=e^{-2ikR'}[1-(\pi\Gamma_\alpha/D)], \quad (2.15)$$

when all magnitudes of the order $(\Gamma_\alpha\Gamma)/D^2$ or $(\Gamma_\alpha/D)kR'$ or smaller are neglected. It is seen in each interval $D_s$ that the main contribution to the average comes from the main resonance. The contribution of neighboring resonances which are expressed by $\eta_0{}^*$ in (2.10) contribute only to expressions which are smaller than (2.14) by a factor of the order $\Gamma/D$ or $kR$.

We now use (2.14) for the calculation of the "shape elastic" scattering and get, with the help of (2.8),

$$\sigma_{se}{}^{(0)}=\pi\lambda\!\!\!^{-2}|(e^{2ikR'}-1)+\pi\Gamma_\alpha/D|^2. \quad (2.16)$$

For small $kR'$, this becomes

$$\sigma_{se}{}^{(0)}=4\pi R'^2[1+(\pi\Gamma_\alpha/2kR'D)^2]. \quad (2.17)$$

The magnitude $[\pi\Gamma_\alpha/2(kR'D)]^2$ is usually rather small. [It is of the order of $10^{-2}$; see, for example, the estimate in Blatt and Weisskopf,[5] Chap. VIII, Eq. (7.14).] Hence, $\sigma_{se}{}^{(0)}$ is very nearly equal to $4\pi R'^2$ for $kR'\ll1$.

We get the cross section for the formation of the compound nucleus according to (2.9),

$$\sigma_c{}^{(0)}=2\pi^2\lambda\!\!\!^{-2}(\Gamma_\alpha/D)(1-\tfrac{1}{2}\pi\Gamma_\alpha/D). \quad (2.18)$$

[5] The appearance of the length $R'$ is a consequence of our general treatment of the nuclear resonance in the Appendix. In the special derivation of the Breit-Wigner formula, as given in Feshbach, Peaslee, and Weisskopf (reference 1) or J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), assumptions are made which make $R'$ constant and equal to the nuclear radius $R$. It is shown in this paper that these assumptions probably are valid only in special cases as in the case of strong coupling, for example.

The average total cross section then becomes

$$\sigma_t^{(0)}=\sigma_{se}^{(0)}+\sigma_c^{(0)}=4\pi R'^2+2\pi^2\lambda\!\!\!^{-2}\Gamma_\alpha/D. \quad (2.19)$$

The second term in this expression is proportional to $(1/v)$.

It is interesting to compare $\sigma_c$ with the average of $\sigma_r$, which, according to (2.12), is

$$\bar{\sigma}_r^{(0)}=2\pi^2\lambda\!\!\!^{-2}(\Gamma_\alpha/D)(\Gamma-\Gamma_\alpha)/\Gamma. \quad (2.20)$$

Hence, the difference between the two, the "compound elastic" scattering, is [neglecting the small factor $\pi\Gamma_\alpha/(2D)$]

$$\sigma_{ce}^{(l)}=2\pi^2\lambda\!\!\!^{-2}\Gamma_n^2/D\Gamma. \quad (2.21)$$

This is just the average of that part of the elastic scattering (2.12) which corresponds to the resonance amplitude only, namely, of

$$\sigma=\pi\lambda\!\!\!^{-2}\frac{(\Gamma_\alpha{}^s)^2}{(\epsilon-\epsilon_s)^2+(\Gamma^s/2)^2}.$$

It is the cross section which one would get for the re-emission into the entrance channel from the Breit-Wigner expression (2.13).

It is significant that expressions (2.15), (2.18), and (2.19) do not contain the total width $\Gamma$ but only the channel width $\Gamma_\alpha$. The "gross" properties (total, shape elastic, and compound nucleus formation cross sections) are independent of the nature of the other exit channels. They would remain unchanged, for example, if the exit channels $\beta\neq\alpha$ were closed. It would only increase $\sigma_{ce}$ at the expense of $\bar{\sigma}_r$, as seen in (2.20) and (2.21). This is connected with the fact that a change of $\Gamma$ with constant $\Gamma_\alpha$ changes only the width of the resonance, but not its area.

At the energies considered here, the cross section for the formation of the compound nucleus contains only magnitudes ($\Gamma_\alpha$ and $D$), which can be determined by studying the neutron resonances. Hence, investigations of slow neutron resonances are useful to check the theoretical predictions of $\sigma_c$ at low energy. The "shape-elastic" scattering, on the other hand, in this energy region is almost entirely given by $4\pi R'^2$ and is therefore essentially independent of the neutron resonance values. Apart from the small correction $\pi^3\lambda\!\!\!^{-2}(\Gamma_\alpha/D)^2$, it is equal to the potential scattering as shown in (2.12) and, therefore, can be measured also by studying the cross sections near and between resonances.

## III. POTENTIAL-WELL MODEL

In this section we shall employ a potential-well model to determine the gross-structure cross sections. We have adopted for the purposes of a preliminary survey the simplest type of potential well:

$$\begin{aligned} V&=-V_0(1+i\zeta), \quad r<R,\\ V&=0, \quad r>R, \end{aligned} \quad (3.1)$$

where $V_0$ and $\zeta$ are constants and $R$ is the nuclear radius. The use of the complex potential is necessary to obtain nonzero values for the cross section for the formation of the compound nucleus. A similar model in which $\zeta\sim 1$ was employed by Bethe.[6] Fernbach, Serber, and Taylor[7] have used the same model in order to describe nuclear scattering at very high energies. A model in which $\zeta=0$ was used by Ford and Bohm[8] in discussing zero-energy cross sections. It is essential that the crudeness of this model be emphasized. We have, for example, omitted any spin-orbit terms which play an important role in the shell model, but which we expect will not affect the over-all qualitative features which we seek here. The constants in (3.1) may well turn out to be energy dependent. We particularly expect this for $\zeta$, since we know that $\bar{\sigma}_r$ is large at high energies, while the success of the shell model indicates that $\zeta$ should be zero for the ground states of nuclei.

We give some of the details of the calculations with potential (3.1). For each $l$ we calculate the value of the logarithmic derivative,

$$f_l=R(u_l'/u_l)_{r=R}. \quad (3.2)$$

The average reflection factor $\bar{\eta}_l$ is then

$$\bar{\eta}_l=e^{-2i\delta_l}\left(1-\frac{2s_l}{M_l+iN_l}\right), \quad (3.3)$$

where

$$\delta_l=\tan^{-1}(-j_l(x)/n_l(x)), \quad (3.4a)$$

$$\Delta_l+is_l=1+xh_l'(x)/h_l(x), \quad (3.4b)$$

$$M_l=s_l-\mathrm{Im}f_l, \quad N_l=-\Delta_l+\mathrm{Re}f_l. \quad (3.4c)$$

The functions $j_l$, $n_l$, and $h_l$ are the spherical Bessel, Neumann, and Hankel functions, respectively, while $x$ is, as usual, $kR$.[9] $h_l'(x)$ is the derivative of $h_l(x)$ with respect to $x$; $\Delta_l$ and $s_l$ are both real magnitudes and are defined as the real and imaginary part of the expression on the right of (3.4b).

For potential (3.1), $f_l$ may be written down directly

$$f_l=1+Xj_l'(X)/j_l(X), \quad (3.5)$$

where

$$X^2=x^2+X_0^2(1+i\zeta), \quad X_0^2=(2m/\hbar^2)V_0R^2.$$

This is, however, not the most convenient form for determining the real (Re) and imaginary (Im) parts of $f_l$. We have instead employed recurrence relations for these quantities based on recurrence relations for $j_l$.

[6] H. Bethe, Phys. Rev. **57**, 1125 (1940).
[7] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).
[8] K. W. Ford and D. Bohm, Phys. Rev. **79**, 745 (1950).
[9] This follows the notation of Morse, Lowan, Feshbach, and Lax, U. S. Navy Department of Research and Inventions Report No. 62.1R, 1945 (unpublished).

For $l=0$, we get

$$f_0=X\cot X,$$
$$\mathrm{Re}f_0=\frac{X_1\sin 2X_1+X_2\sinh 2X_2}{\cosh 2X_2-\cos 2X_1}, \tag{3.6}$$
$$\mathrm{Im}f_0=\frac{X_2\sin 2X_1-X_1\sinh 2X_2}{\cosh 2X_2-\cos 2X_1},$$

where $X=X_1+iX_2$. The recurrence relations which follow from

$$f_l=\frac{X^2}{l-f_{l-1}}-l$$

are

$$\mathrm{Re}f=\frac{(X_1^2-X_2^2)(l-\mathrm{Re}f_{l-1})-2X_1X_2\,\mathrm{Im}f_{l-1}}{(l-\mathrm{Re}f_{l-1})^2+(\mathrm{Im}f_{l-1})^2}-l, \tag{3.7}$$

$$\mathrm{Im}f_l=\frac{(X_1^2-X_2^2)\,\mathrm{Im}f_{l-1}+2X_1X_2(l-\mathrm{Re}f_{l-1})}{(l-\mathrm{Re}f_{l-1})^2+(\mathrm{Im}f_{l-1})^2}. \tag{3.8}$$

The asymptotic expression for $f_l$,

$$f_l\xrightarrow[X\to\infty]{}X\cot(X-\tfrac{1}{2}l\pi), \tag{3.9}$$

unfortunately cannot be generally employed. The fractional error in (3.9) is $l(l+1)/(X\sin 2X)$, from which we learn that (3.9) is not sufficiently accurate for $l\geqslant 2$, while for $l=1$ it will fail for small $X$ or for $X=n\pi$.

The total cross section, as well as the cross section for the formation of the compound nucleus, may be easily obtained

$$\frac{\bar\sigma_t^{(l)}}{\pi R^2}=\frac{4}{x^2}(2l+1)\left[\sin^2\delta_l+s_l\frac{M_l\cos 2\delta_l-N_l\sin 2\delta_l}{M_l^2+N_l^2}\right],$$
$$\frac{\sigma_c^{(l)}}{\pi R^2}=\frac{4}{x^2}(2l+1)s_l\left[\frac{-\mathrm{Im}f_l}{M_l^2+N_l^2}\right]\equiv\frac{(2l+1)T_l}{x^2}, \tag{3.10}$$
$$\bar\sigma_t=\textstyle\sum_l\bar\sigma_t^{(l)},\quad \sigma_c=\sum_l\sigma_c^{(l)},$$

where the $T_l$ may be interpreted as penetrabilities.

These cross sections will have characteristic large-scale resonances, which are present in the experimental data. In the $l=0$ case, these resonances occur when

$$X_1=(X_0^2+x^2)^{\frac{1}{2}}=(n+\tfrac{1}{2})\pi+\frac{X_0^2\zeta^2}{2(2n+1)\pi},$$

where $n$ is an integer and where we have assumed that $\zeta X_0^2/n\pi\ll 1$. The width of the large-scale resonance is $2xh^2/mR^2$, which in the experimental range is of the order of Mev. For a given energy, the $l=0$ cross section will give maxima as a function of $R$. The width of these maxima against changes in $R$ is approximately $(2xR/X^2)$, independent of $R$.

The angular distribution for shape elastic scattering is

$$\frac{d\sigma_{se}}{d\Omega}=\frac{\lambda^2}{4}\left|\sum_l(2l+1)(1-\bar\eta_l)P_l(\cos\theta)\right|^2.$$

Therefore

$$\frac{1}{R^2}\frac{d\sigma_{se}}{d\Omega}=(\mathrm{Re}\sum)^2+(\mathrm{Im}\sum)^2, \tag{3.11}$$

where

$$\mathrm{Im}\sum=\frac{x}{4}\sum_l\frac{\sigma_t^{(l)}}{\pi R^2}P_l(\cos\theta), \tag{3.12}$$
$$\mathrm{Re}\sum=\frac{1}{2x}\sum_l(2l+1)\left[\sin 2\delta_l-2s_l\frac{M_l\sin 2\delta_l+N_l\cos 2\delta_l}{M_l^2+N_l^2}\right].$$

Before we can compare the theory with the experimental data on angular distributions, it is necessary to add the compound elastic scattering. From our general qualitative ideas, we may break the process up into the formation of the compound nucleus and the re-emission of the incident particle into a particular $l$ state, which will naturally have a very definite associated angular distribution. The result is particularly simple in the case of a target nucleus of spin zero and an incident particle of spin zero, since here the angular momentum of the incident particle cannot change in an elastic scattering process. We may therefore write

$$\frac{d\sigma_{ce}}{d\Omega}=\sum\sigma_c^{(l)}|Y_{l0}|^2w_l, \tag{3.13}$$

where $Y_{l0}$ are the normalized spherical harmonics and $w_l$ is the probability that the compound nucleus formed by the absorption of a particle of angular momentum $l$ will decay by emission of the same particle without loss of energy or change in angular momentum.

This simple result cannot be applied to the neutron case because of the possibility of spin changes of the neutron and re-orientation of the spin of the target nucleus without any change in the energy of either the neutron or the target nucleus. The formalism which needs to be used here has been worked out by Hauser and Feshbach[10] and by Wolfenstein.[11] The target nucleus and neutron system is now characterized by the spin of the target nucleus $I$, its $z$ component $m$, and spin of the neutron $i$, the channel spin $s(\mathbf{s}=\mathbf{i}+\mathbf{1})$, the angular momentum of the incident neutron $l$, and its $z$ component which is zero, and of course the parity of the system. The compound nucleus will have a total angular momentum $J$, $z$ component $m$, and will decay into a residual nucleus of spin $I'$ and a particle of spin $i'$. These form a final channel spin $s'(\mathbf{s}'=\mathbf{i}'+\mathbf{1}')$, $z$ component $m-m'$. The system will have an angular momentum $l'$, $z$ component $m'$.

[10] W. Hauser and H. Feshbach, Phys. Rev. **87**, 366 (1952).
[11] L. Wolfenstein, Phys. Rev. **82**, 690 (1951).

To obtain the desired cross section, we must now introduce the assumptions of the statistical nuclear theory. We assume that, upon appropriate averaging, the various $J$ levels do not interfere, that there is no residual interference between the various $l$'s which can form the given compound state $J$, or between the various $l$'s into which it can decay. We then break up the process of compound elastic scattering into the cross section for the formation of the compound nucleus in state $J$, with incident particles of angular momentum $l$, multiplied by the probability that it will decay by emission of a particle of angular momentum $l'$, leaving the residual nucleus in the ground state with spin $I$. The formation process is given first by the cross section for the formation of the compound nucleus which, because of our simple assumption (3.1), depends only on $l$ and is $\sigma_c^{(l)}$. This must be multiplied by the probability of forming the system with angular momentum $J$, using incident particles of angular momentum $l$. Again, because of the absence of any spin-dependent forces, this is simply the square of the Clebsch-Gordan coefficient $|(ls0m|lsJm)|^2$. On the emission side, we will need the probability of forming $J$ with particles of angular momentum $l'$. This is given by $|(l's'm'm-m'|l's'Jm)|^2$. We need the relative probability of decay with emission of $l'$ particles leaving the nucleus in the ground state which we will denote by $w(l')\leqslant 1$. The limitations of the relative probability of different kinds of emission arising from angular momentum conservation are contained in the Clebsch-Gordan coefficients. The function $w$ contains all the other dependence. Because of our assumption of spin-independent forces in Eq. (3.1), it will depend only on $l'$ and the parity of the system. The angular distribution of the emitted particles is $|Y_{l',m'}|^2$. Combining these results, we have

$$\frac{d\sigma_{ce}}{d\Omega}=\frac{1}{(2i+1)(2I+1)}\sum\sigma_c^{(l)}|(ls0m|lsJm)|^2 \times|(l's'm'm-m'|l's'Jm)|^2 w(l')|Y_{l'm'}|^2. \quad (3.14)$$

The indicated sums are over $m$, $m'$, $s$, $s'$, $l$, $l'$, and $J$. The spin factor in front arises from the average over initial spin states, which involves the sum over $m$ and $m'$.

By employing methods due to Racah[12] and discussed by Blatt and Biedenharn,[13] the sums over $m$ and $m'$ may be performed yielding

$$\frac{d\sigma_{ce}}{d\Omega}=\frac{1}{4\pi(2I+1)(2i+1)}\sum\frac{\sigma_c^{(l)}w(l')}{2l+1} \times Z(lJlJ;sL)Z(l'Jl'J;s'L)P_L(\cos\theta), \quad (3.15)$$

where the $Z$ factors are defined by Biedenharn, Blatt, and Rose[14] and for which tables[15] are available. The sums are over $J$, $l$, $l'$, and $L$. Only even $L$ will occur. This result is given in reference 10.[16] We have not introduced a specific notation to describe the role of parity, so it should be understood that parity is conserved both in the formation and in the decay of the compound nucleus.

The total compound elastic cross section may be easily evaluated from (3.15) and gives the expected result

$$\sigma_{ce}=\sum\frac{2J+1}{(2I+1)(2i+1)(2l+1)}\sigma_c^{(l)}w(l').$$

Expression (3.15) simplifies considerably in two special cases (a) $I\gg 1$ and (b) $I=0$. In case (a), it follows from the sum rule (see reference 10),

$$\sum_{s'l'}(2s'+1)|Z(lJlJ;sL)Z(l'Jl'J;s'L)|P_L=(2J+1)^2(2l+1), \quad (3.16)$$

that $d\sigma_{ce}/d\Omega$ is approximately independent of angle. We note that if $I\gg 1$, the factor $2s'+1$ is approximately a constant, the error being of the order of $(1/I)$. In case (b) we note that $s=s'=\frac{1}{2}$ and that $l=l'$ because of parity conservation. We therefore find for this case (placing $i=\frac{1}{2}$)

$$\frac{d\sigma_{ce}}{d\Omega}=\sum_{l,L}\frac{\sigma_c^{(l)}w(l)}{4\pi(4l+2)}[Z^2(l,l+\tfrac{1}{2},l,l+\tfrac{1}{2};\tfrac{1}{2},L) +Z^2(l,l-\tfrac{1}{2},l,l-\tfrac{1}{2};\tfrac{1}{2},L)]P_L. \quad (3.17)$$

The factors $w$ which may be computed as outlined in reference 10 depend on the details of the levels of the residual nucleus. There it is shown that

$$w(l')=T_{l'}(E)/\sum_{pqs}T_p(E_q'). \quad (3.18)$$

The quantity $w(l')$ lies between 0 and 1. The values of $T_p$ are calculated in reference 10 under the assumption of strong coupling. The ideas underlying the present theory would change these factors to those given by Eq. (3.10). Since the compound elastic scattering is not very large compared to the shape elastic, we have only determined the upper limit for $\sigma_{ce}$, which is given by putting $w(l')=1$. We expect $\sigma_{ce}$ to be near this upper limit at energies for which there is little inelastic scattering or capture, and to be near zero when inelastic scattering or other nuclear reactions are appreciable.

[12] G. Racah, Phys. Rev. **61**, 186 (1942); **62**, 438 (1942).
[13] J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. **24** 258 (1952).
[14] Biedenharn, Blatt, and Rose, Revs. Modern Phys. **24**, 249 (1952).
[15] L. C. Biedenharn, Oak Ridge National Laboratory Report ORNL-1501, May 28, 1953 (unpublished).
[16] It is worth while noting that Eq. (3.15) may be derived from the general analysis of Blatt and Biedenharn by combining the definitions of average cross section as given in Sec. II and the statistical assumptions. The chief elements of the latter are (1) nonoverlap of resonances and (2) random phases for the scattering matrix so that, upon averaging over possible ways of forming the compound nucleus, interference terms average to zero.

## IV. ISOLATED RESONANCES

We should like to establish a correspondence between the parameters describing a single compound nucleus resonance and the parameters which describe the average potential (3.1). This is most easily done for the low-energy case. We evaluate the cross sections for very low energy on the basis of the potential (3.1) and by comparing them with the expressions for the average cross sections which were derived in Sec. II in terms of the resonance parameters. The only two resonance parameters entering here are the ratio $\Gamma_\alpha/D$ of neutron width[17] to level distance and the radius $R'$ of the potential scattering.

We start with the evaluation of the results from (3.1). The only contribution comes from $l=0$ and we get from (3.3)

$$\bar{\eta}_0=e^{-2ix}\frac{f_0+ix}{f_0-ix}=e^{-2ix(1-\alpha)}, \tag{4.1}$$

where $\alpha$ is a complex number:

$$\alpha=(1/x)\tan^{-1}(x/f_0)\cong 1/f_0,$$

and $f_0$ is given by the expressions (3.6).

This should be compared with (2.14) in order to express the two relevant magnitudes $R'$ and $\Gamma_\alpha/D$ in terms of $X_1$ and $X_2$. Equating (2.14) and (4.1) gives in the limit of $k\to 0$, a limit which also implies $\Gamma_\alpha/D\to 0$:

$$R'=R(1-\alpha_1), \quad (\pi/2x)(\Gamma_\alpha/D)=\alpha_2,$$

where $\alpha_1$ and $\alpha_2$ are the real and imaginary parts of $\alpha$.

From (3.6) we can easily obtain the following relations:

$$\alpha=f_0^{-1}=\alpha_1+i\alpha_2,$$

$$\alpha_1=\frac{1}{|X|^2}\frac{X_2B-X_1A\sin 2X_1}{B^2+2A\cos^2X_1},$$

$$\alpha_2=\frac{1}{|X|^2}\frac{X_1B-X_2A\sin 2X_1}{B^2+2A\cos^2X_1},$$

with $A=1/(2\cosh^2X_2)$, $B=\tanh X_2$.

We now distinguish two limiting cases: the cases of strong and weak coupling. In the first case the absorption is so strong that the neutron is completely absorbed in a distance of a nuclear radius within nuclear matter: $\exp(-X_2)\ll 1$. In the case of weak coupling we assume $X_2\ll 1$.

Hence we get, for strong coupling: $A\to 0$, $B\to 1$, and

$$\alpha_1=X_2/|X|^2=1/X_2', \quad \alpha_2=X_1/|X|^2=1/X_1',$$

and $R'=R(1-1/X_2')$, $\Gamma_\alpha/D=2x/\pi X_1'$. The length $R'$ is almost equal to $R$ since $(X_2')^{-1}$ is a small magnitude. The expression for $\Gamma_\alpha/D$ is the same as that used by Feshbach, Peaslee, and Weisskopf with the only exception that $X_1'$ replaces $X_1$. The former magnitude is somewhat larger than $X_1$.[18]

Strong coupling therefore leads essentially to the same results as Feshbach, Peaslee, and Weisskopf: The potential scattering length is roughly equal to $R$ and $\Gamma_\alpha/D=2(x/\pi X_1')$.

In the weak coupling approximation we get

$$A=\tfrac{1}{2}, \quad B=X_2\ll 1, \quad |X|^2=X_1^2,$$

and hence

$$R'=R\left(1-\frac{1}{2X_1}\frac{\sin 2X_1}{X_2^2+\cos^2X_1}\right),$$

$$\frac{\Gamma_\alpha}{D}=\frac{2x}{\pi X_1}\beta, \quad \beta=X_2\frac{1-(1/2X_1)\sin 2X_1}{X_2^2+\cos^2X_1}.$$

Here $R'$ and $\beta$ are functions of $X_1$, and hence of $R$, with a characteristic resonance denominator. The shapes of these functions are reminiscent of optical dispersion and absorption curves, respectively. The maximum in $\beta$ occurs when $X_0\cong(n+\frac{1}{2})\pi$ ($n$ integer), the value of $\beta$ being $2/[(n+\frac{1}{2})\pi\zeta]$ and the width of the peak at half-maximum $(n+\frac{1}{2})\pi\zeta$. The minimum value of $\beta$ is about $(n+\frac{1}{2})\pi\zeta/2$. Figure 1 shows both magnitudes plotted as a function of $X_0$ for a value of $\zeta=0.03$.

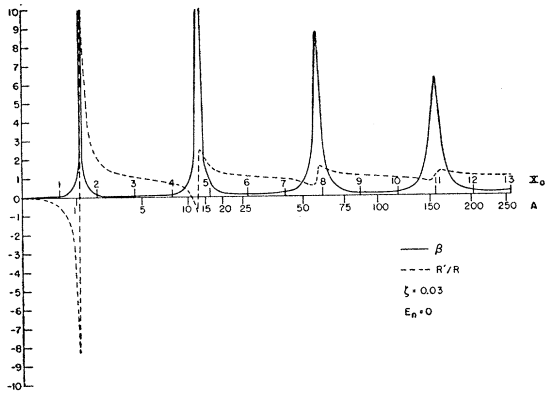


FIG. 1. Potential scattering length $R'$ and the ratio $\Gamma_\alpha/D$ of the neutron width to the level distance at low energy as a function of $X_0=K_0R$ for $\zeta=0.03$. $R'$ is plotted in units of $R$ and $\Gamma_\alpha/D$ is given in the form of the parameter $\beta=(\pi/2)(V_0/\epsilon)^{\frac{1}{2}}(\Gamma_\alpha/D)$, where $\epsilon$ is the energy of the neutron. The atomic-weight scale corresponding to $X_0$ is shown also for a potential-well depth $V_0=42$ Mev and radii $R=1.45\times 10^{-13}A^{\frac{1}{3}}$ cm.

[17] From here on we use the symbol $\Gamma_\alpha$ for "neutron width" since the entrance channel $\alpha$ is a *neutron* channel in all cases which we treat in this paper.

[18] It is plausible to assume that, in the case of strong coupling, the imaginary part of the potential is of the same order as the real part. An imaginary part that is much larger than the real one would imply that the absorption takes place over distances small compared to the wavelength in the interior. Hence $X_1'$ is about twice as large as $X_0$ which leads to a $\Gamma_\alpha/D$ half as large as in Feshbach, Peaslee, and Weisskopf. This strong coupling result is somewhat more consistent than the result in Feshbach, Peaslee, and Weisskopf. In the latter paper the boundary condition was chosen such that the wave inside the nucleus is a sine wave $\sin(Kr-\delta)$, an assumption that is contrary to the idea of strong compound nucleus formation. In fact, a wave $\exp(+\Omega r)\sin(Kr-\delta)$ with $\Omega\sim K$ would be more consistent and does lead to the same result as the one above.

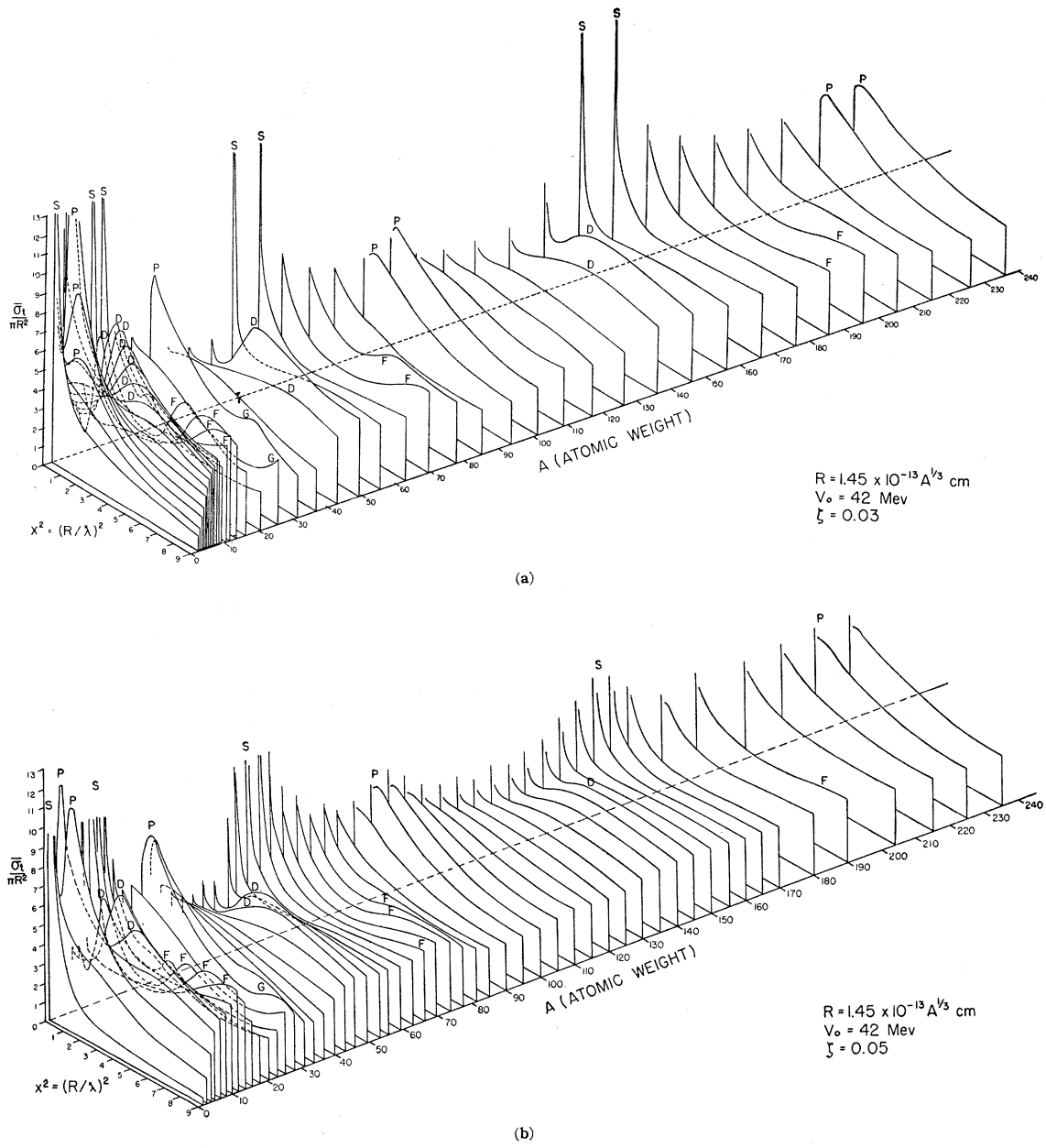


FIG. 2. (a) Calculated neutron total cross sections as a function of energy and mass number, for a well depth $V_0=42$ Mev, radius $R=1.45\times10^{-13}A^{\frac{1}{3}}$, $\zeta=0.03$. The energy $\epsilon$ is expressed in terms of

$$x^2=[A^{\frac{2}{3}}\cdot A/10(A+1)]\epsilon,$$

where $\epsilon$ is in Mev. (b) The same for $\zeta=0.05$.

## V. COMPARISON WITH EXPERIMENTAL RESULTS

Figure 2a shows a profile presentation of the calculations of the neutron total cross sections on the basis of the potential (3.1) with a depth $V_0=42$ Mev and a radius $R=1.45\times10^{-13}A^{\frac{1}{3}}$ cm. The constant $\zeta$ is assumed to be 0.03 which corresponds to an absorption coefficient of $\kappa=4.2\times10^{11}$ cm$^{-1}$ in nuclear matter for neutrons of zero energy in free space. This means that the intensity of a beam of slow neutrons is reduced in nuclear matter to $1/e$ at a distance of $\kappa^{-1}=2.4\times10^{-12}$ cm. The cross sections are plotted as a function of the energy in units of $x^2=(R/\lambda)^2$ and of the atomic weight. The letters denoting the maxima indicate the character of the resonance causing the maximum. Figure 3 contains a profile presentation of the observed cross sections plotted against the same coordinates.

The experimental curves in Fig. 3 are averages over resonances. For higher $A$ and small level distance this

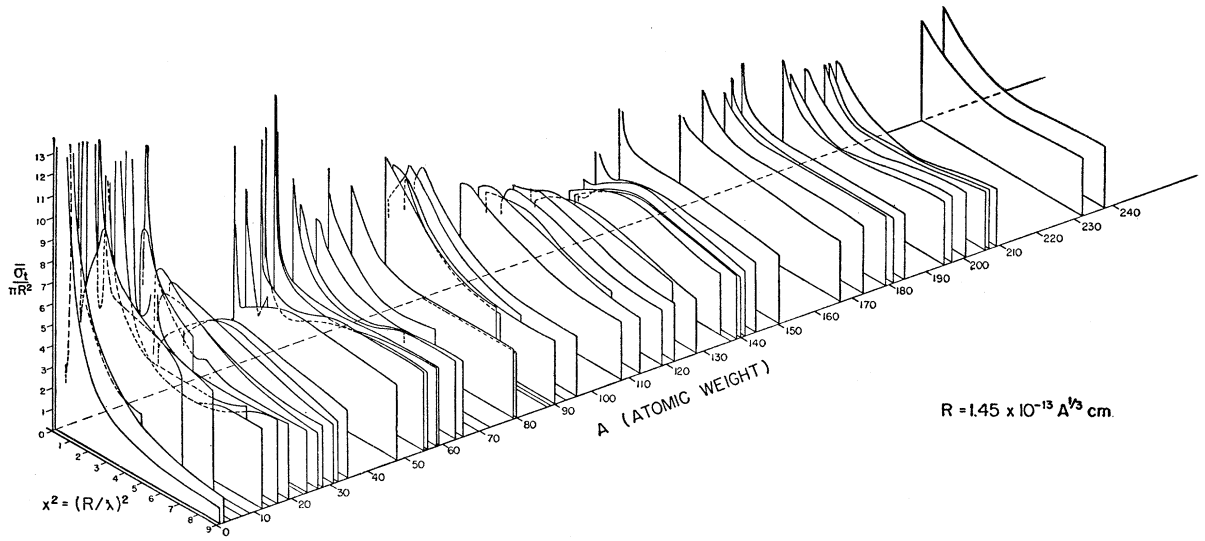


FIG. 3. Observed neutron total cross sections as a function of energy and mass number. The energy is expressed in terms of $x^2$ as in Fig. 2, (a) and (b).

average was done by the measuring apparatus itself, for lower $A$ the averaging was done in the drawing. The theoretical and the experimental curves do not extend to zero energy. They are broken off at an energy of about 50 kev. As is well known, the curves should go approximately as $\epsilon^{-\frac{1}{2}}$ at very low energy. The experimental curves are compiled from measurements by many workers.[19–25]

The comparison of these two figures shows that the theory can account for a number of striking features of the experimental results. In particular, the theory reproduces the drop of the cross sections at low energies in the regions $A\sim40$ and $100<A<140$.

It also reproduces the large cross sections at low energy in the regions $A\sim60$, $A\sim90$, and $A\sim150$. The large values at $A\sim90$ are ascribed to a $P$ resonance; whereas the other two regions are supposed to contain $S$ resonances. $P$ resonances are expected to fall off towards low energies; whereas the $S$ resonances merge directly with the $(1/v)$ rise. The observed energy dependence indicates the $P$-resonance behavior in the region $A\sim90$ and shows typical $S$-resonance behavior at $A\sim60$ and 150. There is an indication of $P$-resonance at low energies for $A\sim30$ as the theory predicts.

The theory also reproduces the type of maxima ($D$ maxima) which are found for energies corresponding to $x^2\sim3$ in the regions $A\sim40$, and $A\sim140$. It seems that the predicted $F$-wave maximum near $A\sim200$ is also observed. It is remarkable that one finds reasonably good agreement in the shape of the curves even at very low atomic numbers: $A<20$.[26]

We are using here a different depth of the potential than in the calculations published previously by the same authors.[27] The previous calculations were based upon a well depth of only $V_0=19$ Mev. The change to $V_0=42$ Mev was suggested by Adair[28] and improves the agreement considerably. At the time of the first calculation only measurements for $A>60$ were used. The similarity between the theoretical results for $V_0=42$ Mev and $V_0=19$ Mev for $A>60$ can be explained as follows:

$S$-wave maxima at low energy occur if $RK_0=r_0A^{\frac{1}{3}}K_0\cong(n+\frac{1}{2})\pi$ and $P$ maxima if $r_0A^{\frac{1}{3}}K_0\cong n\pi$, where $K_0=(2mV_0/\hbar^2)^{\frac{1}{2}}$. For $V_0=19$ Mev and $r_0=1.45\times10^{-13}$ cm, one gets therefore $S$ maxima at $A\sim38$ and 170, and $P$ maxima at $A\sim11$ and 90. For $V_0=42$ and the same $r_0$ one gets $S$ maxima at $A\sim11$, 55, and 150; $P$ maxima at $A\sim27$, 90, and 216. Hence, the $P$ maximum near 90 and the $S$ maximum near 160 are reproduced by both potential depths. The behavior of the curves in the neighborhood of these maxima also must be similar, in particular, the depression at low energy for values of $A$ just below an $S$ maximum. However, the experimental data for nuclei below $A=60$ definitely indicate another $S$ maximum near 55 and a strong low-energy depression for $A\sim40$ as predicted by $V_0=42$. These features are

[19] H. H. Barschall, Phys. Rev. **86**, 431 (1952).
[20] Miller, Adair, Bockelman, and Darden, Phys. Rev. **88**, 83 (1952).
[21] Walt, Becker, Okazaki, and Fields, Phys. Rev. **89**, 1271 (1953).
[22] Okazaki, Darden, and Walton, Phys. Rev. **93**, 461 (1954).
[23] N. Nereson and S. Darden, Phys. Rev. **89**, 775 (1953); Phys. Rev. **94**, 1678 (1954); and unpublished data on Li and B (private communication).
[24] C. F. Cook and T. W. Bonner, Phys. Rev. **94**, 651 (1954); McCrary, Taylor, and Bonner, unpublished data on Li (private communication).
[25] *Neutron Cross Sections*, U. S. Atomic Energy Commission Report AECU-2040 (Technical Information Division, Department of Commerce, Washington, D. C., 1952), and three supplements (unpublished).
[26] See also C. E. Porter, Bull. Am. Phys. Soc. **29**, No. 5, 25 (1954).
[27] Feshbach, Porter, and Weisskopf, Phys. Rev. **90**, 166 (1953).
[28] R. K. Adair, Phys. Rev. **94**, 737 (1954).

not reproduced by the theoretical curves for $V_0=19$. We therefore believe that $V_0=42$ Mev yields a better model. It should be noted that the agreement is not very sensitive to a change of potential $V_0$ with a corresponding change of $r_0$ such that $V_0^{\frac{1}{3}}r_0$ stays constant.

The shapes of the total cross section curves are quite sensitive to the value of the absorption constant $\zeta$. An increase of $\zeta$ flattens the maxima and minima. Strong fluctuations in the calculated curves occur only at lower energies for values of $x<1.5$. This is below 2 Mev at $A\sim60$ and below 0.6 Mev at $A\sim200$. At higher energies the contributions of the numerous angular momenta prevent the appearance of any pronounced maxima or minima. Therefore the determination of $\zeta$ by fitting the calculated curves to the experimental ones only gives the value of $\zeta$ for relatively low energies. We cannot exclude a change of $\zeta$ at energies of, say, more than 1 Mev or fluctuations in $\zeta$ from one value of $A$ to another although below 1 Mev it seems that $\zeta$ cannot vary much as a function of $A$. In the low-energy region the determination is quite accurate. A change of $\zeta$ to 0.05 or to 0.02 would give rise to a worse agreement with experiments. Figure 2(b) shows the total cross sections for $\zeta=0.05$, and it is obvious that the maxima and minima are not as pronounced as in the experimental data.[29]

We now turn to the calculations of the angular distribution of the elastic scattering. Figure 4 shows the experimental results at 1 Mev as measured by Walt and Barschall.[30] The most characteristic features are the flat distributions around $A\sim60$, a very strong forward peaking and a rise at backward angles at $A\sim140$, and the appearance of a second maximum at 90° around $A\sim180$. The calculation of the angular dependence (Fig. 5) is not unambiguous since the amount of compound elastic scattering is difficult to

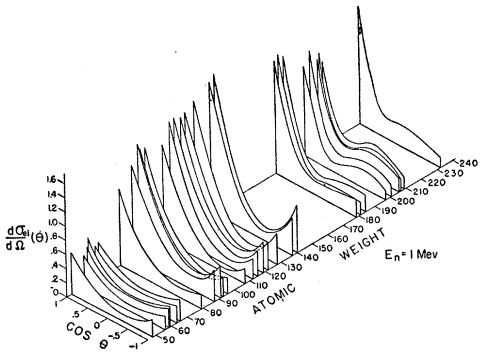


FIG. 4. Observed angular distribution (in barns/sterad) of the elastic 1-Mev neutron scattering as a function of cos$\theta$ and the mass number $A$ as measured by Walt and Barschall.

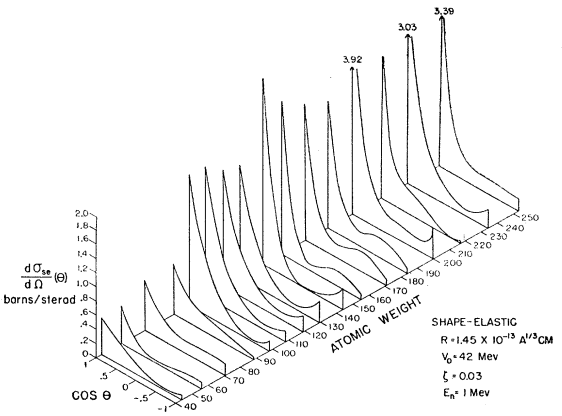


FIG. 5. Calculated angular distribution of the elastic neutron scattering (shape elastic only) as a function of cos$\theta$ and the mass number $A$ for a well $V_0=42$ Mev, $R=1.45A^{\frac{1}{3}}\times10^{-13}$ cm, and $\zeta=0.03$.

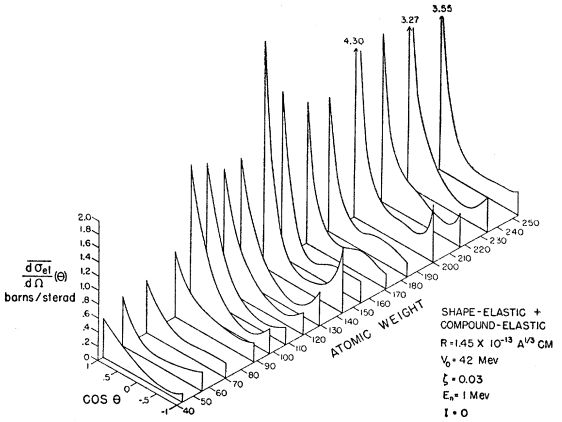


FIG. 6. Calculated angular distribution of the elastic neutron scattering (shape elastic plus maximum compound elastic) as a function of cos$\theta$ and $A$ for a well $V_0=42$ Mev, $R=1.45A^{\frac{1}{3}}\times10^{-13}$ cm, and $\zeta=0.03$.

determine. Furthermore, the angular dependence of the compound elastic scattering depends upon the spin of the target nucleus. We therefore have shown in Fig. 5 the calculated angular distribution of the shape elastic scattering only. In Fig. 6 the compound elastic scattering is added in full which would correspond to the case in which the compound state decays exclusively via the entrance channel. The target spin was assumed to be zero. The actual $d\sigma_{el}/d\Omega$ must lie somewhere between Fig. 5 and Fig. 6. For nuclei with strong inelastic scattering, Fig. 5 should be the better approximation.

It is seen from Figs. 5 and 6 that some of the main features are again reproduced by the theory. The flatness of the distribution around $A\sim60$ comes from the fact that the $P$ contribution is very weak in this region and, at small angles, of opposite phase to the $S$ scattering. This occurs always at values $KR$ somewhat below a $P$ resonance. The second maximum at 90° at high mass numbers is not too well reproduced. The

[29] The disagreement is worse for high values of $A$. This might be an indication of a slight decrease of $\zeta$ with the mass number. If the absorption were concentrated in a surface layer of given thickness, one would expect a similar effect [see M. H. Johnson and E. Teller, Phys. Rev. **93**, 357 (1954)].

[30] M. Walt and H. H. Barschall, Phys. Rev. **93**, 1062 (1954).

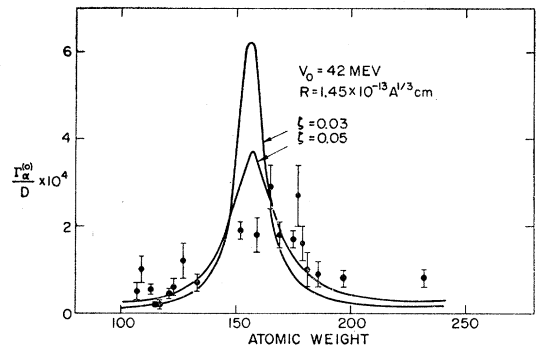


FIG. 7. Ratio $\Gamma_\alpha^{(0)}/D$ of neutron width to level distance for low energies as a function of $A$. Here $\Gamma_\alpha^{(0)}=\Gamma_\alpha(\epsilon^{(0)}/\epsilon)^{\frac{1}{2}}$ is the "reduced" width, and $\epsilon^{(0)}$ is taken to be 1 ev. The curves represent the calculated values for $\zeta=0.03$ and 0.05. The points represent the observed values and the limits of error.

theory shows it only between $A=150$ and $A=200$. The angular dependence above $A=200$ does not seem to agree too well with the experiment. The angular distributions are not very sensitive to the choice of constants. The results with $V_0=19$ Mev are not very different from the ones shown here.

The agreement with experiments is also less satisfactory for the cross section $\sigma_c$ for the formation of the compound nucleus. It is difficult to measure $\sigma_c$ directly since it includes the compound elastic scattering besides the reaction cross section, and the former cannot easily be separated from the shape elastic scattering. At very low energies, however, the formation of the compound nucleus can be measured by studying the individual resonances (see Sec. III). The relevant magnitude is the ratio $\Gamma_\alpha/D$ of the neutron width to the level distance, averaged over a number of neighboring resonances. The theoretical values $\Gamma_\alpha/D$ expected on the basis of $V_0=42$ Mev are shown in Fig. 7 together with a compilation[31] of the measurements[32–41] of $\Gamma_\alpha/D$. Only recently has it been possible to measure the neutron widths of several resonances in one isotope, so that the average $\Gamma_\alpha$ and the level distance can be determined to some degree of reliability. It is seen that the expected maximum of $\Gamma_\alpha/D$ at $A\sim155$ is noticeable, but it is not as strong as the theory predicts for the same value of $\zeta$ which gives the best fit for the total cross sections ($\zeta=0.03$). Also the values off peak are somewhat larger than predicted.

The fact that the resonance at $A\sim155$ is not as strong as expected might be connected with the large deviations from sphericity which are ascribed to the nuclei in this region.[42] If the shape of the potential well is ellipsoidal, one would expect results which roughly represent averages over the spherical results taken over radii which lie between the smallest and the largest axis. This would give rise to a flattening of the maxima and a rise of the wings in the theoretical curves of Fig. 7.[43]

Although no direct measurement of the formation of the compound nucleus is possible, the measurements of inelastic cross sections $\sigma_{in}$ or reaction cross sections $\sigma_r$ can be used to compare with the theoretical predictions of $\sigma_c$. Evidently $\sigma_{in}$ and $\sigma_r$ must be smaller than $\sigma_c$. The difference $\sigma_c-\bar{\sigma}_r$ is the compound elastic cross section which is expected to be rather small if inelastic scattering or other reactions are strong enough to compete for the decay of the compound state.

Walt and Barschall have determined inelastic scattering cross sections $\sigma_{in}$ at 1 Mev by subtracting the elastic scattering from the total scattering. The values of $\sigma_{in}$ should be less than or equal to the theoretical values of $\sigma_c$.

Figure 8 shows a comparison between the observed inelastic cross sections and the calculated $\sigma_c$ at 1 Mev as functions of $A$. The observed values are of the expected order of magnitude, but they do not agree with the theoretical curve. The absence of the maximum at $A=50$ might be explained by the fact that the compound elastic scattering is relatively high for these nuclei. The same fact explains the low value of the inelastic cross section in lead and bismuth. However, the expected maxima at $A=90$ and 150 seem to occur at higher values of $A$. We have no explanation for these discrepancies.

There are many measurements of inelastic cross sections at somewhat higher energies. They all indicate that the values are not too far from $\pi(R+\lambda\!\!\!^{-})^2$, which is

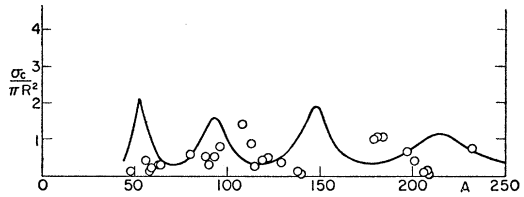


FIG. 8. The calculated cross section $\sigma_c$ for compound nucleus formation at $E_n=1$ Mev and the observed reaction cross section at 1 Mev as determined by Walt and Barschall. $R$ is taken to be $1.45\times10^{-13}A^{\frac{1}{3}}$ cm. In the calculations the parameters $V_0$ and $\zeta$ were taken to be 42 Mev and 0.03, respectively.

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[37] Hughes, Kato, and Levin, Phys. Rev. **92**, 1094 (1953); and private communication.
[38] R. L. Christensen, Phys. Rev. **92**, 1509 (1953).
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[42] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selenskab Mat. fys. Medd. **27**, 16 (1953).
[43] This thought was suggested to us by A. Bohr and B. R. Mottelson.

the value one would expect if the neutron wave were totally absorbed in contrast to our findings of $\zeta \sim 0.03$. Especially the measurements at 14 Mev[3] indicate this fact. On the basis of this evidence one would conclude that the value of $\zeta$ is strongly energy dependent and reaches a value ($\zeta \gtrsim 0.12$) corresponding to almost total absorption in a medium-sized nucleus certainly at 14 Mev but most likely already at energies as low as 4.5 Mev. The latter conclusion is based upon measurements of inelastic cross sections by Lonsjo, Taylor, and Bonner.[44] In this connection it is interesting that the calculations of Morrison, Muirhead, and Rosser[45] also give a very strong increase with energy of the absorption of nucleons in nuclear matter just in the region which corresponds to incident neutrons of 1 Mev. These calculations are based on the Goldberger method[46] of the scattering of free particles with the application of the Pauli principle. The effect of the exclusion principle alone causes a sharp drop of the energy exchange with decreasing energy.[47]

It is apparent that our model is much less successful in reproducing the strength of compound nucleus formation than in reproducing the total and elastic scattering. It gives too much variation with $A$ of $\Gamma_\alpha/D$ at low energies and probably too little compound nucleus production at 1 Mev and higher, although it is possible to explain the discrepancies at higher energy by assuming that $\zeta$ increases with energy above 1 Mev.

The discrepancies may come from two possible sources: (A) The potential $V(r)$ as given by (3.1) may not be the shape best fitted for the model. (B) The attempt of this paper, the description of the gross behavior of a nucleus by a complex one-particle potential, may be unsuccessful. In connection with (A) it must be noted that the potential (3.1) necessarily is an oversimplified version, since it is physically impossible that the potential well actually has a discontinuity in the form of a sudden jump at $r=R$. It might be that a rounding-off of the corners of the potential well will improve the agreement with experiments.

The smoothing of the edges of the square-well potential was of significance for the interpretation of the elastic proton scattering with heavy nuclei. This scattering has been measured with protons of an energy of about 18 Mev by Gugelot,[48] Burkig and Wright,[49] and by Cohen and Neidigh.[50] The results cannot be interpreted on the basis of a potential (3.1) with sharp edge, as shown by Chase and Rohrlich.[51] However, Woods and Saxon[52] have shown recently that a rounding-off even within the small interval of $0.5\times10^{-13}$ cm changes the results considerably and brings them into much better agreement with the experiments.

It is possible, therefore, that the smoothing of the discontinuity of $V$ at $r=R$ would also improve the agreement of theory and experiment in respect to compound nucleus formation. It would decrease the reflection of the neutron wave at the nuclear surface and hence increase the cross section $\sigma_c$ when all other constants ($V_0$,$R$,$\zeta$) are unchanged. It remains to be seen whether the rounding-off of the potential well improves the agreement with respect to $\sigma_c$ and $\Gamma_\alpha/D$ and with respect to the angular distribution of the elastic cross section, without destroying the agreement of the total and elastic cross sections.

Calculations are under way to investigate these possibilities.

It must be pointed out that one should never expect any exact agreement between the predictions based upon a model of this type and the observed cross sections. The very nature of this attempt to describe a complicated many-body problem by a simple one-body potential implies that the model can only contain the main features of the situation. Apart from this general limitation it should be kept in mind that we have used here a potential which has a particularly simple dependence on the radius and on the mass number. We have assumed the same radial dependence for the real and imaginary part which is very probably too strict an assumption. We have neglected spin-dependent forces as observed by Adair and co-workers,[53] and we have excluded any special features connected with the shell structure.

The purpose of the proposed approach is to connect some characteristic salient features of the nuclear cross sections with simple nuclear properties rather than to construct a theory which will produce the exact quantitative details of the observations.

## ACKNOWLEDGMENTS

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We also wish to express our gratitude to those who have participated in the important, but tedious, work of computing. Our thanks are offered to Fern Abrams,

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## APPENDIX

### The Scattering Amplitude at Low Energies

The scattering amplitude $\eta$ is the diagonal element $S_{\alpha\alpha}$ of the scattering matrix, where the index $\alpha$ refers to the entrance channel. The matrix $S$ is given by the following expression [see Blatt and Weisskopf, formula (X, 4.11)]:

$$S_{\alpha\beta}=\exp[-i(k_\alpha+k_\beta)R]S_{\alpha\beta}'. \tag{A.1}$$

Here $R$ is the nuclear radius, and

$$S'=(1+i\mathfrak{R}')/(1-i\mathfrak{R}'), \tag{A.1a}$$

where $\mathfrak{R}'$ is connected with the derivative matrix $\mathfrak{R}$ by

$$\mathfrak{R}_{\alpha\beta}'=(k_\alpha k_\beta)^{\frac{1}{2}}\mathfrak{R}_{\alpha\beta},$$

and $k_\alpha$, $k_\beta$ are the channel wave numbers at the energy $E$. The matrix $\mathfrak{R}$ is defined on page 545 of Blatt and Weisskopf. It can be expressed in the following form [see Sec. X (4.22)]:

$$\mathfrak{R}_{\alpha\beta}=\sum_s \frac{y_{s\alpha}y_{s\beta}}{E_s-E}, \tag{A.2}$$

where $E_s$ are the resonance energies and the $y_{s\alpha}$ are magnitudes which are connected with the channel widths $\Gamma_\alpha{}^s$ (partial widths for the decay via the channel $\alpha$):

$$\Gamma_\alpha{}^s=2k_\alpha y_{s\alpha}{}^2. \tag{A.3}$$

Each $y_{s\alpha}$ is real but its sign might be positive or negative. We make the reasonable assumption that the signs are distributed at random.

In what follows we will assume that we work in an energy region for which, first, $k_\alpha R\ll 1$, and, second, $\Gamma^s\ll D_s$, where $D_s$ is the interval

$$D_s=\tfrac{1}{2}(E_{s+1}-E_{s-1}),$$

which includes the resonance $E_s$ from mid-point to mid-point. We also assume that the values $\Gamma_\alpha{}^s$ and $D_s$ have the same order of magnitude for all resonances in an energy interval $I$ which includes many resonances but which is small compared to energy intervals occurring in single-particle problems (say $I\sim 10$ kev for heavy nuclei).

Let us surround each resonance $E_s$ by an energy interval $D_s$ from $\frac{1}{2}(E_{s-1}+E_s)$ to $\frac{1}{2}(E_s+E_{s+1})$. In this interval we can write the matrix $\mathfrak{R}_{\alpha\beta}'$ in the form

$$\mathfrak{R}_{\alpha\beta}'=\frac{y_{s\alpha}'y_{s\beta}'}{E_s-E}+g_{\alpha\beta}, \tag{A.4}$$

where $y_{s\alpha}'=(k_\alpha)^{\frac{1}{2}}y_{s\alpha}$. The matrix $g_{\alpha\beta}$ has no singularities in $D_s$ and is given by

$$g_{\alpha\beta}=\sum_{t\neq s}\frac{y_{t\alpha}'y_{t\beta}'}{E_t-E}. \tag{A.4a}$$

We now estimate the order of magnitude of $g_{\alpha\beta}$. Here it is important to take into account that the signs of the $y_{t\alpha}'$ are distributed at random over the different resonances $t$ and the different channels $\alpha$.

First we split $g_{\alpha\beta}$ into the contributions of neighboring and far-off levels:

$$\begin{aligned} g_{\alpha\beta}&=g_{\alpha\beta}'+r_{\alpha\beta},\\ g_{\alpha\beta}'&=\sum_{t\neq s}{}'\frac{y_{t\alpha}'y_{t\beta}'}{E_t-E},\\ r_{\alpha\beta}&=\sum_{t\neq s}{}''\frac{y_{t\alpha}'y_{t\beta}'}{E_t-E}, \end{aligned} \tag{A.4b}$$

where the prime on the summation sign means that the sum should be extended only over resonances within the interval $I$, and the double prime means extension over the resonances outside $I$. Because of the random signs of $y_{t\alpha}'$, the terms in the sums (A.4b) have random signs for $\alpha\neq\beta$ and only the immediate-neighbor resonances contribute appreciably to $g_{\alpha\beta}$; on the same grounds $r_{\alpha\beta}$ can be neglected. We, then, obtain the estimate:

$$|g_{\alpha\beta}'|\cong|g_{\alpha\beta}|\sim(\Gamma_\alpha\Gamma_\beta)^{\frac{1}{2}}/D,\quad r_{\alpha\beta}\approx 0,\quad \alpha\neq\beta. \tag{A.5}$$

We understand by $\Gamma_\alpha$ (without superscript) the average value of $\Gamma_\alpha{}^s$ in the interval $I$. For $\alpha=\beta$, all terms have the same sign for $E_t>E$ or $E_t<E$. Since there are roughly an equal number of levels above and below $E_s$ in the interval $I$, we get the following estimate:

$$g_{\alpha\alpha}'\sim\Gamma_\alpha/D. \tag{A.5a}$$

The order of magnitude of the contribution $r_{\alpha\alpha}$ from the faraway levels is quite undetermined. However, the scattering cross section between resonances turns out to be $4\pi R^2(1+r_{\alpha\alpha}/x)^2$. Experimentally, we know that this cross section is of the order of $4\pi R^2$, and hence we conclude $r_{\alpha\alpha}\sim k_\alpha R\equiv x$.

It follows from (A.5) and (A.5a) that the $g_{\alpha\beta}$ are all small compared to unity in the energy region considered, and we proceed to expand (A.1) in powers of $g$. For this purpose we introduce the *factorable* matrix

$$T_{\alpha\beta}=-y_{s\alpha}'y_{s\beta}'/\Delta,\quad \Delta=E-E_s. \tag{A.6}$$

The following relation holds:

$$T^n=B^{n-1}T, \quad n\geqslant 1, \qquad B=\sum_\beta T_{\beta\beta}. \tag{A.7}$$

The quantity $B$ is a number and it is connected with the total width. $\Gamma^s=\sum_\beta \Gamma_\beta{}^s$ of the resonance $s$:

$$B=-\tfrac{1}{2}\Gamma^s/\Delta. \tag{A.8}$$

Hence, if a matrix is a function of $T$ which can be expressed as a power series with *powers larger than zero*

$$A(T)=\sum_{n=1}^{\infty} a_n T^n,$$

we get the relation

$$A_{\alpha\beta}=B^{-1}A(B)T_{\alpha\beta}. \tag{A.9}$$

We may now expand (A.1), noting that $\mathfrak{R}'=T+g$:

$$S'=-1+\frac{2}{1-i(T+g)}$$
$$=-1+\frac{2}{1-iT}+\frac{1}{1-iT}(2ig)\frac{1}{1-iT}$$
$$-\frac{2}{1-iT}g\frac{1}{1-iT}g\frac{1}{1-iT}+\cdots.$$

From (A.9) we have $1/(1-iT)=1+T/(1-iB)$, so that

$$S'=1+\frac{2iT}{1-iB}+2ig-2g^2-2\frac{(gT+Tg)}{1-iB}+S^*,$$

$$S^*=-2i\frac{TgT}{(1-iB)^2}-2i\frac{Tg^2+gTg+g^2T}{(1-iB)}$$
$$-2\frac{TgTg+Tg^2T+gTgT}{(1-iB)^2}+2i\frac{TgTgT}{(1-iB)^3}. \tag{A.10}$$

The scattering amplitude $\eta_0$ is the diagonal element $S_{\alpha\alpha}$. We first note the cross sections which would follow from (A.10) when we put $g=0$: ($x=k_\alpha R$)

$$\eta_0=e^{-2ix}\left(1+\frac{2iT_{\alpha\alpha}}{1-iB}\right)=e^{-2ix}\left(1-\frac{i\Gamma_\alpha{}^s}{\Delta+i\Gamma^s/2}\right), \tag{A.11}$$

and hence we get the well-known expressions:

$$\sigma_{el}{}^{(0)}=\pi\lambda\!\!\!^{-2}\left|2x+\frac{\Gamma_\alpha{}^s}{\Delta+i\Gamma^s/2}\right|^2,$$
$$\sigma_r{}^{(0)}=\pi\lambda\!\!\!^{-2}\frac{\Gamma_\alpha{}^s(\Gamma^s-\Gamma_\alpha{}^s)}{\Delta^2+(\Gamma^s/2)^2}. \tag{A.12}$$

We now proceed to neglect all terms in the expansion (A.10) which would give rise to terms in $\sigma_{el}$ and $\sigma_r$ of the order of $\delta\sigma_r$ and $\delta\sigma_{el}$, respectively, with

$$\delta\sigma_r=\sigma_c{}^{(0)}f, \quad \delta\sigma_{el}=\sigma_c{}^{(0)}f, \tag{A.13a}$$

where $\sigma_c{}^{(0)}=\pi\lambda\!\!\!^{-2}\Gamma_\alpha\Gamma/[\Delta^2+(\Gamma^2/4)]$ and $f$ is a small number,

$$f\sim x \quad \text{or} \quad f\sim\Gamma/D. \tag{A.13b}$$

This means we will neglect cross sections which are small by a factor $\Gamma/D$ or $x$, compared to $\sigma_c{}^{(0)}$, which can be regarded as the one-level value of the cross section for the formation of the compound nucleus.

It will be shown below that all terms of (A.10) contained in $S^*$ give rise to corrections in the cross sections of the order (A.13) or smaller. Hence, within the accuracy (A.13), we can write $S'$ in the form:

$$S'=1+\frac{2iT}{1-iB}+2ig-2g^2-\frac{2(gT+Tg)}{1-iB}. \tag{A.14}$$

To prove this point, we first examine the effect of a small addition $\Delta\eta'$ to $\eta'=S_{\alpha\alpha}{}'$ on the cross sections: We get

$$\Delta\sigma_{el}=\pi\lambda\!\!\!^{-2}2\ \mathrm{Re}[(e^{2ix}-\eta')\Delta\eta'^*]\lesssim 4\pi\lambda\!\!\!^{-2}|\Delta\eta'|,$$
$$\Delta\sigma_r=\pi\lambda\!\!\!^{-2}2\ \mathrm{Re}[\eta'^*\Delta\eta']\leqslant 2\pi\lambda\!\!\!^{-2}|\Delta\eta'|, \tag{A.15}$$

where $\mathrm{Re}(a)$ is the real part of $a$. The former relation follows from the fact that $|e^{2ix}-\eta'|\leqslant 2$ and the latter relation from the fact that $|\eta'|\leqq 1$. As an example, we discuss the omission of the first term in the expression $S^*$ in (A.10). We find according to (A.5) and (A.5a)

$$|\Delta\eta'|\sim|(TgT)_{\alpha\alpha}/(1-iB)^2|\approx\frac{\Gamma_\alpha\Gamma}{\Delta^2+(\Gamma^s/2)^2}[(\Gamma/D)+x],$$

and hence the contributions to the cross section of this term are negligible according to (A.15) and (A.13). Similar considerations show that the other neglected terms in (A.10) contribute the same or less to the cross sections.

We now single out the diagonal element of $S'$ because of its significance for the scattering amplitude. We can write $S_{\alpha\alpha}{}'$ from (A.14) and (A.13b) in the following form:

$$S_{\alpha\alpha}{}'\cong\exp(2ir_{\alpha\alpha})\left[\left(1+\frac{2iT_{\alpha\alpha}}{1-iB}\right)+2ig_{\alpha\alpha}{}'\right.$$
$$\left.-2(g'^2)_{\alpha\alpha}-\frac{4(g'T)_{\alpha\alpha}}{1-iB}\right].$$

This expression differs from the diagonal element of (A.14) by terms which are of the order $S^*$ and therefore negligible, as, for example, $r_{\alpha\alpha}(g'T)_{\alpha\alpha}/(1-iB)$. We

can write it in the form

$$S_{\alpha\alpha}'=\exp(2ir_{\alpha\alpha})\times\left[1+\frac{2iT_{\alpha\alpha}}{1-iB}+\frac{T_{\alpha\alpha}}{1-iB}G_1+iG_2+G_3\right], \quad \text{(A.14a)}$$

or, according to (A.1)

$$S_{\alpha\alpha}=\eta_0=\exp(-2ik_\alpha R')\times\left[1+\frac{2iT_{\alpha\alpha}}{1-iB}+\frac{T_{\alpha\alpha}}{1-iB}G_1+iG_2+G_3\right], \quad \text{(A.14b)}$$

where

$$R'=R-r_{\alpha\alpha}/k_\alpha. \quad \text{(A.16)}$$

$$G_1=4\sum_{t\neq s}'\sum_\beta\frac{y_{t\alpha}'y_{t\beta}'(y_{s\beta}'/y_{s\alpha}')}{E_t-E},$$
$$G_2=2g_{\alpha\alpha}', \quad \text{(A.17)}$$
$$G_3=-2(g'^2)_{\alpha\alpha}.$$

This is the form which is used in the text. The orders of magnitude of these real functions are given by (2.11).

The following simplification can be used if one calculates the scattering cross section $\sigma_{el}$ and the transfer cross sections $\sigma_{\alpha\beta}$ (cross section of the reaction $\alpha\rightarrow\beta$):

$$\sigma_{\alpha\beta}^{(0)}=\pi\lambda\!\!\!^{-2}|S_{\alpha\beta}'|^2 \quad \text{(A.18)}$$

within the limits of accuracy (A.13). It turns out that the last two terms in (A.14) give rise to nonnegligible contributions only to $\sigma_r$ when expression (2.3a) is used. In expression (2.3) for $\sigma_{el}$ and in expression (A.18) for $\sigma_{\alpha\beta}$, the two last terms of (A.14) give rise to contributions which can be neglected according to (A.13).[54] Hence for the calculation of $\sigma_{\alpha\beta}^{(0)}$ and $\sigma_{el}^{(0)}$ we may use the shorter form

$$S'=1+[2iT/(1-iB)]+2ig,$$

or

$$S_{\alpha\beta}'\simeq\delta_{\alpha\beta}\exp(2ir_{\alpha\alpha})+2i\sum_r'\frac{y_{r\alpha}'y_{r\beta}'}{E-E_r+i\Gamma^r/2}, \quad \text{(A.19)}$$

where the sum is extended over all resonances within the interval $I$. Actually the imaginary part $+i\Gamma^r/2$ in the denominator of (A.16) should be found only in the term $r=s$, but the addition in the other terms leads only to errors smaller than (A.13). We then get for the cross section $\sigma_{\alpha\beta}$

$$\sigma_{\alpha\beta}=\pi\lambda\!\!\!^{-2}|S_{\alpha\beta}'|^2=4\pi\lambda\!\!\!^{-2}\left|\sum_r'\frac{y_{r\alpha}'y_{r\beta}'}{E-E_r+i\Gamma^r/2}\right|^2, \quad \text{(A.20)}$$

and for the scattering cross section,

$$\sigma_{el}=\pi\lambda\!\!\!^{-2}|e^{2ix}-S_{\alpha\alpha}'|^2,$$
$$=\pi\lambda\!\!\!^{-2}\left|\exp(2ik_\alpha R')-1+\sum_r\frac{i\Gamma_\alpha{}^r}{E-E_r+i\Gamma^r/2}\right|^2. \quad \text{(A.21)}$$

According to (A.19), the value of $\sigma_{\alpha\beta}$ goes to zero between two resonances $E_s$ and $E_{s+1}$ if the sign of $y_{s\alpha}'y_{s\beta}'$ is the same for both resonances. If the sign is opposite, no zero occurs. Note that this statement is good only to the accuracy (A.13). At the zero of (A.17) the actual cross section might still be of the order (A.13).

We note in (A.21) that the potential scattering amplitude $\exp(2ikR')-1$ corresponds to the scattering by an impenetrable sphere of radius $R'$ as given by (A.16). The quantity $R'$ itself is a function of the energy, which is slowly varying and changes only over intervals much larger than $D$.

The forms (A.15) and (A.16) correspond to the Breit-Wigner formulas used in the literature before the more exact investigations by Wigner and Eisenbud.[55] The amplitudes contain characteristic sums over the contributions of the different resonances with the imaginary contribution $i\Gamma^r/2$ in the denominator. It has been pointed out repeatedly that the forms (A.17) and (A.18) are not exactly correct. We have shown, however, that they are valid within the errors given by (A.13).

[54] At first sight this seems puzzling since $\sigma_r=\Sigma_\beta\sigma_{\alpha\beta}$. It must be remembered that Eq. (2.3a) uses the diagonal element of $S$; whereas (A.18) uses off-diagonal elements. The connection between these elements is established by the unitary nature of $S$: $1-|S_{\alpha\alpha}|^2=\Sigma_{\beta\neq\alpha}(S_{\alpha\beta})^2$. In order to insure the validity of this equation up to the order $g^2$, one must include the last two terms of (A.14) in $S_{\alpha\alpha}$, but it is not necessary to include them in $S_{\alpha\beta}'$.

[55] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).