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Some Properties of Nuclear Normal Modes*

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Energy levels and antisymmetric eigenfunctions are calculated, with the aid of normal coordinates, for a nuclear Hamiltonian containing inter-nucleon potentials of the Hooke's law type. Such a nuclear model exhibits the same shell structure as do the same nucleons moving without interaction in a common harmonic-oscillator central field. Modifications in the Hamiltonian—use of force parameters which depend on A, cutting off of forces at finite range, inclusion of spin-orbit forces—are discussed in connection with shell structure and the nuclear photoeffect.

I. INTRODUCTION

DESPITE the more fundamental status of meson theory, it remains interesting to study approximate nuclear Hamiltonians which depend on nucleon variables only. Even in this approximation the complexity of the many-body problem has led to the use of various simplifications, such as the single-particle (or central-field) model, the alpha-particle model, the liquid-drop model, and various other collective models. The occurrence of nuclear shells has refocused attention on the single-particle model,[1] whose recent successes do not seem to be wholly understood.

The purpose of this paper is to discuss a nuclear model first proposed by Houston,[2] studied by Margenau and Warren,[3] and, more recently, by Krook, by Rosen, by Post, and independently by the present authors.[4] The nuclear forces are assumed, in first approximation, to be ordinary central forces, obeying Hooke's law, between all pairs of nucleons. These forces do not exhibit saturation, and in fact correspond to infinite binding energy. The one important advantage of studying them is that one does not need to ignore the fact that the nucleus is a many-body system; the eigenfunctions and eigenvalues of such a Hamiltonian can be readily calculated with complete accuracy.

II. TRANSFORMATION TO NORMAL COORDINATES

The Hamiltonian used as a first approximation is

$$H_0=\sum_{k=1}^{A}\frac{1}{2m}p_k^2+\sum_{k>j=1}^{A}\left[\tfrac{1}{2}b_{kj}(\mathbf{r}_k-\mathbf{r}_j)^2-D_{kj}\right], \tag{1}$$

where $\mathbf{p}_k$ and $\mathbf{r}_k$ are, respectively, the momentum and the position vectors of the kth nucleon, D_{kj} is a positive constant, m is the mass of a neutron or a proton, A is the nuclear mass number, and $b_{kj}=b_n$, b_p, or b_{np} according as nucleons k and j are both neutrons, both protons, or one of each.

It is possible in general to find normal coordinates ξ_α, η_α, ζ_α $(\alpha=1\cdots A)$, linear homogeneous functions of the Cartesian coordinates x_k, y_k, z_k of the nucleons, such that H_0 is a sum of squares of the normal coordinates and their conjugate momenta;[5] a possible form is

$$H_0=\sum_{\alpha=1}^{A}\frac{\pi_\alpha^2}{2m}+\sum_{\alpha=1}^{A}\tfrac{1}{2}\beta_\alpha\rho_\alpha^2-D, \tag{2}$$

where the β_α are constants, D is the sum of all the D_{kj}, $\rho_\alpha^2=\xi_\alpha^2+\eta_\alpha^2+\zeta_\alpha^2$, and π_α^2 is the sum of squares of the momenta conjugate to ξ_α, η_α, ζ_α. One can deal with this Hamiltonian by converting the x_k to the ξ_α, the y_k to the η_α, and the z_k to the ζ_α all by the same unitary transformation, which can thus be written

$$\mathbf{r}_k=\sum_{\alpha=1}^{A}T_{k\alpha}\boldsymbol{\rho}_\alpha, \tag{3}$$

with its inverse

$$\boldsymbol{\rho}_\alpha=\sum_{k=1}^{A}\mathbf{r}_kT_{k\alpha}, \tag{4}$$

and the orthonormality conditions

$$\sum_\alpha T_{k\alpha}T_{j\alpha}=\sum_\alpha T_{\alpha k}T_{\alpha j}=\delta_{jk}.^6 \tag{5}$$

One of the constants β_α, say β_1, vanishes; $\boldsymbol{\rho}_1$ is then proportional to the position vector of the center of mass. Another of the β's is nondegenerate; it and the corresponding $\boldsymbol{\rho}$ are given by

$$\beta_{Z+1}=Ab_{np}\equiv\beta_{np}, \tag{6}$$

and

$$\boldsymbol{\rho}_{Z+1}=\left(\frac{N}{AZ}\right)^{\frac{1}{2}}\sum_{k=1}^{Z}\mathbf{r}_k-\left(\frac{Z}{AN}\right)^{\frac{1}{2}}\sum_{k=Z+1}^{A}\mathbf{r}_k, \tag{7}$$

where the protons are numbered from 1 to Z, and the neutrons from $Z+1$ to $A=Z+N$. The remaining $(A-2)$

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[1] M. G. Mayer, Phys. Rev. 75, 1969 (1949); Haxel, Jensen, and Suess, Phys. Rev. 75, 1766 (1949); Feenberg, Hammack, and Nordheim, Phys. Rev. 75, 1968 (1949).

[2] W. V. Houston, Phys. Rev. 47, 942 (1935); 49, 206 (1936).

[3] H. Margenau and D. T. Warren, Phys. Rev. 52, 790 (1937); D. T. Warren and H. Margenau, Phys. Rev. 52, 1027 (1937).

[4] Max Krook, 1951 (private communication); Nathan Rosen, 1953 (private communication); H. R. Post, Proc. Phys. Soc. (London) A66, 649 (1953); I. Bloch, Phys. Rev. 80, 138 (1950); I. Bloch and Y. C. Hsieh, Phys. Rev. 91, 240 (1953).

[5] H. Goldstein, *Classical Mechanics* (Addison-Wesley Press, New York, 1950), Chap. 10.

[6] In the most general form of the homogeneous normal-coordinate transformation, the $T_{k\alpha}$ are tensors of second rank.

normal-mode oscillators have only two distinct β's:

$$\beta_2=\beta_3=\cdots=\beta_Z=Zb_p+Nb_{np}\equiv\beta_p,$$
and
$$\beta_{Z+2}=\beta_{Z+3}=\cdots=\beta_A=Zb_{np}+Nb_n\equiv\beta_n. \tag{8}$$

The corresponding normal-mode vectors $\boldsymbol{\varrho}$ are thus highly arbitrary. It develops that the sum of coefficients in each of them vanishes, that $\boldsymbol{\varrho}_2\cdots\boldsymbol{\varrho}_Z$ depend only on proton coordinates, and that $\boldsymbol{\varrho}_{Z+2}\cdots\boldsymbol{\varrho}_A$ depend only on neutron coordinates.

III. WAVE FUNCTIONS AND ENERGY LEVELS

The Hamiltonian in Eq. (2) is easily separable; the eigenfunctions can be taken as products of plane-wave functions of ξ_1, η_1, ζ_1 by Hermite functions of the variables $a_2\xi_2$, $a_2\eta_2$, $a_2\zeta_2$, $\cdots a_A\zeta_A$, where $a_\alpha=(m\beta_\alpha/\hbar^2)^{\frac{1}{4}}$. The energy values are sums of the energies inherent in the separate normal coordinates with their conjugate momenta. The internal energy of the nucleus, excluding energy of motion of the center of mass, is

$$E=\sum_{\alpha=2}^{A}(l_\alpha+m_\alpha+n_\alpha+\tfrac{3}{2})\hbar(\beta_\alpha/m)^{\frac{1}{2}}-D, \tag{9}$$

where the $(3A-3)$ quantum numbers l_α, m_α, n_α are non-negative integers. These energies are, of course, independent of spins. Any energy eigenfunction can be made spin dependent through being multiplied by a product of single-particle spin functions.

It is expected that the exclusion principle rules out many of the lower energies permitted in Eq. (9). This matter has been investigated by means of antisymmetrized wave functions, which can be obtained with the aid of the generating function for the Hermite functions:[7]

$$f(u,a\xi)=\exp(-u^2+2ua\xi-\tfrac{1}{2}a^2\xi^2)$$
$$=\sum_{n=0}^{\infty}\exp(-\tfrac{1}{2}a^2\xi^2)H_n(a\xi)u^n/n!. \tag{10}$$

Such generating functions involving the normal coordinates associated solely with protons $(2\leq\alpha\leq Z)$ can be multiplied together and multiplied by proton spin functions; they then generate a complete set of spin-dependent eigenfunctions of the part of H_0 involving protons alone. The most general procedure is to use one independent auxiliary variable (like u above) with each of the normal coordinates ξ_2, η_2, $\cdots\zeta_Z$. The neutron states can be dealt with similarly. These two product generating functions can then be multiplied together to generate the product eigenfunctions for all the normal-mode oscillators except No. 1 (center of mass) and No. $Z+1$ (n *vs* p).

[7] Obviously one would like, if possible, to choose the normal coordinates in such a way that interchange of two similar particles is equivalent to interchange of two normal-mode oscillators, perhaps with some of the normal coordinates also changed in sign. E. B. Shanks of the Vanderbilt Mathematics Department has proven that such choices are impossible except in a few physically uninteresting cases.

The antisymmetrizing procedure is applied to the product generating functions rather than the wave functions themselves. The omitted oscillators (1 and $Z+1$) do not need to be included because their coordinates are invariant under permutation of similar particles. To antisymmetrize the product generating function for (say) the protons, one forms the usual sum of products with permuted nucleon coordinates and spins. The determinant form is convenient; e.g., for the proton oscillators,

$$G_p=\exp\left[-\sum_{\alpha=2}^{Z}(U_\alpha^2+\tfrac{1}{2}a_\alpha^2\rho_\alpha^2)\right]\Delta, \tag{11}$$

where Δ is a $Z\times Z$ determinant with elements

$$\Delta_{mn}=s_n(m)\exp(2\sum_{\alpha,\beta=2}^{Z}T_{m\beta}T_{n\alpha}a_\beta\mathbf{U}_\alpha\cdot\boldsymbol{\varrho}_\beta). \tag{12}$$

Here $\mathbf{U}_\alpha$ is the vector whose Cartesian components are the independent auxiliary variables u_α, v_α, w_α. The spin function $s_n(m)$ is the nth (i.e., first or second) of two independent spin functions for the mth particle; for example, $s_1(m)$ and $s_2(m)$ might correspond to the spin of the mth particle having a z component of $\hbar/2$ and $-\hbar/2$, respectively.

The generating function G_p for the proton normal modes $2\cdots Z$ is so defined as to be antisymmetric under interchange of space and spin coordinates of any two protons. Thus, if G_p is expressed as a power series in the components of the U_α, the coefficient of each separate product of powers of all these components is an (unnormalized) antisymmetric energy eigenfunction for the normal modes $2\cdots Z$, with the corresponding energy

$$E_p=\sum_{\alpha=2}^{Z}(l_\alpha+m_\alpha+n_\alpha+\tfrac{3}{2})\hbar(\beta_\alpha/m)^{\frac{1}{2}}-\sum_{k>j=1}^{Z}D_{kj}. \tag{13}$$

The quantum numbers l_α, m_α, n_α are, respectively, the powers of the u_α, v_α, and w_α in the term in question. Because of the β_α here are all equal [Eq. (8)], the energy E_p depends only on the sum K_p of these quantum numbers and is in general degenerate.

If the power series representing G_p is regarded as a Taylor series in the u_α, v_α, w_α, one can see that any one of the antisymmetrized eigenfunctions mentioned above is proportional to one of the derivatives of G_p with respect to the u_α, v_α, w_α evaluated with these variables all equal to zero; the order of differentiation with respect to any one of the variables is equal to one of the quantum numbers l_α, m_α, n_α; the total order of the derivative is thus the total excitation of the proton oscillators $2\cdots Z$ above their apparent ground-state energy, in units of $\hbar(\beta_p/m)^{\frac{1}{2}}$. Examination of the derivatives of G_p reveals that all vanish below a certain total order K_p, which depends on how many protons are present and on how equally they are divided between the two spin functions. For a given number of protons with given spin functions, K_p is just the same as it would be if the protons moved independently as isotropic harmonic oscillators. A similar result holds for the neutron oscillators $Z+2, \cdots, A$. The center-of-mass oscillator (No. 1) and the neutron-proton oscillator $(Z+1)$ are unaffected by antisymmetry.

The fact that the Pauli exclusion principle affects the ground-state energy of the set of interacting nucleons in just the same way as if they did not interact, suggests that nuclear shell behavior may be a property of independent collective modes of motion rather than of independent particles moving in an average central field. However, the Hamiltonian H_0 (which gives closed shells at N or $Z=2$, 8, 20, 40, 70, and 112) needs to be modified if it is to give the correct shells, perhaps through inclusion of spin-orbit interactions of the type $\mathbf{r}\times\mathbf{p}\cdot\mathbf{s}$, or tensor forces.

IV. PARITIES, ANGULAR MOMENTA, AND MAGNETIC MOMENTS

It is easily seen that any energy eigenfunction for the internal oscillators (excluding center-of-mass functions) has a parity which is that of the sum of all the quantum numbers, $\sum_{\alpha=2}^{A}(l_\alpha+m_\alpha+n_\alpha)$. Also, from Eqs. (3)–(6), although the "orbital angular momentum" $\boldsymbol{\mu}_\alpha\equiv\boldsymbol{\rho}_\alpha\times\boldsymbol{\pi}_\alpha$ of a given normal-mode oscillator cannot be expressed in terms of the orbital angular momenta $\mathbf{M}_k=\mathbf{r}_k\times\mathbf{p}_k$ of the nucleons, the total orbital angular momentum of the nucleus with respect to a fixed origin can be expressed either as $\sum_{k=1}^{A}\mathbf{M}_k$ or as $\sum_{\alpha=1}^{A}\boldsymbol{\mu}_\alpha$, for these sums are equal.

Furthermore, if $\mathbf{M}$ is the angular momentum of the nucleus with respect to its center of mass,

$$\mathbf{M}=\sum_{\alpha=2}^{A}\boldsymbol{\mu}_\alpha=\mathbf{M}_p+\mathbf{M}_n+\mathbf{M}_{np}, \tag{14}$$

where $\mathbf{M}_p\equiv\sum_{\alpha=2}^{Z}\boldsymbol{\mu}_\alpha$ and $\mathbf{M}_n\equiv\sum_{\alpha=Z+2}^{A}\boldsymbol{\mu}_\alpha$ are the angular momenta of protons and neutrons about their respective centers of mass, and $\mathbf{M}_{np}\equiv\boldsymbol{\mu}_{Z+1}$ is the angular momentum of relative motion of neutrons and protons.

It is well known that a single three-dimensional isotropic harmonic oscillator with Cartesian quantum numbers l, m, n is in a mixture of angular momentum states with eigenvalues given by $L=l+m+n$, $l+m+n-2$, $l+m+n-4$, $\cdots 0$ or 1. Therefore, an energy eigenfunction for the proton normal modes $2\cdots Z$, with total quantum number K_p, is in general a mixture of eigenfunctions of $M_p{}^2$ and M_{pz} (with eigenvalues $L_p\cdot(L_p+1)\cdot\hbar^2$, and $L_{pz}\hbar$, respectively), with L_p assuming all integral values from 0 to K_p, and L_{pz} ranging between $\pm L_p$. A complete set of simultaneous eigenfunctions of proton energy, $M_p{}^2$, and M_{pz} can be formed as linear combinations of proton oscillator eigenfunctions with a common K_p but various values of the separate l_α, m_α, n_α. For any given Z, the ground state splits into the same eigenstates of $M_p{}^2$ and M_{pz} as if the protons moved independently in a harmonic-oscillator well; likewise, $M_n{}^2$ and M_{nz} for a given N. $M_{np}{}^2=0$ in the ground state.

If the spin functions s_n are chosen to correspond to spin "up" and spin "down," the wave functions generated are eigenfunctions of S_z, the z component of total spin. Linear combinations of generating functions which differ through interchange of spin functions can be chosen so that they generate eigenfunctions of S_z and S^2. The antisymmetrized ground state of H_0 always has S_p and S_n each equal to 0 or $\frac{1}{2}\hbar$, according as Z and N are even or odd.

Thus parity, and orbital and spin angular momenta, for the ground state of any nucleus are just the same as if the nucleons of each type moved without interacting in their own harmonic-oscillator central fields. Because H_0 does not lead to interaction moments,[8] the same statement applies to ground-state magnetic moments.

V. RADIATIVE TRANSITIONS

The n *vs* p oscillator $(Z+1)$ in the present treatment resembles the oscillator discussed by Goldhaber and Teller *et al.*[9] in connection with the nuclear photoeffect. In fact, the only nonvanishing electric-dipole matrix elements from the ground state are those which correspond to single excitation of this oscillator. To first order, then, the electric dipole transitions have the integrated cross section,

$$\int\sigma dE=(\pi e^2h/mc)(NZ/A), \tag{15}$$

with the excited state being at an energy above ground of

$$E-E_g=\hbar(Ab_{np}/m)^{\frac{1}{2}}. \tag{16}$$

If one calculates b_{np} and D for the deuteron so as to get the correct binding energy (2.23 Mev) and a reasonable wave function (mean separation of 3.16×10^{-13} cm as in a square well of radius 2×10^{-13} cm), one obtains

$$\begin{aligned}&b_{np}=3.37\times10^{25}\text{ Mev/cm}^2,\\&D=10.15\text{ Mev},\\&V=0\text{ at }r=7.76\times10^{-13}\text{ cm},\\&h\nu=5.28\text{ Mev}.\end{aligned} \tag{17}$$

The fact that $h\nu$ is in the neighborhood of the calculated energy of maximum photoelectric cross section[10] (4.46 Mev) is little better than a coincidence in the case of the deuteron, in which no transfers of energy need to occur before disintegration (i.e., the lifetime of the intermediate state is extremely short).

[8] R. G. Sachs and N. Austern, Phys. Rev. **81**, 705 (1951).
[9] M. Goldhaber and E. Teller, Phys. Rev. **74**, 1046 (1948); J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950); Ferentz, Gell-Mann, and Pines, Phys. Rev. **92**, 836 (1953).
[10] See, for example, H. A. Bethe, *Elementary Nuclear Theory* (John Wiley and Sons, Inc., New York, 1947).

Equation (16) implies that the resonant energy of the photoeffect varies as $(Ab_{np})^{\frac{1}{2}}$. Experiment[11] reveals a proportionality to approximately $A^{-1/6}$ in the medium and heavy range, so one must assume that b_{np} is approximately proportional to $A^{-4/3}$, i.e., that many-body forces are present. Inasmuch as the form of H_0 rules out exchange forces and velocity-dependent forces with saturation properties, it is not surprising that many-body forces should need to be invoked here, even though their form is oversimplified. In keeping with the assumed form, one should take D_{kj} for each pair of nucleons to be a decreasing function of A, also, so that the ground states of heavy nuclei are not too negative in energy (zero energy being defined as the lowest energy at which one or more nucleons can escape to infinity from an actual nucleus).

A nucleus with the Hamiltonian H_0, having absorbed a photon, is unable to disintegrate; it can only return to ground by re-emission of the photon. The fact that actual nuclei emit particles or undergo fission upon absorbing gamma rays indicates not only that the Hooke's-law forces should be cut off at some finite range, but also that the normal-mode oscillators interact with each other strongly enough to permit the *n vs p* oscillator to share its energy with the others in most cases before emitting a photon. Possibly the same perturbation can produce both effects. A complete account of the process would explain how the energy goes into the particular mixture of normal-mode states which corresponds to emission of one nucleon, emission of two nucleons, or fission. The fact that in an emission process the center of mass of the neutrons moves away from that of the protons indicates that the *n vs p* oscillator remains excited with a certain probability. One would expect the same effect in photofission to produce an unequal division of charge between the fragments. A perturbation now being studied is

$$H'=\sum_{k>j=1}^{A}\{D_{kj}-\tfrac{1}{2}b_{kj}(\mathbf{r}_k-\mathbf{r}_j)^2 -\tfrac{1}{2}B_{kj}\exp[(\mathbf{r}_k-\mathbf{r}_j)^2/\lambda_{kj}^2]\}, \quad (18)$$

which replaces the quadratic inter-nucleon potentials by Gaussian potentials. The effect of such a perturbation is difficult to calculate because the Gaussian terms involve all the products of the nucleon position vectors or of the normal-mode vectors. It is tempting to cut off the forces by means of an H' which, like the original V_0, is a sum of terms for the different normal-mode oscillators, or to start with a V which is such a sum. However, it does not seem easy to find such a sum which is still symmetric under interchange of like particles, and even if available, such a potential would not contain the interaction of normal-mode oscillators needed to account for emission of one or two particles.

According to the nuclear model under discussion, by far the most probable photon absorption raises the *n-p* oscillator to its first excited state. If, as seems likely, *n-n*, *n-p*, and *p-p* forces are approximately equal, the Hooke's-law nucleus has no other excited levels much below this one. But actually the energy of this level is positive, or in the continuum. Thus it would appear that H_0, unmodified, cannot possibly account for the many excited states of nuclei which are stable against nucleon emission. Possibly the cutting-off procedure discussed above will sufficiently lower some of the other excited states in relation to the first excited state of the *n-p* oscillator, or maybe the spin-orbit forces needed to produce the correct shells will resolve much of the degeneracy of the present ground energy and thus lead to the proper excited states.

The fact that neutron capture by a nucleus is more often followed by several gamma rays in cascade than by a single gamma ray to ground[12] suggests that the photon-excited nucleus from which a neutron is emitted is in a state orthogonal to some of those produced by neutron capture. Possibly some of the latter have their excess energy mainly in excitation of other oscillators than the *n-p* oscillator. If so, their gamma rays do not arise from electric dipole transitions. In fact, because the *n-p* oscillator is the only one which can ever undergo an electric dipole transition, and its excited states are in the continuum, the present model indicates that $E1$ transitions will not be observed at all, except in conjunction with emission or capture of one or more nucleons.[13] This is true, however, only if the relative energies of the nuclear states are not much changed by perturbations. If H_0 is perturbed as in Eq. (18), or by spin-orbit couplings (e.g., $\sum_{k=1}^{A}\mathbf{s}_k\cdot\mathbf{M}_k$ or $\mathbf{S}_p\cdot\mathbf{M}_p$), some of the ground states of H_0 may be raised and some of the "first excited" states (of opposite parity) may be lowered to negative energies, so that a nucleus which cannot emit a particle can still emit an $E1$ γ ray, followed by another γ ray which is associated with no change of parity.

VI. CONCLUSIONS

It appears that ordinary inter-nucleon forces obeying Hooke's law lead to a nuclear model having the same ground-state energy, parity, angular momentum, and magnetic moment as the single-particle harmonic-oscillator model. The model exhibits saturation if the force parameters depend properly on A. Cutting off of the forces at finite range and addition of spin-orbit coupling seem likely to give correct shells, may also yield correct excited bound states, and will affect magnetic moments in ways as yet unknown. Possibly, too, such considerations may provide a new point of view toward such other phenomena as surface vibrations, fission, and nuclear reactions and scattering.

[11] Montalbetti, Katz, and Goldemberg, Phys. Rev. **91**, 659 (1953).

[12] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

[13] See M. G. Mayer, Phys. Rev. **78**, 16 (1950).