Derivation of Magnetostriction and Anisotropic Energies for Hexagonal, Tetragonal, and Orthorhombic Crystals

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In order to determine the measurements necessary to characterize the anisotropic energy and the saturation magnetostriction in hexagonal cobalt, a phenomenological derivation has been given for the equations which characterize the effects. Out to fourth rank tensors, the results are the same as those for circular symmetry and it requires two constants to specify the anisotropic energy and four to specify the magnetostriction. When sixth rank tensors are evaluated, a characteristic hexagonal symmetry appears. It requires four constants to characterize the anisotropic energy and nine to characterize the magnetostriction. These constants can be measured by using two oriented slabs. Four of the constants can be determined by measurements parallel to the saturation magnetization, four when the magnetostriction is perpendicular to the magnetization and one when they are 45° apart.

In the Appendix the first approximations for the magnetostrictive and anisotropy energies are derived for tetragonal and orthorhombic crystals.

I. INTRODUCTION

In order to determine what measurements have to be taken to characterize the saturation magnetostriction and the magnetic anisotropic energy of a hexagonal cobalt crystal (class $6/m$, $2/m$, $2/m$), which are discussed in an accompanying paper by R. M. Bozorth, a phenomenological derivation has been given for the equations which describe these effects. These equations can be derived from a general thermodynamic function of which there are several depending on which variables are considered the fundamental ones. In order to give directly the measured magnetostriction constants and the magnetic anisotropic energy at constant stress (the ordinarily measured values), it is better to use the stresses, intensities of magnetization, and entropy as the fundamental variables. Furthermore, since we are not interested in temperature effects, we can assume the entropy constant and introduce only the stresses and intensity of magnetization as the fundamental variables.

A general expression has been derived in a previous paper[1] for a crystal with a center of symmetry. Since cobalt belongs to the crystal class $6/m$, $2/m$, $2/m$ (Herman-Mauguin) or D_{6h} (Schönflies), it has a center of symmetry and hence the expression derived previously is valid. In tensor notation, this expression can be written in the form

$$2H_1=-[s_{ijkl}{}^{I}T_{ij}T_{kl}+R_{ijklno}I_iI_jT_{kl}T_{no}+M_{ijmn}I_iI_jT_{mn}+N_{ijklmn}I_iI_jI_kI_lT_{mn}]+K_{mn}{}^{T}I_mI_n+K_{mnop}{}^{T}I_mI_nI_oI_p+K_{mnopqr}{}^{T}I_mI_nI_oI_pI_qI_r. \quad (1)$$

In this equation T_{ij} are the stresses expressed as a second rank tensor with i interchangeable with j, I_i are the three magnetic intensities along the three rectangular coordinates, $s_{ijkl}{}^{I}$ are the elastic compliances, R_{ijklno} are the terms added to the elastic compliances when the crystal is magnetized in various directions (neglected in the present paper), M_{ijmn} are the first-order magnetostriction terms, N_{ijklmn} the second-order magnetostriction terms, and the three K terms are the first-, second-, and third-order anisotropic energy terms, all measured for constant stress.

All that is required to evaluate these terms is a knowledge of the components of the tensors in the sym-

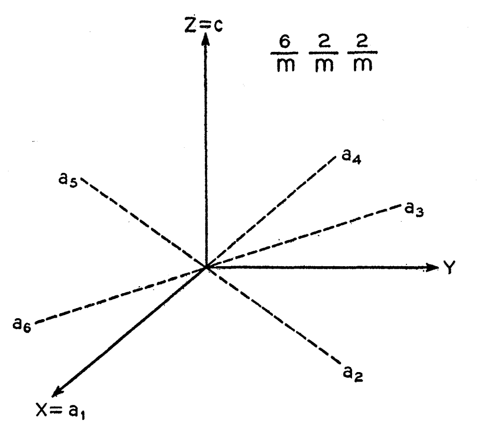

FIG. 1. Relation between x, y, and z rectangular axes and the crystallographic axes.

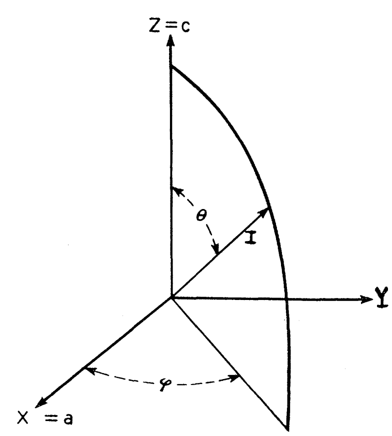

FIG. 2. Relation between spherical coordinate angles and rectangular x, y, and z axes.

[1] W. P. Mason, Phys. Rev. 82, 715 (1951).

metry system $6/m$, $2/m$, $2/m$. Up to fourth rank tensors, the number and kinds of terms are well known and are given in Sec. II.

The components of sixth rank tensors for this symmetry have been derived only recently.[2] As discussed in Sec. III, these components can be used to evaluate the morphic R constants, the second-order (N) magnetostriction terms and the third-order anisotropic energy terms. In all the sixth rank tensors, the characteristic hexagonal symmetry appears and the results are different from those derived on the basis of circular symmetry.

II. ELASTIC COMPLIANCES, FIRST-ORDER MAGNETOSTRICTION, AND FIRST- AND SECOND-ORDER ANISOTROPIC ENERGY TERMS IN COBALT

The matrices of the second and fourth rank tensors using the engineering shearing strains, i.e., ($S_{ij}=\partial u_i/\partial x_j+\partial u/\partial x_i$) rather than the tensor strains, can be written in the form[3]

$$K_{mn}=\begin{vmatrix} K_{11} & 0 & 0 \\ 0 & K_{11} & 0 \\ 0 & 0 & K_{33} \end{vmatrix};\quad s_{ijkl}=\begin{vmatrix} s_{1111} & s_{1122} & s_{1133} & 0 & 0 & 0 \\ s_{1122} & s_{1111} & s_{1133} & 0 & 0 & 0 \\ s_{1133} & s_{1133} & s_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{1111}-s_{1122}) \end{vmatrix};$$

$$K_{mnop}=\begin{vmatrix} K_{1111} & K_{1122} & K_{1133} & 0 & 0 & 0 \\ K_{1122} & K_{1111} & K_{1133} & 0 & 0 & 0 \\ K_{1133} & K_{1133} & K_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{1133} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{1133} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{1111}-K_{1122} \end{vmatrix}; \qquad (2)$$

$$M_{ijmn}=\begin{vmatrix} M_{1111} & M_{1122} & M_{1133} & 0 & 0 & 0 \\ M_{1122} & M_{1111} & M_{1133} & 0 & 0 & 0 \\ M_{3311} & M_{3311} & M_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{1111}-M_{1122} \end{vmatrix}.$$

Since the first two and last two numbers of the subscripts can be interchanged with each other, it has become customary to replace two interchangeable indices by single indices according to the convention

$$1=11;\quad 2=22;\quad 3=33;\quad 4=23;\quad 5=13;\quad 6=12.$$

With this substitution we can write the expression for H_1 as

$$\begin{aligned} 2H_1=&-[s_{11}{}^I(T_1{}^2+T_2{}^2)+2s_{12}{}^IT_1T_2+2s_{13}{}^I(T_1+T_2)T_3 \\ &+s_{33}{}^IT_3{}^2+s_{44}{}^I(T_4{}^2+T_5{}^2)+2(s_{11}{}^I-s_{12}{}^I)T_6{}^2 \\ &+2M_{11}(I_1{}^2T_1+I_2{}^2T_2)+2M_{12}(I_1{}^2T_2+I_2{}^2T_1) \\ &+2M_{13}(I_1{}^2T_3+I_2{}^2T_3)+2M_{31}[I_3{}^2(T_1+T_2)] \\ &+2M_{33}I_3{}^2T_3+4M_{44}(I_2I_3T_4+I_1I_3T_5) \\ &+4(M_{11}-M_{12})I_1I_2T_6]+K_1{}^T(I_1{}^2+I_2{}^2)+K_3{}^TI_3{}^2 \\ &+K_{11}{}^T(I_1{}^4+2I_1{}^2I_2{}^2+I_2{}^4)+6K_{13}{}^T(I_1{}^2+I_2{}^2)I_3{}^2 \\ &+K_{33}{}^TI_3{}^4, \end{aligned} \qquad (3)$$

where the z or 3 axis coincides with the c hexagonal axis as shown by Fig. 1, and the x axis coincides with one of the 6 hexagonal a axes.

These equations hold for a crystal with a large number of domains when the directions of the domains are uncorrelated, for then the components of magnetization are independent. For a single domain or for a saturated crystal, the intensity of magnetization has a fixed value I_s, and only the direction of magnetism can be changed. If α_1, α_2, α_3 are the direction cosines of the intensity of magnetization with respect to the crystal axes, one has

$$I_1=\alpha_1I_s;\quad I_2=\alpha_2I_s;\quad I_3=\alpha_3I_s, \qquad (4)$$

where $\alpha_1{}^2+\alpha_2{}^2+\alpha_3{}^2=1$.

Furthermore, the energy H_1 is usually expressed as a change from the energy of the demagnetized crystal. For cobalt the direction of easy magnetization is along the hexagonal axis with $\alpha_3=\pm1$. Hence, on the average,

$$\sum_1^n\alpha_3=0;\quad \frac{1}{n}\sum_1^n\alpha_3{}^2=1.$$

With these substitutions the expression for $2H_1$ becomes

$$\begin{aligned} 2H_1=&-[s_{11}{}^I(T_1{}^2+T_2{}^2)+2s_{12}{}^IT_1T_2+2s_{13}{}^I(T_1+T_2)T_3 \\ &+s_{33}{}^IT_3{}^2+s_{44}{}^I(T_4{}^2+T_5{}^2)+2(s_{11}{}^I-s_{12}{}^I)T_6{}^2] \\ &-[2(M_{11}-M_{12})I_s{}^2(\alpha_1{}^2T_1+\alpha_2{}^2T_2+2\alpha_1\alpha_2T_6) \\ &+2(M_{33}-M_{13})I_s{}^2(\alpha_3{}^2-1)T_3+2(M_{31}-M_{12}) \\ &\times I_s{}^2(\alpha_3{}^2-1)(T_1+T_2)+4M_{44}I_s{}^2(\alpha_2\alpha_3T_4+\alpha_1\alpha_3T_5)] \\ &+[(K_1-K_3)I_s{}^2+(2K_{11}-6K_{13})I_s{}^4][1-\alpha_3{}^2] \\ &+[4K_{13}-K_{11}-K_{33}]I_s{}^4[1-\alpha_3{}^4]. \end{aligned} \qquad (5)$$

[2] R. Fieschi and F. G. Fumi, Nuovo cimento **10**, 865 (1953). Hexagonal and trigonal systems were also considered by Dr. H. Wondratschek and are in course of publication.

[3] The K_{mn} tensor is similar to the dielectric tensor and the s_{ijkl} tensor is given in W. P. Mason, *Piezoelectric Crystals and Their Application to Ultrasonics* (D. Van Nostrand Company, Inc., New York, 1950). The K_{mnop} tensor can be obtained from the s_{ijkl} tensor by interchanging all the subscripts. The M_{ijkl} tensor is similar to the photoelastic tensor of Pockels as corrected by Bhagavantum.

Since

$$(1-\alpha_3^2)=1-\cos^2\theta=\sin^2\theta,\quad (1-\alpha_3^4)=(1-\cos^4\theta)=2\sin^2\theta-\sin^4\theta,\quad (6)$$

where θ is the angle of the saturation magnetization from the hexagonal axis, as shown by Fig. 2, the anisotropic energy can be written in the form

$$A\sin^2\theta+B\sin^4\theta,\quad (7)$$

where

$$A=\tfrac{1}{2}(K_1-K_3)I_s^2+(K_{13}-K_{33})I_s^4,\quad B=\tfrac{1}{2}(K_{11}+K_{33}-4K_{13})I_s^4.$$

This is a well-known result for cobalt.[4]

The form of the saturation magnetostriction, which has not previously been derived, can be obtained from the equation for the magnetostrictive strain in any direction which is given by the formula

$$\lambda=-\beta_1^2\frac{\partial H_1}{\partial T_1}-\beta_2^2\frac{\partial H_1}{\partial T_2}-\beta_3^2\frac{\partial H_1}{\partial T_3}-\beta_2\beta_3\frac{\partial H_1}{\partial T_4}-\beta_1\beta_3\frac{\partial H_1}{\partial T_5}-\beta_1\beta_2\frac{\partial H_1}{\partial T_6},\quad (8)$$

where β_1, β_2, β_3 are the direction cosines of the direction for magnetostrictive strain with respect to the x, y, and z axes defined above and λ is the saturation magnetostriction. Performing the above differentiation and collecting terms,

$$\lambda=(M_{11}-M_{12})I_s^2(\alpha_1\beta_1+\alpha_2\beta_2)^2+(M_{12}-M_{31})\times I_s^2(1-\alpha_3^2)(1-\beta_3^2)+(M_{13}-M_{33})I_s^2(1-\alpha_3^2)\beta_3^2+2M_{44}I_s^2(\alpha_1\beta_1+\alpha_2\beta_2)\alpha_3\beta_3.\quad (9)$$

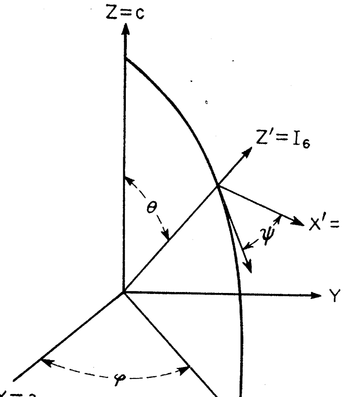

FIG. 3. Relation between rotated coordinate system x', y', and z' and original rectangular coordinate system x, y, and z.

[4] R. M. Bozorth, *Ferromagnetism* (D. Van Nostrand Company, Inc., New York, 1951), p. 564.

To evaluate the M constants in terms of the measured values of magnetostriction, we first put the field in the same direction as the magnetostriction. Then $\alpha_1=\beta_1$, $\alpha_2=\beta_2$, $\alpha_3=\beta_3$, and

$$\lambda=(M_{11}-M_{31})I_s^2(1-\beta_3^2)^2+(-M_{33}+M_{13}+2M_{44})I_s^2(1-\beta_3^2)\beta_3^2.\quad (10)$$

Hence, only two constants can be obtained under these conditions. These are obtained by setting $\alpha_1=\beta_1=1$, giving λ_A and $\alpha_1=\beta_1=1/\sqrt{2}$, $\alpha_3=\beta_3=1/\sqrt{2}$, giving λ_D. From Eq. (10),

$$\lambda_A=(M_{11}-M_{31})I_s^2;\quad \lambda_D=[\tfrac{1}{4}(M_{11}-M_{31})+\tfrac{1}{4}(-M_{33}+M_{13}+2M_{44})]I_s^2.\quad (11)$$

To determine the other two constants, we have to determine the expansion in a different direction from the field. The two simplest directions are when the field is perpendicular to the hexagonal axis and the saturation magnetostriction is measured at right angles to the field in the basal plane, and perpendicular to the basal plane. Calling $\lambda_B(\alpha_1=1,\beta_2=1)$ and $\lambda_C(\alpha_1=1,\beta_3=1)$ the two values, we find for these

$$\lambda_B=(M_{12}-M_{31})I_s^2;\quad \lambda_C=(M_{13}-M_{33})I_s^2.\quad (12)$$

With these values, the four constants become

$$(M_{11}-M_{12})I_s^2=\lambda_A-\lambda_B,\quad (M_{12}-M_{31})I_s^2=\lambda_B,\quad (M_{13}-M_{33})I_s^2=\lambda_C,\quad 2M_{44}I_s^2=(-\lambda_A+4\lambda_D-\lambda_C).\quad (13)$$

Inserting these values in the magnetostrictive equation, we find

$$\lambda=\lambda_A[(\alpha_1\beta_1+\alpha_2\beta_2)^2-(\alpha_1\beta_1+\alpha_2\beta_2)\alpha_3\beta_3]+\lambda_B[(1-\alpha_3^2)(1-\beta_3^2)-(\alpha_1\beta_1+\alpha_2\beta_2)^2]+\lambda_C[(1-\alpha_3^2)\beta_3^2-(\alpha_1\beta_1+\alpha_2\beta_2)\alpha_3\beta_3]+4\lambda_D(\alpha_1\beta_1+\alpha_2\beta_2)\alpha_3\beta_3.\quad (14)$$

These four constants have been evaluated for cobalt in the following paper by Bozorth, who finds the following values:

$$\lambda_A=-45\times10^{-6},\quad \lambda_B=-95\times10^{-6},\quad \lambda_C=+110\times10^{-6},\quad \lambda_D=-100\times10^{-6}.\quad (15)$$

The properties of polycrystalline materials can be calculated to this order of approximation by assuming that all orientations of the crystal grains are equally probable. The two quantities of interest are the magnetostriction $\lambda_{\parallel}$ along a rod in the direction of the applied field and the change in dimension, $\lambda_{\perp}$, perpendicular to the applied field. The volume magnetostriction ω is then equal to

$$\omega=\lambda_{\parallel}+2\lambda_{\perp}.\quad (16)$$

If we saturate along a given direction, the increase in length along this direction for any orientation in the crystal is given by

$$\lambda=\lambda_A[(1-\alpha_3^2)^2-(1-\alpha_3^2)\alpha_3^2]+4\lambda_D(1-\alpha_3^2)\alpha_3^2, \quad (17)$$

where α_3 is the direction cosine between the direction of the magnetic field and the Z axis of the crystal. If we call this angle θ, the expression for λ is

$$\lambda=\lambda_A[\sin^4\theta-\sin^2\theta\cos^2\theta]+4\lambda_D\sin^2\theta\cos^2\theta. \quad (18)$$

To obtain $\lambda_{\parallel}$, we have to average this expression over all possible orientations. The average values of $\sin^4\theta$ and $\sin^2\theta\cos^2\theta$ over a unit sphere are given by the integrals

$$\int_0^{\pi/2}\sin^5\theta d\theta=\frac{8}{15}, \quad \int_0^{\pi/2}\sin^3\theta\cos^2\theta d\theta=\frac{2}{15}. \quad (19)$$

Hence

$$\lambda_{\parallel}=\tfrac{2}{5}\lambda_A+(8/15)\lambda_D. \quad (20)$$

The volume magnetostriction of a single crystal material can be calculated by taking a three-axis system x', y', z' located in any orientation and adding the magnetostrictive strains in all three systems. If we take the axes of the three-axis system with respect to the axes for which the direction of magnetization has the direction cosines α_1, α_2, α_3 as shown by Fig. 3, the direction cosines for the x', y' and z' axes are given by

$$\begin{aligned} x'&: \beta_1=\cos\theta\cos\varphi;\ \beta_2=\cos\theta\sin\varphi;\ \beta_3=-\sin\theta;\\ y'&: \beta_1=-\sin\varphi;\ \beta_2=\cos\varphi;\ \beta_3=0; \quad (21)\\ z'&: \beta_1=\sin\theta\cos\varphi;\ \beta_2=\sin\theta\sin\varphi;\ \beta_3=\cos\theta. \end{aligned}$$

Using these values in Eq. (14), leaving α_1, α_2, α_3 arbitrary, and adding all these values of λ, we find

$$(\lambda_{x'}+\lambda_{y'}+\lambda_{z'})=\omega=(\lambda_A+\lambda_B+\lambda_C)(1-\alpha_3^2)=(\lambda_A+\lambda_B+\lambda_C)\sin^2\theta, \quad (22)$$

where θ is the angle of magnetization with respect to the hexagonal axis. Hence, irrespective of the orientation of the crystallite considered, the volume magnetostriction depends only on the direction of the intensity of magnetization with respect to the hexagonal axis.

For a polycrystalline material, this angle is averaged over all directions and hence, since

$$\int_0^{\pi/2}\sin^3\theta d\theta=\tfrac{2}{3}, \quad (23)$$

we find that the volume magnetostriction for a polycrystalline material is

$$\bar{\omega}=\tfrac{2}{3}(\lambda_A+\lambda_B+\lambda_C). \quad (24)$$

From Eqs. (16) and (20) we find

$$\lambda_{\perp}=(2/15)\lambda_A+\tfrac{1}{3}(\lambda_B+\lambda_C)-(4/15)\lambda_D. \quad (25)$$

III. ANISOTROPIC ENERGY AND SATURATION MAGNETOSTRICTION TAKING ACCOUNT OF SIXTH RANK TENSORS

When one takes account of the sixth rank tensors of Eq. (1), additional terms are added to the anisotropic energy terms and the saturation magnetostriction, and these reveal the hexagonal properties of the crystal. The simplest term to consider is the anisotropy energy term

$$K_{mnopqr}I_mI_nI_oI_pI_qI_r. \quad (26)$$

According to Fieschi and Fumi,[2] the energy product has the form

$$K_{111}I_1^6+15K_{112}I_1^4I_2^2+15K_{122}I_1^2I_2^4+15K_{113}I_1^4I_3^2+15K_{223}I_2^4I_3^2+90K_{123}I_1^2I_2^2I_3^2+15K_{133}I_1^2I_3^4+15K_{233}I_2^2I_3^4+K_{333}I_3^6. \quad (27)$$

In the hexagonal system $6/m$, $2/m$, $2/m(D_{6h})$, there are a number of relations between the constants. For the case where all six indices can be interchanged, in three-index symbols, these relations take the form

$$5K_{112}=-2K_{111}+3K_{222};\quad 5K_{122}=3K_{111}-2K_{222};\quad K_{113}=3K_{123};\quad K_{113}=K_{223};\quad K_{133}=K_{233}. \quad (28)$$

Introducing these values, the expression for the anisotropic energy becomes

$$K_{111}[I_1^6-6I_1^4I_2^2+9I_1^2I_2^4]+K_{222}[I_2^6-6I_2^4I_1^2+9I_2^2I_1^4]+15K_{113}[I_1^2+I_2^2]^2I_3^2+15K_{133}(I_1^2+I_2^2)I_3^4+K_{333}I_3^6. \quad (29)$$

Introducing the values

$$I_1=\alpha_1I_s,\quad I_2=\alpha_2I_s,\quad I_3=\alpha_3I_s, \quad (30)$$

and subtracting out the demagnetization energy, the sixth rank term becomes

$$K_{111}I_s^6(\alpha_1^6-6\alpha_1^4\alpha_2^2+9\alpha_1^2\alpha_2^4)+K_{222}I_s^6(\alpha_2^6-6\alpha_2^4\alpha_1^2+9\alpha_2^2\alpha_1^4)+15K_{113}I_s^6(\alpha_1^2+\alpha_2^2)^2\alpha_3^2+15K_{133}I_s^6(\alpha_1^2+\alpha_2^2)\alpha_3^4+K_{333}I_s^6(\alpha_3^6-1). \quad (31)$$

If we go to spherical coordinates as shown by Fig. 2,

$$\alpha_1=\sin\theta\cos\varphi;\quad \alpha_2=\sin\theta\sin\varphi;\quad \alpha_3=\cos\theta. \quad (32)$$

Introducing these values and combining and reducing terms, and adding the lower order terms of (7), the total anisotropic energy can be written in the form

$$A\sin^2\theta+B\sin^4\theta+\sin^6\theta(C+D\cos6\varphi), \quad (33)$$

where

$$\begin{aligned} A&=\tfrac{1}{2}(K_1-K_3)I_s^2+(K_{13}-K_{33})I_s^4-\tfrac{3}{2}K_{333}I_s^6;\\ B&=\tfrac{1}{2}(K_{11}+K_{33}-4K_{13})I_s^4+\tfrac{1}{2}(15K_{113}-30K_{133}+3K_{333})I_s^6;\\ C&=[\tfrac{1}{2}(-15K_{113}+15K_{133}-K_{333})+\tfrac{1}{4}(K_{111}+K_{222})]I_s^6;\\ D&=\tfrac{1}{4}(K_{111}-K_{222})I_s^6. \end{aligned} \quad (34)$$

The sixth rank magnetostriction tensor containing the second-order magnetostrictive terms can be evaluated in a similar manner. The energy products become

$$N_{111}I_1^4T_1+(N_{112}I_1^4T_2+6N_{121}I_1^2I_2^2T_1+8N_{166}I_1^3I_2T_6)$$
$$+(N_{221}I_2^4T_1+6N_{122}I_1^2I_2^2T_2+8N_{266}I_1I_2^3T_6)$$
$$+(N_{113}I_1^4T_3+6N_{131}I_1^2I_3^2T_1+8N_{155}I_1^3I_3T_5)$$
$$+(N_{331}I_3^4T_1+6N_{133}I_1^2I_3^2T_3+8N_{355}I_1I_3^3T_5)$$
$$+(N_{223}I_2^4T_3+6N_{232}I_2^2I_3^2T_2+8N_{244}I_2^3I_3T_4)$$
$$+(N_{332}I_3^4T_2+6N_{233}I_2^2I_3^2T_3+8N_{344}I_3^3I_2T_4)$$
$$+(6N_{123}I_1^2I_2^2T_3+6N_{132}I_1^2I_3^2T_2+6N_{231}I_2^2I_3^2T_1$$
$$+24N_{144}I_1^2I_2I_3T_4+24N_{255}I_1I_2^2I_3T_5$$
$$+24N_{366}I_1I_2I_3^2T_6)+N_{222}I_2^4T_2+N_{333}I_3^4T_3. \quad (35)$$

For a hexagonal symmetry $6/m$, $2/m$, $2/m$ there are a number of relations between these constants. On account of the form of Eq. (1), we can interchange the first four indices or, in terms of three-index terms,

$$N_{abc}=N_{bac}. \quad (36)$$

From this relationship, the equivalences given by Fieschi and Fumi[2] take the form

$$N_{112}=-2N_{111}+3N_{222}-4N_{166};$$
$$3N_{121}=-2N_{111}+3N_{222}-2N_{166};$$
$$N_{122}=-N_{111}+2N_{222}-2N_{166}-2N_{121};$$
$$N_{266}=N_{111}-N_{222}+N_{166};$$
$$N_{221}=2N_{222}-N_{111}-4N_{166};$$
$$N_{113}=3N_{123}=N_{223};\quad N_{131}=N_{132}+2N_{366}=N_{232};$$
$$N_{133}=N_{233};\quad N_{213}=N_{123};$$
$$N_{331}=N_{332};\quad N_{355}=N_{344};\quad N_{132}=N_{231};$$
$$N_{155}=3N_{144}=N_{244}. \quad (37)$$

These 16 relations reduce the number of constants appearing in the equation to 11. Introducing these values, the sixth rank tensor, after subtracting the demagnetized value $N_{331}I_s^4(1-\beta_3^2)+N_{333}I_s^4\beta_3^2$, becomes

$$N_{111}I_s^4[\alpha_1^4\beta_1^2+2\alpha_1^4\beta_2^2-8\alpha_1^3\alpha_2\beta_1\beta_2+3\alpha_2^4\beta_1^2+6\alpha_1^2\alpha_2^2\beta_2^2]$$
$$+N_{222}I_s^4[\alpha_2^4\beta_2^2-3\alpha_1^4\beta_2^2+12\alpha_1^3\alpha_2\beta_1\beta_2-4\alpha_2^4\beta_1^2$$
$$-6\alpha_1^2\alpha_2^2\beta_2^2+4\alpha_1\alpha_2^3\beta_1\beta_2]+6N_{121}I_s^4(\alpha_1\beta_2-\alpha_2\beta_1)^2(1-\alpha_3^2)$$
$$+3N_{123}I_s^4\beta_3^2(1-\alpha_3^2)^2+6N_{131}I_s^4\alpha_3^2(\alpha_1\beta_1+\alpha_2\beta_2)^2$$
$$+6N_{132}I_s^4\alpha_3^2(\alpha_1\beta_2-\alpha_2\beta_1)^2+8N_{155}I_s^4\alpha_3\beta_3(1-\alpha_3^2)$$
$$\times(\alpha_1\beta_1+\alpha_2\beta_2)+N_{331}I_s^4(\alpha_3^4-1)(1-\beta_3^2)$$
$$+6N_{133}I_s^4\alpha_3^2\beta_3^2(1-\alpha_3^2)+8N_{344}I_s^4\alpha_3^3\beta_3(\alpha_1\beta_1+\alpha_2\beta_2)$$
$$+N_{333}I_s^4(\alpha_3^4-1)\beta_3^2. \quad (38)$$

By combining these terms with the first-order terms of Eq. (19), and combining and reducing terms, the sum can be written in the form of a nine-constant equation:

$$\lambda=A[2\alpha_1\alpha_2\beta_1+(\alpha_1^2-\alpha_2^2)\beta_2]^2+B\alpha_3^2[(\alpha_1\beta_1+\alpha_2\beta_2)^2$$
$$-(\alpha_1\beta_2-\alpha_2\beta_1)^2]+C[(\alpha_1\beta_1+\alpha_2\beta_2)^2-(\alpha_1\beta_2-\alpha_2\beta_1)^2]$$
$$+D(1-\alpha_3^2)(1-\beta_3^2)+E\alpha_3^2\beta_3^2(1-\alpha_3^2)+F\alpha_3^2(1-\alpha_3^2)$$
$$+G\beta_3^2(1-\alpha_3^2)+H\alpha_3\beta_3(\alpha_1\beta_1+\alpha_2\beta_2)$$
$$+I\alpha_3^3\beta_3(\alpha_1\beta_1+\alpha_2\beta_2), \quad (39)$$

where

$$A=(N_{222}-N_{111})I_s^4;$$
$$B=[3(N_{131}-N_{132}+N_{121})+N_{111}-2N_{222}]I_s^4;$$
$$C=[(2N_{222}-N_{111}-3N_{121})I_s^4+\tfrac{1}{2}(M_{11}-M_{12})I_s^2];$$
$$D=[(2N_{111}-2N_{222}+3N_{121}-N_{331})I_s^4$$
$$+\tfrac{1}{2}(M_{11}+M_{12}-2M_{31})I_s^2];$$
$$E=[2(N_{111}-N_{222})+3(N_{121}-N_{131}-N_{132}-N_{123})$$
$$+6N_{133}+N_{331}-N_{333}]I_s^4;$$
$$F=[2(N_{222}-N_{111})-N_{331}+3(N_{131}+N_{132}-N_{121})]I_s^4;$$
$$G=[(3N_{123}-N_{333})I_s^4+(M_{13}-M_{33})I_s^2];$$
$$H=[8N_{155}I_s^4+2M_{44}I_s^2];$$
$$I=8(N_{344}-N_{155})I_s^4.$$

To evaluate the 9 constants requires the measurement of 9 independent orientations. The simplest measurements are those for which the magnetization is parallel or perpendicular to the saturation magnetostriction. Eight independent constants can be determined from these measurements. The remaining constant has to be determined when the magnetization and magnetostriction are in directions not 0° or 90° with respect to each other. For the parallel case,

$$\alpha_1=\beta_1;\quad \alpha_2=\beta_2;\quad \alpha_3=\beta_3, \quad (40)$$

and the equation becomes

$$\lambda_{\parallel}=A[3\alpha_1^2\alpha_2-\alpha_2^3]^2+B\alpha_3^2(1-\alpha_3^2)^2$$
$$+(C+D)(1-\alpha_3^2)^2+(E+I)\alpha_3^4(1-\alpha_3^2)$$
$$+(F+G+H)\alpha_3^2(1-\alpha_3^2). \quad (41)$$

Introducing spherical coordinates as shown by Fig. 2, we find that

$$\alpha_1=\sin\theta\cos\varphi;\quad \alpha_2=\sin\theta\sin\varphi;\quad \alpha_3=\cos\theta. \quad (42)$$

If these values are introduced in (41) and all the values are expressed in even powers of $\sin\theta$, Eq. (41) reduces to

$$\lambda_{\parallel}=A'\sin^2\theta+B'\sin^4\theta+C'\sin^6\theta+D'\sin^6\theta\cos6\varphi, \quad (43)$$

where

$$A'=E+F+G+H+I;$$
$$B'=B+C+D-2E-2I-F-G-H;$$
$$C'=\tfrac{1}{2}A-B+E+I;\quad D'=-\tfrac{1}{2}A.$$

The first three constants can be determined by measuring the magnetostriction when the values are measured in the x-z plane with λ_1 90° from z, λ_2 60° from z and λ_3 30° from z, while the fourth value is determined by measuring the saturation magnetostriction when $\theta=90°$, $\varphi=90°$, i.e., for measurements in the y-z plane with the direction of measurement 90° from the z axis as shown by Fig. 4(a). These four values are determined from the equations

$$\begin{aligned}\lambda_1&=A'+B'+C'+D';\\ \lambda_2&=\tfrac{3}{4}A'+(9/16)B'+(27/64)(C'+D');\\ \lambda_3&=\tfrac{1}{4}A'+(1/16)B'+(1/64)(C'+D');\\ \lambda_4&=A'+B'+C'-D'.\end{aligned}\tag{44}$$

Solving for these values, we get

$$\begin{aligned}A'&=8\lambda_3-(8/3)\lambda_2+\lambda_1;\\ B'&=-(56/3)\lambda_3+(40/3)\lambda_2-(16/3)\lambda_1;\\ (C'+D')&=(32/3)\lambda_3-(32/3)\lambda_2+(16/3)\lambda_1;\\ D'&=\tfrac{1}{2}(\lambda_1-\lambda_4).\end{aligned}\tag{45}$$

Hence 4 of the 9 constants can be evaluated by measurements for which the magnetostriction is in the same direction as the saturation magnetization.

These constants also determine the value of the saturation magnetostriction for a polycrystalline material, since as shown by Eq. (17), all the contributions to the polycrystalline material are in the direction of the saturation magnetization. Since the average values of $\sin^2\theta$, $\sin^4\theta$, and $\sin^6\theta$ over the sphere are given by the integrals

$$\int_0^{\pi/2}\sin^3\theta d\theta=\tfrac{2}{3};\quad \int_0^{\pi/2}\sin^5\theta d\theta=\frac{8}{15};\quad \int_0^{\pi/2}\sin^7\theta d\theta=\frac{48}{105},\tag{46}$$

the average value for a magnetostrictive cobalt rod is

$$\begin{aligned}\lambda_{\parallel}&=\frac{2}{3}A'+\frac{8}{15}B'+\frac{48}{105}C'=\frac{2}{3}\left(8\lambda_3-\frac{8}{3}\lambda_2+\lambda_1\right)\\ &+\frac{8}{15}\left(-\frac{56}{3}\lambda_3+\frac{40}{3}\lambda_2-\frac{16}{3}\lambda_1\right)\\ &+\frac{48}{105}\left(\frac{32}{3}\lambda_3-\frac{32}{3}\lambda_2+\frac{16}{3}\lambda_1\right)-\frac{48}{105}(\lambda_1-\lambda_4)\\ &=\frac{16}{63}\lambda_3+\frac{16}{35}\lambda_2+\frac{2}{63}\lambda_1+\frac{8}{35}\lambda_4.\end{aligned}\tag{47}$$

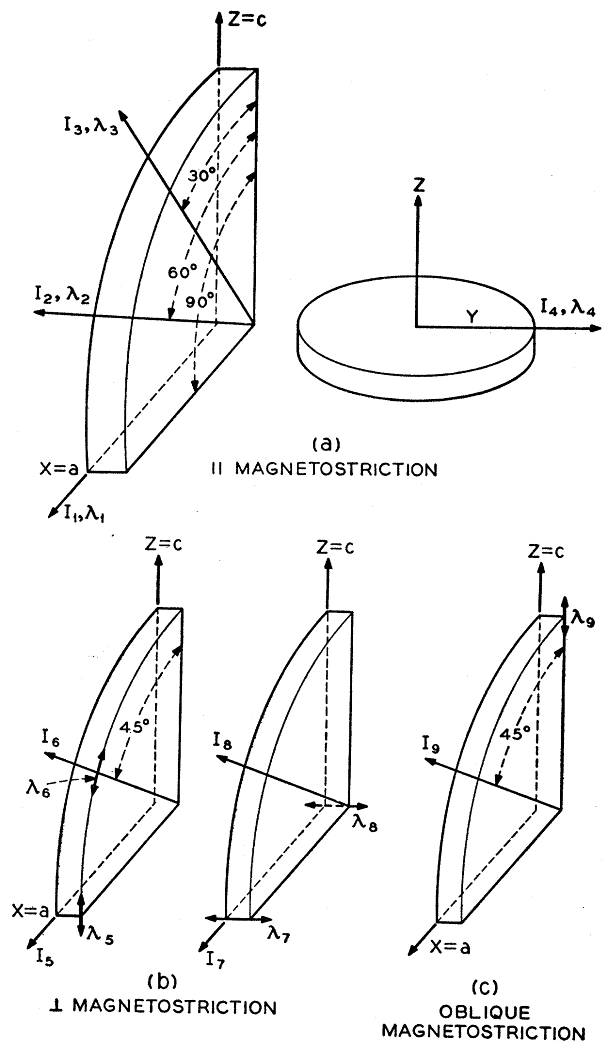

FIG. 4. Crystal cuts for measuring nine independent constants in magnetostriction equations for hexagonal crystal.

When the field is perpendicular to the direction of measurement,

$$\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=0.\tag{48}$$

With this simplification,

$$\begin{aligned}\lambda_{\perp}=&A[(\alpha_1^2-3\alpha_2^2)\beta_2-2\alpha_2\alpha_3\beta_3]^2+(B-I)\alpha_3^4\beta_3^2\\ &-B\alpha_3^2(\alpha_1\beta_2-\alpha_2\beta_1)^2+(C-H)\alpha_3^2\beta_3^2-C(\alpha_1\beta_2-\alpha_2\beta_1)^2\\ &+D(1-\alpha_3^2)(1-\beta_3^2)+E\alpha_3^2\beta_3^2(1-\alpha_3^2)\\ &+F\alpha_3^2(1-\alpha_3^2)+G\beta_3^2(1-\alpha_3^2).\end{aligned}\tag{49}$$

If we introduce spherical coordinates as shown by Fig. 3, with the magnetization directed along the z' axis and the magnetostriction measured along x', the direction cosines become

$$\alpha_1=\sin\theta\cos\varphi;\quad \alpha_2=\sin\theta\sin\varphi;\quad \alpha_3=\cos\theta;\tag{50}$$

$$\beta_1=\cos\theta\cos\varphi\cos\psi-\sin\varphi\sin\psi;\quad \beta_2=\cos\theta\sin\varphi\cos\psi+\cos\varphi\sin\psi;\quad \beta_3=-\sin\theta\cos\psi.$$

By inserting these in Eq. (49) and reducing and collecting terms, the perpendicular magnetization can be written in the form

$$\lambda_\perp = A\sin^4\theta[\cos\theta\sin\varphi\cos\psi(3\cos^2\varphi-\sin^2\varphi) +\sin^2\psi\cos\varphi(\cos^2\varphi-3\sin^2\varphi)]^2+\sin^2\theta \times[(B+C+D+F-H-I)\cos^2\psi+(D+F-B-C) \times\sin^2\psi]+\sin^4\theta[(2I+H+E+G-2B-C-D-F) \times\cos^2\psi+(B-F)\sin^2\psi]+\sin^6\theta[B-E-I]\cos^2\psi. \quad (51)$$

A has previously been derived from parallel measurements, and since the last term is $-(C'+D')$ this equation determines four more constants. By designating

$$B+C+D+F-H-I=E', \quad 2I+H+E+G-2B-C-D-F=F', \quad D+F-B-C=G', \quad B-F=H', \quad (52)$$

these four constants can be related to four new magnetostrictive measurements. As shown by Fig. 4(b) and (c), all of these measurements can be made on a single slab in the x-z plane. The constants λ_5 and λ_6, measured when $\psi=0$, are measured when the saturation magnetization is 90° and 45° from the z axis and the magnetostriction is measured in the plane perpendicular to the magnetization. The other two constants, λ_7 and λ_8, are measured along the plane thickness for the same directions of magnetization. In all these measurements, $\varphi=0$. For the first set, $\psi=0$ and the equations become

$$\lambda_5=E'+F'-(C'+D'); \lambda_6=\tfrac{1}{2}E'+\tfrac{1}{4}F'-\tfrac{1}{8}(C'+D'). \quad (53)$$

The second set are obtained when $\psi=90°$, so that

$$\lambda_7=G'+H'+A; \quad \lambda_8=\tfrac{1}{2}G'+\tfrac{1}{4}(H'+A). \quad (54)$$

Since $A=\lambda_4-\lambda_1$, we can solve these equations simultaneously for E', F', G', and H', obtaining

$$E'=4\lambda_6-\lambda_5-\tfrac{1}{2}(C'+D'); \; F'=2\lambda_5-4\lambda_6+\tfrac{3}{2}(C'+D'); \; G'=4\lambda_8-\lambda_7; \quad H'=2\lambda_7-4\lambda_8-A. \quad (55)$$

Since $C'+D'=(32/3\lambda_3-(32/3)\lambda_2+(16/3)\lambda_1$ and $A=\lambda_4-\lambda_1$, these values are related to the measured magnetostriction values by the equations

$$E'=4\lambda_6-\lambda_5-(16/3)\lambda_3+(16/3)\lambda_2-(8/3)\lambda_1;$$
$$F'=2\lambda_5-4\lambda_6+(48/3)\lambda_3-(48/3)\lambda_2+(24/3)\lambda_1;$$
$$G'=4\lambda_8-\lambda_7; \quad H'=2\lambda_7-4\lambda_8+\lambda_1-\lambda_4. \quad (56)$$

If we introduce the A to I values in the expression for A' to H' of Eqs. (43) and (56) and solve simultaneously for the unprimed values, we determine A, C, D, and G uniquely and have four more relations between the other five constants.

To determine the other constant requires a measurement for which the magnetization and magnetostriction are not parallel or perpendicular. The simplest method for determining the measurement to make is to introduce a vector for the saturation magnetization having the direction cosines

$$\alpha_1=\sin\theta_1\cos\varphi_1; \quad \alpha_2=\sin\theta_1\sin\varphi_1; \quad \alpha_3=\cos\theta_1, \quad (57)$$

and another vector for the magnetostriction having the direction cosines

$$\beta_1=\sin\theta_2\cos\varphi_2; \quad \beta_2=\sin\theta_2\cos\varphi_2; \quad \beta_3=\cos\theta_2. \quad (58)$$

If these values are introduced into the general expression, Eq. (39), the equation becomes

$$\lambda=A\sin^4\theta_1\sin^2\theta_2\sin^2(2\varphi_1+\varphi_2) +\tfrac{1}{4}B\sin^2 2\theta_1\sin^2\theta_2[\cos(\varphi_2-\varphi_1)+\sin(\varphi_2-\varphi_1)]^2 +C\sin^2\theta_1\sin^2\theta_2[\cos(\varphi_2-\varphi_1)+\sin(\varphi_2-\varphi_1)]^2 +D\sin^2\theta_1\sin^2\theta_2+\tfrac{1}{4}E\sin^2 2\theta_1\cos^2\theta_2+\tfrac{1}{4}F\sin^2 2\theta_1 +G\sin^2\theta_1\cos^2\theta_2+\tfrac{1}{4}H\sin 2\theta_1\sin 2\theta_2\cos(\varphi_2-\varphi_1) +\tfrac{1}{4}I\cos^2\theta_1\sin 2\theta_1\sin 2\theta_2\cos(\varphi_2-\varphi_1). \quad (59)$$

An examination of this equation shows that if

$$\theta_1=45°; \quad \theta_2=0°; \quad \varphi_1=\varphi_2=0; \quad \text{i.e.,} \quad \alpha_1=\alpha_3=1/\sqrt{2}; \quad \alpha_2=0; \quad \beta_1=\beta_2=0; \quad \beta_3=1, \quad (60)$$

the measured magnetostriction λ_9 will equal

$$\lambda_9=\tfrac{1}{4}(E+F)+\tfrac{1}{2}G, \quad \text{or} \quad E+F=4\lambda_9-2\lambda_5. \quad (61)$$

This orientation is shown by Fig. 4(c). This gives enough relations to solve for all the constants in terms of the measured values, and we find

$$A=\lambda_4-\lambda_1; \quad B=-2\lambda_9-2\lambda_8+\lambda_7+2\lambda_6-\tfrac{1}{2}\lambda_4 +(4/3)\lambda_3+(4/3)\lambda_2-(5/6)\lambda_1; \quad C=\tfrac{1}{2}(\lambda_4-\lambda_7);$$
$$D=\lambda_1+\tfrac{1}{2}(\lambda_7-\lambda_4); \quad E=6\lambda_9-2\lambda_8+\lambda_7-2\lambda_6-2\lambda_5 -\tfrac{1}{2}\lambda_4-(4/3)\lambda_3-(4/3)\lambda_2+(11/6)\lambda_1;$$
$$F=-2\lambda_9+2\lambda_8-\lambda_7+2\lambda_6+\tfrac{1}{2}\lambda_4+(4/3)\lambda_3 +(4/3)\lambda_2-(11/6)\lambda_1; \quad G=\lambda_5;$$
$$H=4\lambda_9-4\lambda_6-\lambda_5-(16/3)\lambda_3+(16/3)\lambda_2-(5/3)\lambda_1;$$
$$I=-8\lambda_9+4\lambda_6+2\lambda_5+(40/3)\lambda_3 -(24/3)\lambda_2+(8/3)\lambda_1. \quad (62)$$

These values can be checked by direct substitution. As shown by Fig. 4(a), (b), (c), all of these nine constants can be measured from two oriented slabs, one in the x-z plane and the other in the basal x-y plane.

Another relation of interest is the value of the volume magnetostriction in terms of these constants and the direction of the saturation magnetization. This can be calculated by introducing the direction cosines of Eq. (21) into the more general expression, Eq. (39), and carrying out the summation. This results in the equation

$$\bar{\omega}=(2D+E+3F+G)\sin^2\theta+(A-E-3F)\sin^4\theta. \quad (63)$$

Introducing the value of these constants in terms of the λ's, we find

$$\omega=\{4[\lambda_8+\lambda_6+\tfrac{2}{3}(\lambda_2+\lambda_3)-\tfrac{1}{6}\lambda_1]-[\lambda_1+\lambda_5+\lambda_7]\}\sin^2\theta +\{-4[\lambda_8+\lambda_6+\tfrac{2}{3}(\lambda_2+\lambda_3)-\tfrac{1}{6}\lambda_1] +2(\lambda_1+\lambda_5+\lambda_7)\}\sin^4\theta. \quad (64)$$

Since $\frac{2}{3}(\lambda_2+\lambda_3)-\frac{1}{6}\lambda_1=\lambda_D$ of Sec. II, this equation reduces to

$$\omega=[4(\lambda_8+\lambda_6+\lambda_D)-(\lambda_1+\lambda_5+\lambda_7)]\sin^2\theta +[-4(\lambda_8+\lambda_6+\lambda_D)+2(\lambda_1+\lambda_5+\lambda_7)]\sin^4\theta. \quad (65)$$

The average value for a polycrystalline material is

$$\bar{\omega}=\tfrac{2}{3}(2D+E+3F+G)+(8/15)(A-E-3F) =(8/15)(\lambda_8+\lambda_6+\lambda_D)+\tfrac{2}{3}(\lambda_1+\lambda_5+\lambda_7). \quad (66)$$

The perpendicular component then is

$$\lambda_\perp=\tfrac{1}{2}(\bar{\omega}-\lambda_{\parallel})=(4/15)(\lambda_6+\lambda_8)+(1/5)(\lambda_5+\lambda_7) +(16/315)\lambda_3-(16/315)\lambda_2 +(51/315)\lambda_1-(4/35)\lambda_4. \quad (67)$$

The morphic R constants, which determine the change in elastic constants due to the change in symmetry resulting from magnetization can be determined in a similar manner, but since no measurements have been made for cobalt, they will not be derived here.

IV. ANISOTROPY ENERGY DUE TO MAGNETOSTRICTION

The measurements of the anisotropic energy are carried out at constant stress so that the lattice is allowed to deform under the action of the magnetostriction forces. Hence, part of the anisotropic energy is due to magnetostrictive strains. Since it is desirable to determine how much anisotropic energy is inherent in the lattice and how much occurs due to magnetostriction, a calculation is given for the first-order magnetostrictive terms of Sec. II.

This value can be determined by evaluating the magnetostrictive energy that is required to go from a condition of constant stress to constant strain and subtract this from the anisotropic energy at constant stress. From Eq. (5), we find that the six components of strain are given by the equations

$$\begin{aligned}
S_1&=s_{11}{}^IT_1+s_{12}{}^IT_2+s_{13}{}^IT_3+\lambda_A\alpha_1{}^2+\lambda_B\alpha_2{}^2,\\
S_2&=s_{12}{}^IT_1+s_{11}{}^IT_2+s_{13}{}^IT_3+\lambda_A\alpha_2{}^2+\lambda_B\alpha_1{}^2,\\
S_3&=s_{13}{}^I(T_1+T_2)+s_{33}{}^IT_3+\lambda_C(1-\alpha_3{}^2),\\
S_4&=s_{44}{}^IT_4+(-\lambda_A-\lambda_C+4\lambda_D)\alpha_2\alpha_3,\\
S_5&=s_{44}{}^IT_5+(-\lambda_A-\lambda_C+4\lambda_D)\alpha_1\alpha_3,\\
S_6&=2(s_{11}{}^I-s_{12}{}^I)T_6+2(\lambda_A-\lambda_B)\alpha_1\alpha_2.
\end{aligned} \quad (68)$$

Hence, in the absence of any applied stresses T_1 to T_6, the crystal will be strained by the values on the right-hand side of each equation. To determine the anisotropic energy due to magnetostriction, we have to calculate the energy required to distort the crystal so that the resultant strains are all zero. To perform this calculation, it is desirable to express the stresses in terms of the strains, which can be done by solving Eq. (68) for the T's when all the magnetostrictive strains are zero. This results in the equations

$$T_1=c_{11}{}^IS_1+c_{12}{}^IS_2+c_{13}{}^IS_3;\quad T_2=c_{12}{}^IS_1+c_{11}{}^IS_2+c_{13}{}^IS_3;$$
$$T_3=c_{13}{}^I(S_1+S_2)+c_{33}{}^IS_3;\quad T_4=c_{44}{}^IS_4;$$
$$T_5=c_{44}{}^IS_5;\quad T_6=\tfrac{1}{2}(c_{11}{}^I-c_{12}{}^I)S_6; \quad (69)$$

where

$$2c_{11}{}^I=\frac{s_{33}{}^I}{\alpha}+\frac{1}{s_{11}{}^I-s_{12}{}^I};\quad 2c_{12}{}^I=\frac{s_{33}}{\alpha}-\frac{1}{s_{11}{}^I-s_{12}{}^I};$$
$$c_{13}{}^I=\frac{-s_{13}{}^I}{\alpha};\quad c_{33}{}^I=\frac{s_{11}{}^I+s_{12}{}^I}{\alpha};\quad c_{44}{}^I=\frac{1}{s_{44}{}^I};$$
$$\frac{c_{11}{}^I-c_{12}{}^I}{2}=\frac{1}{2(s_{11}{}^I-s_{12}{}^I)};\quad \alpha=s_{33}{}^I(s_{11}{}^I+s_{12}{}^I)-2s_{13}I^2.$$

The total energy required is calculated from the formula

$$E_A{}^T-E_A{}^S=\tfrac{1}{2}[T_1S_1+T_2S_2+T_3S_3 +T_4S_4+T_5S_5+T_6S_6], \quad (70)$$

since the strain changes from the value given in Eq. (68) to zero during the motion. Introducing the negative of the values of S_1 to S_6 of Eq. (68), when the T's are zero, into Eq. (69) to determine the values of T and inserting both the S and T values in Eq. (70), the energy required to erase the magnetostrictive strains is

$$E_A{}^T-E_A{}^S=\tfrac{1}{2}\{(1-\alpha_3{}^2)^2[c_{11}{}^I(\lambda_A{}^2+\lambda_B{}^2) +2c_{12}{}^I\lambda_A\lambda_B+2c_{13}{}^I(\lambda_A+\lambda_B)\lambda_C+c_{33}{}^I\lambda_C{}^2] +c_{44}{}^I(1-\alpha_3{}^2)\alpha_3{}^2(-\lambda_A-\lambda_C+4\lambda_D)^2\}. \quad (71)$$

Hence, since $(1-\alpha_3{}^2)^2=\sin^4\theta$, $(1-\alpha_3{}^2)\alpha_3{}^2=\sin^2\theta\cos^2\theta =\sin^2\theta-\sin^4\theta$, the anisotropic energy at constant strain can be written in the form

$$E_A{}^S=[\tfrac{1}{2}(K_1{}^T-K_3{}^T)+K_{13}{}^T-K_{33}{}^T-\tfrac{1}{2}c_{44}{}^I(-\lambda_A-\lambda_C+4\lambda_D)^2]\sin^2\theta+\{\tfrac{1}{2}(K_{11}{}^T+K_{33}{}^T-4K_{13}{}^T)-\tfrac{1}{2}[c_{11}{}^I(\lambda_A{}^2+\lambda_B{}^2)+2c_{12}{}^I\lambda_A\lambda_B+2c_{13}{}^I(\lambda_A+\lambda_B)\lambda_C+c_{33}{}^I\lambda_C{}^2-c_{44}{}^I(-\lambda_A-\lambda_C+4\lambda_D)^2]\}\sin^4\theta. \quad (72)$$

ACKNOWLEDGMENTS

The writer is indebted to Dr. R. M. Bozorth for suggesting this problem. Dr. P. W. Anderson has independently derived the expression for the magnetostriction in a tetragonal crystal as discussed in the appendix. Dr. J. A. Lewis has independently derived the first-order constants for a crystal with circular symmetry.

APPENDIX A. CALCULATIONS FOR TEGRAGONAL AND ORTHORHOMBIC CRYSTALS

The first-order approximations discussed in Sec. II have been extended to tetragonal and orthorhombic crystals, since cobalt ferrite, heat-treated in a magnetic field, probably crystallizes in one of these two systems. Since the procedures necessary to calculate these constants have already been discussed in Sec. II, only the final results are given. In agreement with experiment, it is assumed that the easy direction of magnetization lies along the z axis, and all the formulas given are for the difference between the saturated conditions for any direction and the demagnetized condition with equal numbers of domains directed along $\pm z$.

For tetragonal crystals,[5] the anisotropy energy E_A, magnetostriction λ_S, and difference between anisotropy energy at constant stress and constant strain, $E_A{}^T-E_A{}^S$, are:

$$E_A{}^T=K_1\sin^2\theta+K_2\sin^4\theta+K_3\sin^4\theta\sin^2\varphi\cos^2\varphi, \quad (73)$$

$$\lambda=\tfrac{1}{2}\lambda_1[(\alpha_1\beta_1-\alpha_2\beta_2)^2-(\alpha_1\beta_2+\alpha_2\beta_1)^2+(1-\beta_3{}^2)(1-\alpha_3{}^2)-2\alpha_3\beta_3(\alpha_1\beta_1+\alpha_2\beta_2)]+4\lambda_2\alpha_3\beta_3(\alpha_1\beta_1+\alpha_2\beta_2)+4\lambda_3\alpha_1\alpha_2\beta_1\beta_2+\lambda_4[\beta_3{}^2(1-\alpha_3{}^2)-a_3\beta_3(\alpha_1\beta_1+\alpha_2\beta_2)]+\tfrac{1}{2}\lambda_5[(\alpha_1\beta_2-\alpha_2\beta_1)^2-(\alpha_1\beta_1+\alpha_2\beta_2)^2+(1-\beta_3{}^2)(1-\alpha_3{}^2)], \quad (74)$$

where

$$\lambda_1=\lambda(\alpha_1=1,\beta_1=1);\quad \lambda_2=\lambda(\alpha_1=\beta_1=\alpha_3=\beta_3=1/\sqrt{2});$$
$$\lambda_3=\lambda(\alpha_1=\alpha_2=\beta_1=\beta_2=1/\sqrt{2});$$
$$\lambda_4=\lambda(\alpha_1=1,\beta_3=1);\quad \lambda_5=\lambda(\alpha_1=1,\beta_2=1);$$

$$E_A{}^T-E_A{}^S=\{\tfrac{1}{2}c_{44}(4\lambda_2-\lambda_1-\lambda_4)^2\sin^2\theta+[c_{11}(\lambda_1{}^2+\lambda_5{}^2)+c_{33}\lambda_4{}^2+2c_{12}\lambda_1\lambda_5+2c_{13}(\lambda_1+\lambda_5)\lambda_4-c_{44}(4\lambda_2-\lambda_1-\lambda_4)^2]\sin^4\theta+[-2(c_{11}-c_{12})(\lambda_1-\lambda_5)^2+c_{66}(4\lambda_3-2(\lambda_1+\lambda_5))^2]\sin^4\theta\sin^2\varphi\cos^2\varphi\}. \quad (75)$$

For orthorhombic crystals,

$$E_A{}^T=\sin^2\theta[K_1\cos^2\varphi+K_2\sin^2\varphi]+\sin^4\theta[K_3\cos^4\varphi+K_4\sin^2\varphi\cos^2\varphi+K_5\sin^4\varphi]+\sin^2\theta\cos^2\theta[K_6\cos^2\varphi+K_7\sin^2\varphi], \quad (76)$$

$$\lambda=\lambda_1[\alpha_1{}^2\beta_1{}^2-\alpha_1\alpha_2\beta_1\beta_2-\alpha_1\alpha_3\beta_1\beta_3]+\lambda_2[\alpha_2{}^2\beta_1{}^2-\alpha_1\alpha_2\beta_1\beta_2]+\lambda_3[\alpha_1{}^2\beta_3{}^2-\alpha_1\alpha_2\beta_1\beta_2]+\lambda_4[\alpha_2{}^2\beta_2{}^2-\alpha_1\alpha_2\beta_1\beta_2-\alpha_2\alpha_3\beta_2\beta_3]+\lambda_5[\alpha_1{}^2\beta_3{}^2-\alpha_1\alpha_3\beta_1\beta_3]+\lambda_6[\alpha_2{}^2\beta_3{}^2-\alpha_2\alpha_3\beta_2\beta_3]+4\lambda_7(\alpha_1\alpha_2\beta_1\beta_2)+4\lambda_8\alpha_1\alpha_3\beta_1\beta_3+4\lambda_9\alpha_2\alpha_3\beta_2\beta_3, \quad (77)$$

where

$$\lambda_1=\lambda(\alpha_1=1,\beta_1=1);\quad \lambda_2=\lambda(\alpha_2=1,\beta_1=1);$$
$$\lambda_3=\lambda(\alpha_1=1,\beta_2=1);\quad \lambda_4=\lambda(\alpha_2=1,\beta_2=1);$$
$$\lambda_5=\lambda(\alpha_1=1,\beta_3=1);\quad \lambda_6=\lambda(\alpha_2=1,\beta_3=1);$$
$$\lambda_7=\lambda(\alpha_1=\beta_1=\alpha_2=\beta_2=1/\sqrt{2});$$
$$\lambda_8=\lambda(\alpha_1=\alpha_3=\beta_1=\beta_3=1/\sqrt{2});$$
$$\lambda_9=\lambda(\alpha_2=\alpha_3=\beta_2=\beta_3=1/\sqrt{2});$$

$$E^T-E^S=\tfrac{1}{2}(\sin^2\theta-\sin^4\theta)[c_{55}(4\lambda_8-(\lambda_1+\lambda_5))^2\cos^2\varphi+c_{44}(4\lambda_9-(\lambda_4+\lambda_6))^2\sin^2\varphi]+\tfrac{1}{2}\sin^4\theta\{(c_{11}\lambda_1{}^2+2c_{12}\lambda_1\lambda_3+2c_{13}\lambda_1\lambda_5+c_{22}\lambda_3{}^2+2c_{23}\lambda_3\lambda_5+c_{33}\lambda_5{}^2)\cos^4\varphi+2[c_{11}\lambda_1\lambda_2+c_{12}(\lambda_1\lambda_4+\lambda_2\lambda_3)+c_{13}(\lambda_1\lambda_6+\lambda_2\lambda_5)+c_{22}\lambda_3\lambda_4+c_{23}(\lambda_3\lambda_6+\lambda_4\lambda_5)+c_{33}\lambda_5\lambda_6+\tfrac{1}{2}c_{66}(4\lambda_7-(\lambda_1+\lambda_2+\lambda_3+\lambda_4))^2]\sin^2\varphi\cos^2\varphi+(c_{11}\lambda_2{}^2+2c_{12}\lambda_2\lambda_4+2c_{13}\lambda_2\lambda_6+c_{22}\lambda_4{}^2+2c_{23}\lambda_4\lambda_6+c_{33}\lambda_6{}^2)\sin^4\varphi\}. \quad (78)$$

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[5] For tetragonal crystals the magnetostriction formula was first calculated by P. W. Anderson with results similar to those given here.